The Benefit of Hybrid Lateral Transshipments under Periodic Replenishment

Abstract

Lateral transshipments are a method of responding to shortages of stock in a network of inventory-holding locations. Conventional *reactive* approaches only transship to meet immediate shortages. The paper proposes *hybrid* transshipments which exploit economies of scale in moving additional stock between locations to prevent future shortages in addition to meeting immediate ones. The setting we consider is motivated by the case of retailers who operate networks of outlets supplying car parts via a system of periodic replenishment. It is pioneering in the literature in featuring a stochastic model for demand which is non-stationary and permits general patterns of dependence between the many item types available for sale. It also allows for the application of both lost sales and backorder penalty costs. The generality of our setting makes the work widely applicable. We develop an easy-to-compute quasi-myopic heuristic for determining how enhanced transshipments should be made. We obtain simple characterisations of the heuristic and demonstrate its strong cost performance in both small and large networks in an extensive numerical study.

Keywords: Multi-Item Inventory Control, Lateral Transshipments, Dynamic Programming

1 Introduction

In modern inventory networks increased information offers inventory managers the opportunity to pool risk through cooperation between replenishment points. Lateral transshipments are stock movements between locations in the same echelon of an inventory system. This transportation of goods can be used to rebalance stock proactively across the system or to address shortages reactively as they occur.

When a *reactive* transshipment is triggered in response to a shortage, conventional policies only meet the immediate shortfall. Stock is moved from a location with a surplus to the one experiencing the shortage. However, in many practical scenarios a large proportion of the associated vehicle, fuel and labour costs are independent of the amount transshipped. When this is the case, conventional reactive approaches ignore potential economies of scale which may make it beneficial to transship more than is required to meet the immediate shortfall in order to reduce the risk of future shortages. The rebalancing of stock achieved thereby has hitherto been associated with *proactive* approaches to transshipment. The latter have conventionally reallocated the entire network's inventory at isolated time points. While such approaches can be beneficial, the infrastructure required to support a large number of locations interacting simultaneously can make this costly, especially in networks with a large geographical spread. Focussing stock rebalancing on pairs of locations significantly reduces this burden. The paper proposes a *hybrid* approach which exploits the above potential for cost savings by using transshipments to rebalance stock between pairs of locations when a shortage occurs in one of them. Since the triggering mechanism is as for conventional reactive transshipments, the additional implementation overhead is minimal.

Transshipments have been studied in a variety of scenarios. An application context of particular interest to us is the sale of car parts including tyres, shock absorbers and exhaust systems from networks of depots which typically also have facilities for undertaking repairs. Common features of such networks include the following: stock replenishment is from a central depot which sends out large trucks on tours to resupply parts of the network. The determination of such tours will involve many problem features which lie outside of our model and we shall suppose throughout that the periodicity of each location's replenishment is fixed. The nature of the replenishment process means that it is unlikely that all depots in the network would be replenished on the same day. Depots are usually located in centres of population where rents are high and space for inventory is limited. Hence space constraints may be such that inventory levels are forced lower than an (unconstrained) economic analysis might indicate. This feature in part fuels the need for the pooling of inventory across the network. Demand for items is likely non-stationary as well as stochastic. For example the demand pattern at weekends may well be different from that during the working week. Further, individual customers are unlikely to require a single item. Individual demands will typically be for several of each of several item types.

The model considered in the paper, which assumes the periodic review and replenishment of stock, develops the reactive transshipment model of Archibald et al. (2010) in a way which captures the features mentioned in the preceding paragraph. It is pioneering in the literature in the generality of its characterisation of demand. Demand instances are assumed to occur in a non-stationary manner, while individual customer requirements (how many of each item type) are drawn from a general joint distribution. This is a huge advance over the customary assumption of a stationary Poisson process of singleton demands for one item type only. Hence the work not only delivers significant cost savings over current proposals but is also relevant to a wider range of practical scenarios.

In summary, key contributions of the paper include:

1. The introduction of a hybrid approach to transshipment which meets shortages and rebalances stock in a periodic review setting;
2. Its pioneering consideration of non-stationary compound stochastic demand for several item types;
3. Characterisations of the structure of the hybrid transshipment policies developed;
4. A numerical lower bound on achievable costs for some cases in which locations are replenished simultaneously;
5. An extensive numerical study in which considerable cost benefits from both the hybrid approach and from modelling generality are evidenced. Our numerical evidence suggests that our hybrid proposal closes a large part of the suboptimality gap left by purely reactive approaches.

In Section 2 we review the existing literature in the context of our model which we describe in detail in Section 3. Section 4 is the analytical heart of the paper. There we develop our hybrid transshipment heuristic in our non-stationary multi-item type setting. We also develop some simple characterisations of the structure of the policy and discuss the issue of replenishment levels. Further, an easily computed numerical lower bound is developed for the cost rate achievable in some contexts in which all locations are replenished simultaneously. The simulation study designed to evaluate the performance of the new policy is outlined in Section 5. Results elucidate the considerable performance gain achieved over existing approaches.

2 Literature

Research on transshipments in inventory networks has primarily focussed on their use in the context of stationary demand for a single item type. The broad approach taken to transshipping has been *reactive, proactive* or a *hybrid* of the two. We now consider these approaches in turn.

Much of the literature on reactive transshipments assumes periodic replenishment. Krishnan and Rao (1965) assume demand is met at the end of each review period, so transshipments can be arranged after all demand for the period has been observed. Taking a similar approach to the modelling of demand, Robinson (1990) shows that an order-up-to policy is optimal while Lien et al. (2011) explore optimal network configurations. In many situations customers require, or at least value, immediate service. An assumption that demand for a period can be observed before transshipments are planned is plainly not always appropriate. Archibald et al. (1997) allow multiple transshipments in each review period. A location makes a transshipment request whenever a shortage occurs, but transshipment requests are not always met (a situation known as partial pooling). The form of an optimal replenishment and transshipment policy is established for networks with two locations. Çömez et al. (2012) characterise an optimal transshipment policy for two locations in a similar setting with positive replenishment and transshipment lead times. Archibald (2007) and Archibald et al. (2010) also consider reactive transshipments whenever a shortage occurs, but develop heuristic policies that can be applied to networks of any size. The current work extends the latter *inter alia* by introducing a proactive element into the transshipment policies considered and through its much more general setting of non-stationary demand for many item types.

There are several other inventory problems that are related to the reactive transshipment problem. These generally arise in a multi-product setting where substitution with products of higher specification (Rao et al., 2004; Xu et al., 2011) and allocation of stocks of unfinished products

(Swaminathan and Tayur, 1998) or common components (Gerchak and Henig, 1986) serve a similar purpose to transshipments. Of particular relevance to the current work, Xu et al. (2011) use a non-stationary compound Poisson process to model demand for two mutually substitutable products. Products in this stream of work correspond to locations in transshipment research, so the problems addressed correspond to single item transshipment problems. In contrast, the current work considers a multiple item transshipment problem. A further fundamental distinction between this literature and the current work is that the nature of the problems considered provide no incentive for proactive action and decisions about which items to 'transship' and in what quantity are purely reactive. They thus stand in some contrast to the exploration of hybrid transshipment which is the focus of the current paper.

Research on proactive transshipment focuses predominantly on periodic replenishment. It is possible to think of proactive transshipments as including an element of reactive transshipment in the sense that they aim to rebalance inventory to best satisfy existing shortages and future demand (Lee et al., 2007). However in most cases, transshipment is only allowed at fixed points in each review period (Gross, 1963; Lee et al., 2007). The approach of Agrawal et al. (2004) is closest in spirit to the current work as the timing of transshipment is determined dynamically. However, in contrast to the current work, only one proactive transshipment is allowed per period and inventory is redistributed across all locations. These papers all demonstrate some benefit from stock rebalancing which purely reactive approaches do not exploit.

Reactive and proactive transshipments have also been considered in the context of continuous review replenishment, but this is of less relevance to the current work which focuses on periodic review. For a more detailed review of the transshipment literature, the reader is referred to Paterson et al. (2010).

Zhao et al. (2008) consider reactive and proactive transshipment in a single model. Their production based model uses a conventional reactive transshipment when shortages occur but also separately allocates new stock when it is produced. Hybrid transshipments of the type considered in this paper have previously only been considered by Paterson et al. (2012). That study had rather different applications in mind and featured an approach to stock replenishment based on continuous review of stock levels rather than the periodic review featured here. The methods used are quite different from those in the current work. Further, Paterson et al. (2012) utilised a simple compound Poisson model for the demand of a single item type only.

We are unaware of any contributions in the literature which match the generality of our modelling of demand. Very few consider either non-stationary demand or many item types. Herer and Tzur (2001, 2003) do consider time-varying demand but it is deterministic. Hence they can plan for fully known future demand in a manner which is not possible in a stochastic setting. In Archibald et al. (1997), replenishment decisions for many item types are linked via a constraint on storage space while in Wong et al. (2005) and Kranenburg and van Houtum (2009) there is a linking constraint on average service time. The concluding paragraphs of the Introduction emphasise the generality of our stochastic non-stationary multi-item model for the demand experienced at locations in the network. This generality both represents a huge advance on previous work and also has great relevance for applications.

3 Inventory System Model

We consider a network with N locations, each of which carries an inventory of X distinct item types. Locations are replenished periodically from a central depot. The review period for location i is T_i and hence all item types at i are replenished at times $t_i^* + nT_i, n \in \mathbb{N}$, where $t_i^* \in [0, T_i)$ is the time of the first replenishment at i after 0. For reasons given in the Introduction the replenishment periods $T_i, 1 \leq i \leq N$, will be taken as given and fixed throughout the paper. Distinct locations across the network may have different review periods and so locations are not assumed to replenish simultaneously. We deploy the notation $t_i(t)$ for the time from some arbitrary $t \in \mathbb{R}^+$ until the next replenishment at i. Should a replenishment epoch for location i occur at some time $t_i^* + nT_i$, the inventory of each item type x is restored to the level $S_{ix}(t_i^* + nT_i)$. The dependence of the replenishment levels upon the time at which replenishment occurs can be exploited in cases where the non-stationarity of the demand is very strong. See Section 4.4 for further comments on the determination of the order-up-to levels $S_{ix}(t_i^* + nT_i), n \in \mathbb{N}$. Until then, we shall regard them as fixed. Due to the dependence of the replenishment levels on time, it is theoretically possible for the inventory level of an item at a replenishment epoch to exceed the intended replenishment level. For the purposes of the model we develop, it is assumed that any excess inventory at a location is removed during "replenishment". In practice this situation would be extremely rare and so this assumption will not have a significant impact on the performance of the heuristics developed.

Each location experiences stochastic demand. At location i customers arrive according to a non-homogeneous Poisson process with rate at time t given by $\lambda_i(t)$. We assume that successive demands at location i are independent and identically distributed. We shall use $\mathbf{D}_i \equiv (D_{i1}, D_{i2}, \ldots, D_{iX})$ for the random X-vector denoting a single customer demand at location i, with D_{ix} denoting the size of a single customer's demand for item type x. Plainly $P\left(\sum_{x=1}^{X} D_{ix} \geq 1\right) = 1$. We shall use the notation $f_{i\mathbf{d}} \equiv P(\mathbf{D}_i = \mathbf{d})$ for the multivariate probability mass function for location i demands and write

$$f_{ix} \equiv P(D_{ix} \geq 1) = \sum_{\{\mathbf{d}:d_x \geq 1\}} f_{i\mathbf{d}} \tag{3.1}$$

for the probability that a single customer at location i demands (at least) one item of type x. Should such a demand occur, we refer to the customer as an x-customer. A customer may be an x-customer for several distinct x. The probability mass function for the size of demands for item type x from x-customers is denoted by

$$f_{ixd} \equiv P(D_{ix} = d \mid D_{ix} \geq 1) = \frac{\sum_{\{\mathbf{d}:d_x = d\}} f_{i\mathbf{d}}}{f_{ix}}. \tag{3.2}$$

The above implies that x-customers arrive at location i according to a non-homogeneous Poisson process whose rate at time t is $f_{ix}\lambda_i(t)$ with the size of x-demand from individual x-customers determined by the mass function $\{f_{ixd}, d \geq 1\}$, the latter having finite mean and variance μ_{ix} and σ_{ix}^2 respectively. Additionally, we use f_{ixd}^n for the derived probability that n x-customers together demand exactly d of item type x at location i.

A consequence of allowing composite multivariate demand is that shortages may be of more than one piece of inventory and/or of more than one item type. However, in the description of our methodology in the next section we shall assume, in line with previous work, that transshipments come from a single location. This constraint derives principally from practice in that coordinating movements from more than one location can considerably complicate operating the policy. Further, we shall allow transshipments which meet only part of a current shortage. For some specific cases such as vehicle tyres, a complete set may be required or none at all. An indication will be given in Section 4 following (4.19) of how our methodology may be extended to allow transshipments of more complex structure and/or to meet an 'all or nothing' demand requirement.

Several costs are involved in the operation of an inventory network and most play a role in determining the potential benefit of a transshipment. The only cost exogenous to a transshipment decision is the initial cost to purchase a piece of inventory and we assume this to be constant for each item type across the system. Holding costs are incurred at location i for items of type x at a rate h_{ix} per unit of stock and per unit of time. Further, penalty costs are incurred whenever demand cannot be met immediately. Two methods of penalising unmet demand are considered. If unmet demand is assumed to be lost from the system, then a one-off cost of L_{ix} per unit of unmet demand of item type x is incurred at location i. Alternatively, the demand can be backordered with a penalty cost b_{ix} which is incurred per unit of item type x and per unit of time the item remains out of stock. We are able to address both cost structures. Finally, the cost associated with each transshipment from location i to location j has two elements: a fixed cost per transshipment R^f_{ji}, and a cost per unit of item type x transshipped, R^u_{jix}.

4 Development and analysis of the hybrid transshipment heuristic

To develop a heuristic for transshipment decisions (from where and how much) we broadly follow Axsäter (2003) and Paterson et al. (2012) in their espousal of a quasi-myopic approach to an otherwise intractable problem. This is explained in Section 4.2. Under this approach all decisions are taken in the light of an assumed future for the system which has no transshipments. Expressed technically, the dynamic transshipment policy produced is obtained by performing a single dynamic programming policy improvement step from a no transshipment policy. In order to give effect to this we need to be able to compute the expected costs incurred under a no transshipment policy. We proceed to this in the next subsection.

4.1 Expected costs incurred in the absence of transshipments

In what follows we use IL_{ix} for the inventory level of item type x at location i at some arbitrary time $t \in \mathbb{R}^+$ (deemed the current epoch) and $t + s, s \leq t_i(t)$ for some future time no later than location i's next replenishment. We write $\mathbf{IL}_i \equiv (IL_{i1}, IL_{i2}, \ldots, IL_{iX})$ for the vector of inventory levels of all item types at i and denote by $v_i \{\mathbf{IL}_i, t, s\}$ the expected inventory costs incurred at location i during the time interval $(t, t+s)$ *under an assumption that no transshipments are made in the interim.*

Before proceeding further, we note that, notwithstanding the fact that demands across distinct item types may well be correlated, the fact that the expectation operator is linear means that we have an additive decomposition of total costs into terms which give contributions from individual item types. Hence, we have

$$v_i \{\mathbf{IL}_i, t, s\} = \sum_{x=1}^{X} v_{ix} \{IL_{ix}, t, s\}, \tag{4.1}$$

where

$$v_{ix} \{IL_{ix}, t, s\} = v_{ix} \{IL_{ix}, t, s; \text{hold}\} + v_{ix} \{IL_{ix}, t, s; \text{lost}\} + v_{ix} \{IL_{ix}, t, s; \text{back}\} \tag{4.2}$$

expresses the decomposition of the total costs per item type into costs due to the holding of inventory (first term on the rhs of (4.2)) and costs associated with not being able to meet demand (second and third terms). In the case of the latter, unmet demand incurs a lost sales cost per item and/or a cost per item and time unit if demand is backordered. We adopt the model (4.2) for generality and simplicity. In practice we either have $L_{ix} > 0, b_{ix} = 0 \; \forall i, x$ (lost sales model) or $L_{ix} = 0, b_{ix} > 0$ $\forall i, x$ (backordered sales model). Please note that if $b_{ix} > 0$ any inventory level IL_{ix} may be negative, this corresponding to a number of currently backordered items, we use $IL_{ix}^+ \equiv \max(IL_{ix}, 0)$ and $IL_{ix}^- = \max(-IL_{ix}, 0)$.

In order to compute $v_{ix} \{IL_{ix}, t, s; \text{hold}\}$ we further disaggregate into a sum with a contribution from each of the IL_{ix} units of stock of type x present at location i at time t, considered in the order in which they are demanded to obtain

$$
\begin{aligned}
v_{ix}\{IL_{ix}, t, s; \text{hold}\} &= \sum_{j=1}^{IL_{ix}} E\Big(\text{holding cost from the } j^{\text{th}} \text{ unit of type } x \text{ inventory}\Big) \\
&= \sum_{j=1}^{IL_{ix}} \sum_{n=1}^{j} E\Big(\text{holding cost from the } j^{\text{th}} \text{ unit of type } x \text{ inventory} \mid \\
&\qquad\qquad j^{\text{th}} \text{ unit demanded by the } n^{\text{th}}\ x\text{-customer}\Big) \cdot P^{n}_{ixj},
\end{aligned}
\tag{4.3}
$$

where we use P^{n}_{ixj} for the probability that the j^{th} unit of type x inventory is demanded by the n^{th} x-customer at location i after time t. Please note that the quantities P^{n}_{ixj} may be easily recovered from the quantities f^{m}_{ixd} defined in Section 3.

Now choose a time $\tau \in (t, t+s)$. Recall that the number of x-customers arriving at location i during the interval (t, τ) has a Poisson distribution with mean

$$
\Lambda_{ix}(t,\tau) \equiv f_{ix} \int_{t}^{\tau} \lambda_i(u)\, \mathrm{d}u. \tag{4.4}
$$

It follows that the probability that the n^{th} x-customer after t arrives during the interval $(\tau, \tau+\delta\tau)$ has the form $q_{ix}(n,t,\tau)\,\delta\tau + o(\delta\tau)$ where

$$
q_{ix}(n,t,\tau) = f_{ix}\lambda_i(\tau) \frac{\left(\Lambda_{ix}(t,\tau)\right)^{n-1}}{(n-1)!} \exp\{-\Lambda_{ix}(t,\tau)\}. \tag{4.5}
$$

We can now evaluate the expression in (4.3) by conditioning on the time at which the n^{th} x-customer after t arrives to obtain

$$
\begin{aligned}
&E\Big(\text{holding cost from the } j^{\text{th}} \text{ unit of type } x \text{ inventory} \mid \\
&\qquad j^{\text{th}} \text{ unit demanded by the } n^{\text{th}}\ x\text{-customer}\Big) = h_{ix} \cdot \Big(A_{ix}(n,t,s) + s \cdot B_{ix}(n,t,s)\Big),
\end{aligned}
$$

where

$$
A_{ix}(n,t,s) = \int_{t}^{t+s} (\tau - t) \cdot q_{ix}(n,t,\tau)\ \mathrm{d}\tau \tag{4.6}
$$

and

$$
B_{ix}(n,t,s) = 1 - \int_{t}^{t+s} q_{ix}(n,t,\tau)\ \mathrm{d}\tau = \sum_{m=0}^{n-1} \frac{\left(\Lambda_{ix}(t,t+s)\right)^{m}}{m!} \exp\{-\Lambda_{ix}(t,t+s)\}. \tag{4.7}
$$

Substituting into (4.3) we now have that

$$v_{ix}\left\{IL_{ix}, t, s; \text{hold}\right\} = \sum_{j=1}^{IL_{ix}} \sum_{n=1}^{j} h_{ix} \cdot \Big(A_{ix}(n,t,s) + s \cdot B_{ix}(n,t,s)\Big) \cdot P^{n}_{ixj}. \tag{4.8}$$

A similar analysis readily yields that

$$v_{ix}\left\{IL_{ix}, t, s; \text{lost}\right\} = \sum_{j=IL^{+}_{ix}+1}^{\infty} \sum_{n=1}^{j} L_{ix} \cdot \Big(1 - B_{ix}(n,t,s)\Big) \cdot P^{n}_{ixj}. \tag{4.9}$$

Write $D_{ix}(t, s)$ for the demand for x-items at location i between times t and $t+s$. It is straightforward to show that the expression in (4.9) may alternatively be expressed as

$$\sum_{j=IL^{+}_{ix}+1}^{\infty} L_{ix} \cdot P\Big(D_{ix}(t,s) \geq j\Big),$$

which may be well approximated by a corresponding finite sum $\sum_{j=IL^{+}_{ix}+1}^{M_{ix}(t,s)}$ where $M_{ix}(t, s)$ is chosen to make $P\Big(D_{ix}(t,s) \geq M_{ix}(t,s)\Big)$ sufficiently small. In practice we choose

$$M_{ix}(t,s) = E\Big(D_{ix}(t,s)\Big) + 3\sqrt{var\Big(D_{ix}(t,s)\Big)} = \mu_{ix} \cdot \Lambda_{ix}(t, t+s) + 3\sqrt{\Big(\mu^{2}_{ix} + \sigma^{2}_{ix}\Big) \cdot \Lambda_{ix}(t, t+s)}.$$

For the backorder costs a similar argument to that involving the above calculation of holding costs is needed to compute $v_{ix}\left\{IL_{ix}, t, s; \text{back}\right\}$. Each unit of potential excess x-demand incurs a backorder cost over the period between the corresponding x-customer arrival time and $t + s$. Further, x-items already on backorder at t incur backorder costs over the entire period. This gives

$$v_{ix}\left\{IL_{ix}, t, s; \text{back}\right\} = b_{ix} \cdot \left[\sum_{j=IL^{+}_{ix}+1}^{\infty} \left[s - \sum_{n=1}^{j} \Big(A_{ix}(n,t,s) + s \cdot B_{ix}(n,t,s)\Big) \cdot P^{n}_{ixj}\right] + s \cdot IL^{-}_{ix}\right], \tag{4.10}$$

which may also be well approximated by a finite sum. We can now use (4.8) - (4.10) to obtain (4.2) and hence recover the key quantity $v_i\left\{\mathbf{IL}_i, t, s\right\}$ from (4.1).

4.2 Development of the hybrid heuristic via DP policy improvement

We consider a scenario in which the system has inventory levels $\{(IL_{jx}), 1 \leq x \leq X, 1 \leq j \leq N\}$ at some time $t \in \mathbb{R}^{+}$ when a demand $\mathbf{d}_i$ which cannot be fully met from local stock arises at location i. Hence $d_{ix} > IL_{ix}$ for some x. We denote by $\mathbf{z}_i$ the vector of excess demand at i, namely $(IL_{ix} - d_{ix})^-, 1 \le x \le X$. Our range of actions is considerable. We may transship from any single location with stock and we may transship any quantities which do not exceed the stock levels at the sending location. Alternatively, we may choose not to transship at all and incur costs for lost sales and/or backordered demand at i. Our definition of excess demand includes any outstanding backorders at location i (i.e. when $IL_{ix} < 0$). We assume that items in a transshipment are used to clear backorders and/or meet the current demand before building inventory to help in meeting future demand. One minor constraint we impose is that we never transship so much stock of any type that the corresponding inventory level at the receiving location exceeds its next replenishment level. Our approach to decision-making is to choose the sending location/inventory type quantities (if any) for the transshipment to minimise the expected costs incurred over *any* large horizon H under an assumption that no transshipments are made following the current decision. This is in the spirit of Axsäter (2003) and Paterson et al. (2012).

To proceed we fix horizon H to be any real number in excess of $\max_j T_j$. If the current excess demand $\mathbf{z}_i$ at location i occurring at time t is met in whole or in part through a transshipment of $u_{jix} \le IL_{jx}$ units of type x stock from $j, 1 \le x \le X$, then the costs to be incurred at both i and j over horizon H may be computed under the assumption of no transshipments beyond the current one. For sending location j this total expected cost over horizon H, namely over the time interval $(t, t + H)$, is given by the expression

$$R^f_{ji} + \sum_{x=1}^{X} \Big[R^u_{jix} \cdot u_{jix} + v_{jx} \Big\{ IL_{jx} - u_{jix},\ t,\ t_j(t) \Big\} + v_j \Big\{ t + t_j(t),\ t + H \Big\}, \tag{4.11}$$

where the quantities $v_{jx}\{\cdot, \cdot, \cdot\}$ are computed as in subsection 4.1 and $v_j \left\{ t + t_j(t),\ t + H \right\}$ is the expected cost incurred at location j under no transshipments from the time of the first replenishment after t (at time $t + t_j(t)$) until the end of the horizon (at $t + H$). Please note that this latter quantity is independent of the decision made at the current epoch t. The expression in (4.11) disaggregates the total expected cost incurred at location j over horizon H into the immediate cost of the transshipment (first two terms), the subsequent expected inventory cost until the first replenishment (third term) and the expected cost from the first replenishment to the end of the horizon (fourth term). Similarly, the total expected cost incurred over the horizon H at location i may be expressed as

$$\sum_{x=1}^{X} \left[L_{ix} \cdot \left(IL_{ix}^{+} - d_{ix} + u_{jix} \right)^{-} + v_{ix} \left\{ \widetilde{IL}_{ix}(u_{jix}),\ t,\ t_i(t) \right\} + v_i \left\{ t + t_i(t),\ t + H \right\}, \tag{4.12}$$

where $\widetilde{IL}_{ix}(u_{jix})$ represents the inventory level of item x at location i after demand and transshipment. Under the lost sales model $\widetilde{IL}_{ix}(u_{jix}) = \left(IL_{ix} + u_{jix} - d_{ix} \right)^{+}$. However, under the backordered sales model, inventory levels are not restricted to be positive and $\widetilde{IL}_{ix}(u_{jix}) = IL_{ix} + u_{jix} - d_{ix}$. The first term in (4.12) is the one-off cost associated with any unmet demand following transshipment under the lost sales model. Under the backorder sales model, backorder costs for each item of unmet demand apply for the remaining time until the next replenishment and these are absorbed into the second term in (4.12).

Finally, for any location $k \neq i, j$ which is not a party to the transshipment, the total expected cost over the horizon H may be written

$$\sum_{x=1}^{X} v_{kx} \left\{ IL_{kx}, t, t_k(t) \right\} + v_k \left\{ t + t_k(t),\ t + H \right\}. \tag{4.13}$$

Hence, aggregating over locations using (4.11) - (4.13), the total expected cost incurred across the entire network over horizon H may be expressed as

$$\begin{aligned} R_{ji}^{f} + \sum_{x=1}^{X} \Big[R_{jix}^{u} \cdot u_{jix} + L_{ix} \cdot \left(IL_{ix}^{+} - d_{ix} + u_{jix} \right)^{-} + v_{ix} \left\{ \widetilde{IL}_{ix}(u_{jix}),\ t,\ t_i(t) \right\} + v_{jx} \left\{ IL_{jx} - u_{jix},\ t,\ t_j(t) \right\} \\ - v_{ix} \left\{ IL_{ix}, t, t_i(t) \right\} - v_{jx} \left\{ IL_{jx}, t, t_j(t) \right\} \Big] + \sum_{k=1}^{N} \left[\sum_{x=1}^{X} v_{kx} \left\{ IL_{kx}, t, t_k(t) \right\} + v_k \left\{ t + t_k(t),\ t + H \right\} \right]. \end{aligned} \tag{4.14}$$

Note that the corresponding total expected cost of making no transshipment at t and subsequently none throughout $(t, t + H)$ is

$$\begin{aligned} \sum_{x=1}^{X} \left[L_{ix} \cdot (IL_{ix}^{+} - d_{ix})^{-} + v_{ix} \left\{ \widetilde{IL}_{ix}(0),\ t,\ t_i(t) \right\} - v_{ix} \left\{ IL_{ix}, t, t_i(t) \right\} \right] \\ + \sum_{k=1}^{N} \left[\sum_{x=1}^{X} v_{kx} \left\{ IL_{kx}, t, t_k(t) \right\} + v_k \left\{ t + t_k(t),\ t + H \right\} \right]. \end{aligned} \tag{4.15}$$

Our decision will be taken to secure the smallest possible value of the costs in (4.14) or (4.15).

To express this more succinctly, we develop the index $\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right)$ to reflect the benefit of making a transshipment of size $\mathbf{u}_{ji} \equiv \left\{u_{jix}, 1 \leq x \leq X\right\}$ at time t from j to i when a demand $\mathbf{d}_i$ results in a shortage at i and the inventory levels at i and j are $\mathbf{IL}_i$ and $\mathbf{IL}_j$ respectively. We write

$$\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right) = R^f_{ji} + \sum_{x=1}^{X} \Big[R^u_{jix} \cdot u_{jix} + L_{ix} \cdot \left(IL^+_{ix} - d_{ix} + u_{jix}\right)^- + v_{ix}\left\{\widetilde{IL}_{ix}(u_{jix}),\, t,\, t_i(t)\right\} + v_{jx}\left\{IL_{jx} - u_{jix},\, t,\, t_j(t)\right\} - v_{ix}\left\{IL_{ix},\, t,\, t_i(t)\right\} - v_{jx}\left\{IL_{jx},\, t,\, t_j(t)\right\}\Big] \quad (4.16)$$

and

$$\Delta\left(\mathbf{0} \mid \mathbf{d}_i, \mathbf{IL}_i, t\right) = \sum_{x=1}^{X} \left[L_{ix} \cdot \left(IL^+_{ix} - d_{ix}\right)^- + v_{ix}\{\widetilde{IL}_{ix}(0),\, t,\, t_i(t)\} - v_{ix}\{IL_{ix},\, t,\, t_i(t)\} \right] \quad (4.17)$$

for the no transshipment index. Our hybrid transshipment heuristic mandates a decision at t to achieve

$$\min\left\{ \min_{j,\mathbf{u}_{ji}} \left\{ \Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right); \Delta\left(\mathbf{0} \mid \mathbf{d}_i, \mathbf{IL}_i, t\right) \right\} \right\}, \quad (4.18)$$

where the choice of $\mathbf{u}_{ji}$ in the second minimisation in (4.18) is constrained by both the stock levels at j and by the requirement that the stock levels at i should not go above $S_{ix}\left(t + t_i(t)\right)$. Hence we require that

$$0 \leq u_{jix} \leq \min\left\{IL_{jx}, S_{ix}\left(t + t_i(t)\right) - IL_{ix} + d_{ix}\right\}, 1 \leq x \leq X, \text{ and } 0 < u_{jix} \text{ for some } x. \quad (4.19)$$

If the minimum in (4.18) is achieved by $\Delta\left(\mathbf{0} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right)$ then no transshipment is made. Otherwise, the transshipment uses the pair $(j^*, \mathbf{u}^*_{ji})$ achieving the inner minimisation.

The above approach is flexible and can accommodate a range of important model variants. We can, for example, easily extend the above to allow transshipments from more than a single location while in Sections 4.3 and 4.5 we shall suppose that transshipments may be constrained by the number or weight of items which may be included. Further, the possible 'all or nothing' nature of demand mentioned in Section 3 may be easily incorporated into the above by modifying costs in the analysis to reflect the fact that the demand $\mathbf{d}_i$ will not be lost *in total* following a shortage if and only if the triggered transshipment $\mathbf{u}_{ji}$ satisfies $IL_{ix} + u_{jix} \geq d_{ix}, 1 \leq x \leq X$.

In practice the above heuristic can be obtained with modest computational effort, especially so when X, the number of item types, is small. We recommend an online implementation of the minimisation in (4.18) which computes the key quantities $\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right)$ and $\Delta\left(\mathbf{0} \mid \mathbf{d}_i, \mathbf{IL}_i, t\right)$ as needed. An offline approach which created a library of such values up front would be hugely wasteful. In the event of a shortage at some location i, the relevant values of $\mathbf{d}_i$, $\mathbf{IL}_i$ and t are fixed and a search is prosecuted over locations $j \neq i$ and transshipment profiles $\mathbf{u}_{ji}$. The building blocks for the computation of $\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right)$ are the availability of appropriate quantities of the form $v_{ix}\left\{IL_{ix}, t, t_i(t)\right\}$ and $v_{jx}\left\{IL_{jx}, t, t_j(t)\right\}$. To obtain the complexity of computing these quantities, we write $\hat{\Lambda}_{ix} := \max_n \Lambda_{ix}(t_i^* + nT_i, t_i^* + (n+1)T_i)$ for the maximum mean x-demand at location i during any review period, with $\widehat{M}_{ix} := \mu_{ix} \cdot \hat{\Lambda}_{ix} + 3\sqrt{\left(\mu_{ix}^2 + \sigma_{ix}^2\right) \cdot \hat{\Lambda}_{ix}}$ and $\widehat{M} := \max_{i,x} \widehat{M}_{ix}$. The discussion of the computation of the quantities $v_{ix}\left\{IL_{ix}, t, t_i(t)\right\}$ following (4.9) yields the conclusion that their complexity is no worse than $O\left(\widehat{M}^2\right)$. Further, from (4.16) we see that $O(X)$ such quantities are needed to compute $\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right)$ for a single pair $(j, \mathbf{u}_{ji})$. We now write $\hat{S}_x := \max_{i,n} S_{ix}(t_i^* + nT_i)$ for the maximal replenishment level for items of type x at any location and time. It is straightforward that to compute all of the quantities in (4.18) and to implement the minimisation requires no more than $O\left(X\left(N\widehat{M}\prod_{x=1}^{X}\hat{S}_x\right)^2\right)$ computations. In practice constraints on, for example, the size of vehicles available to prosecute transshipments will mean that the number of feasible $\mathbf{u}_{ji}$ (where $\mathbf{u}_{ji}$ is *feasible* if u_{jix} x-items, $1 \leq x \leq X$, can be carried in a single transshipment from j to i) is much smaller than that calculation implies. Should F_{ji} be the number of feasible transshipments $\mathbf{u}_{ji}$ from j to i and $\hat{F} := \max_{j,i} F_{ji}$ then no more than $O\left(X\left(N\widehat{M}\hat{F}\right)^2\right)$ computations would be needed to implement an appropriate form of (4.18).

4.3 Characterisations of the hybrid heuristic

In a setup as complex as considered here, it is perhaps unsurprising that simple characterisations of effective heuristics are challenging to develop. This subsection gives a brief account of some simple and intuitive features of the hybrid heuristic which are reasonably straightforward to establish.

Theorem 1 states that our hybrid rule is monotone in the stock levels at the sending location. Hence, if in order to meet some shortage summarised by the triple $(i, \mathbf{z}_i, t)$, the rule mandates a transshipment summarised by the pair $(j^*, \mathbf{u}^*_{ji})$ when the stock levels at j are given by $\mathbf{IL}_j$ then in meeting the same shortage when the stock levels at j are uniformly above $\mathbf{IL}_j$ (with other features of the network unchanged) the rule continues to mandate a transshipment from j with the stock transshipped uniformly no less that $\mathbf{u}^*_{ji}$. In the theorem's statement we use $\preceq$ to denote the componentwise weak ordering of two X-vectors. The proof of Theorem 1 may be found in the paper's online appendix.

Theorem 1 *(The hybrid heuristic is monotone in the stock levels of the sending location)*

(a) The index $\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}_j, t\right)$ is nonincreasing componentwise in $\mathbf{IL}_j$ for all fixed values of $\mathbf{u}_{ji}$, $\mathbf{d}_i$, $\mathbf{IL}_i$ and t.

(b) If the minimisation in (4.18) is achieved by the pair $(j^*, \mathbf{u}^*_{ji})$ and if $\mathbf{IL}_j \preceq \mathbf{IL}'_j$, then the minimisation

$$\min\left\{\min_{j,\mathbf{u}_{ji}}\left\{\Delta\left(\mathbf{u}_{ji} \mid \mathbf{d}_i, \mathbf{IL}_i, \mathbf{IL}'_j, t\right); \Delta\left(\mathbf{0} \mid \mathbf{d}_i, \mathbf{IL}_i, t\right)\right\}\right\}$$

is achieved by some pair $(j^*, \mathbf{u}'_{ji})$ where $\mathbf{u}^*_{ji} \preceq \mathbf{u}'_{ji}$.

It is possible to develop this result further as follows: Suppose now that we enhance the constraint set (4.19) by adding a linear constraint of the form

$$\sum_{x=1}^{X} w_x u_{jix} \leq W_j. \tag{4.20}$$

Hence, for example, we could take $w_x = 1\ \forall x$ with W_j then the maximum number of items which can be carried in a single transshipment from j. Alternatively, w_x could be the weight of a single x-item with W_j then the maximum total weight which can be carried in a single transshipment from j. Plainly part (a) of Theorem 1 continues to hold. However, we now have a weakened form of part (b) which states that if IL_{jx} increases (with all else fixed) from a value at which the heuristic solution takes the form $(j^*, \mathbf{u}^*_{ji})$ then the supply location chosen by the heuristic will remain j^* while the amount of item x supplied will not decrease.

It is also of interest to ask how transshipment decisions made by the hybrid heuristic change as the time to the next replenishment at locations increase. The situation is complex but suppose we simplify matters by taking $X = 1$ and by supposing that all individual demands are for single items. Now consider how the transshipment decision made as a result of a shortage at i might change as the time to the next replenishment at location j increases from $t_j(t)$ to $t_j(t) + \delta$. The j-term in an appropriate form of the expression in (4.16) now changes from $v_j\{IL_j - u_{ji}, t, t_j(t)\} - v_j\{IL_j, t, t_j(t)\}$ to $v_j\{IL_j - u_{ji}, t, t_j(t) + \delta\} - v_j\{IL_j, t, t_j(t) + \delta\}$. For small δ, this change in the value of the index $\Delta\left(u_{ji} \mid d_i, IL_i, IL_j, t\right)$ can be shown to be positive if and only if IL_j is less than some quantity $\psi_j\left(t_j(t), u_{ji}\right)$ whose dependence on $t_j(t)$ may be quite complex but which is increasing in u_{ji}. The interpretation of this is that location j becomes less attractive as a potential supplier as its replenishment recedes if its inventory is sufficiently small. In this case the risk of future shortage costs at j exceeds the benefit of reduced holding costs. In general if the review periods T_i, $1 \leq i \leq N$, are increased while keeping other aspects of the system unchanged, then this increases the importance of an effective approach to the pooling of inventory. This is also the case in the analogous situation in which space constraints force replenishment levels $S_{ix}(\cdot)$ to be reduced (to below economic optima) while leaving all else the same. Our numerical results in Section 5 indicate that for the latter the cost advantage of using the hybrid approach is enhanced.

4.4 On the setting of replenishment levels

In the discussion above, replenishment levels are assumed given. In this subsection, we first give a brief account of the economically optimal setting of replenishment levels when there is no pooling of inventory between locations, i.e., no transshipments. No pooling means that distinct locations operate independently. This, together with our global assumption from Section 3 that replenishment will always be required when available, mean that it is sufficient to myopically consider how best to replenish a single location to minimise expected inventory costs incurred over a single review period. At the end of the subsection, we then describe how we deploy this analysis to establish an approach to the setting of replenishment levels in the context of the numerical study of the hybrid proposal in Section 5.

We can without loss of generality consider the optimal replenishment under no pooling of a single item (x) at a single location (i) and drop the identifier ix from the notation. In particular, we consider the choice of replenishment level S to minimise expected inventory costs $v\{S, 0, T\}$ incurred during the review period $[0, T]$. We write S^* for the optimal replenishment level, satisfying $v\{S^*, 0, T\} = \min_{S \in \mathbb{Z}^+} \left\{ v\{S, 0, T\} \right\}$. We also use D_τ for the number of items demanded during $[0, \tau]$, F_τ for its distribution function, defined by $F_\tau(n) = P(D_\tau < n)$, $n \in \mathbb{Z}^+$, and F_T^{-1} for its inverse distribution function, namely $F_T^{-1}(\beta) = \max\left\{n; F_T(n) < \beta\right\}$, $\beta \in [0, 1]$. A proof of the following result may be found in the paper's online appendix.

Proposition 1

(a) The optimal replenishment level S^* in the absence of transshipments is given by

$$S^* = \max\left\{ S \in \mathbb{Z}^+; \int_0^T (h+b) \cdot F_\tau(S)\, d\tau - L + L \cdot F_T(S) - bT < 0 \right\}. \tag{4.21}$$

(b) S^* is bounded above as follows:

$$S^* \leq F_T^{-1}\left(1 - \frac{hT}{hT + bT + L}\right). \tag{4.22}$$

(c) If $L > hT > 0$ then S^* is bounded below as follows:

$$S^* \geq F_T^{-1}\left(1 - \frac{hT}{L}\right). \tag{4.23}$$

We shall refer to the upper and lower bounds on S^* given in the above result as $\overline{S}$ and $\underline{S}$ respectively. We readily conclude that for cases of the lost sales model for which $hT \ll L$, $\overline{S}$ will be reasonably tight since then we have

$$\overline{S} = F_T^{-1}\left(1 - \frac{hT}{hT + bT + L}\right) = F_T^{-1}\left(1 - \frac{hT}{hT + L}\right) \simeq F_T^{-1}\left(1 - \frac{hT}{L}\right) = \underline{S}.$$

As we shall see, this will be enough for our purposes. We now continue by developing approximations to the upper bound $\overline{S}$ based on the normal distribution.

We can use the central limit theorem to develop a normal approximation to the distribution of the total demand D_T under the condition that the expectation $E(D_T)$ is moderately large. Recall that we use μ_d and σ_d^2 respectively for the mean and variance of the number of units demanded by a single individual. It then follows that $E(D_T) = \mu_d \Lambda(T)$ and $Var(D_T) = \left(\mu_d^2 + \sigma_d^2\right)\Lambda(T) = \left(\frac{\mu_d^2+\sigma_d^2}{\mu_d}\right)E(D_T)$. From the above we can conclude that the upper bound $\overline{S}$ on the optimal replenishment level under no pooling is well approximated by

$$\overline{S} \approx E(D_T) + \Phi^{-1}\left(1 - \frac{hT}{hT + bT + L}\right)\sqrt{\left(\frac{\mu_d^2 + \sigma_d^2}{\mu_d}\right)E(D_T)}. \tag{4.24}$$

We now restore the item/location identifier ix. Features which will be present in the numerical examples discussed in Section 5 are those of a repeating demand pattern on a weekly cycle for all items at all locations and of a review period for all items at all locations equal to an integer number of weeks. While these are certainly not necessary for the method they simplify things considerably. Replenishment levels $S_{ix}, 1 \le i \le N, 1 \le x \le X$, now need to be tailored to individual locations i and item-types x but not to the times at which the replenishments are made. From the above analysis a natural approach to the determination of replenishment levels would be to conduct an appropriate search using the above upper bounds for no pooling as a starting point. We would certainly expect that optimal replenishment levels under inventory pooling via transshipments to be somewhat lower than for no transshipments. Our numerical studies confirm this. Further, it is not unreasonable to assume common characteristics for inventory costs and for the nature of individual demands across locations. It is thus not unreasonable to suppose that replenishment levels take the form

$$S_{ix} = E(D_{ixT}) + \alpha_x\sqrt{E(D_{ixT})}, 1 \le i \le N, 1 \le x \le X \tag{4.25}$$

and conduct a search over common $\alpha_x, 1 \le x \le X$, to achieve costs which are close to minimising.

The above discussion notwithstanding, our envisaged application domain frequently features city centre locations where rents are high and space is limited. Hence it may not be possible to replenish at the levels suggested by the analysis of the cost model, as above. Indeed, a key motivation for considering transshipments at all is to operate in a way which mitigates the impact of depressed inventories at individual locations by a scheme of pooling inventory across the network. In light of this, it will be important to consider the impact of our heuristic transshipment policies when replenishment levels are set at a somewhat lower level than cost optimal. In Section 5 we shall consider the performance of our hybrid transshipment heuristic for *both* cases when replenishment levels are set in a cost minimising fashion as in the previous paragraph *and* when rather lower levels are assumed because of space constraints.

4.5 A lower bound on achievable costs when all locations replenish simultaneously

The intractability of our decision problem means that it is only possible to compare the cost performance of our proposed heuristic directly with optimal in small problems, $N \leq 3$ say. This will indeed be done as part of the numerical study in Section 5. For certain cases, we are able to further strengthen our analyses by developing lower bounds on the expected cost rate achievable under *any* policy. Such is the complexity of our setup that we can only achieve simple and effective bounds for cases in which (i) all locations are replenished simultaneously (at times $nT, n \in \mathbb{N}$, where T is a common review period), (ii) all locations share a common holding cost rate for each item type, namely $h_{ix} = h_x$, $1 \leq i \leq N$, $1 \leq x \leq X$, and (iii) a constraint of the form in (4.20) delimits transshipments from each location. To illustrate the approach simply we shall take $X = 1$ and drop the item identifying subscript x in what follows. We shall also focus on the lost sales model. Extensions to $X > 1$ and/or to backorder costs are straightforward.

We shall obtain a lower bound $LB(\mathbf{S}, T)$ on the costs achievable under any policy in a single review period of length T and with replenishment levels given by the N-vector $\mathbf{S}$. The corresponding bound on the achievable cost rate is $T^{-1}LB(\mathbf{S}, T)$. We obtain $LB(\mathbf{S}, T)$ by developing in turn lower bounds on the two elements of inventory costs, namely holding costs and those incurred through stock shortages.

To obtain a lower bound on holding costs, we imagine the network operating as a single location with aggregate replenishment level $S_{tot} := \sum_{i=1}^{N} S_i$ operating under the aggregate demand rate $\lambda_{tot}(t) := \sum_{i=1}^{N} \lambda_i(t)$. This approach to accounting for stock gives a lower bound on the actual stock present (and hence on the corresponding holding cost rate) at all time points since it defers lost sales to the last possible moment. Expressed differently, items leave the system soonest possible.

Using the quantities S_{tot} and $\lambda_{tot}(t)$ in (4.8), we obtain the quantity

$$v_{tot}\Big\{S_{tot}, 0, T; \text{hold}\} = \sum_{j=1}^{S_{tot}} \sum_{n=1}^{j} h \cdot \Big(A_{tot}(n, 0, T) + T \cdot B_{tot}(n, 0, T)\Big) \cdot P_j^n \tag{4.26}$$

as a lower bound on holding costs for the network.

To obtain a lower bound on shortage costs, we first use R^f_{*i} and R^u_{*i} as respectively the smallest fixed and unit costs associated with transshipments to location i. Further, using (4.20) with $X = 1$ we have an inequality $wu_{ji} \leq W_j$ delimiting the size of transshipments from j to i and $W := \max_j \frac{W_j}{w}$ as the maximum quantity which can be handled by a single transshipment. We also recall from subsection 4.4 that the material around (**??**) facilitates the computation of the probability distribution of D_{iT}, the total demand at i in a single review period. Condition now on the event that location i faces an *aggregate* shortage z over a single review period. This shortage will incur costs which are a combination of those due to transshipments and lost sales. It is not difficult to see that when $W < \infty$, the quantity

$$\rho_i(z) := \min_{0 \leq u \leq z} \left\{ \left\lceil \frac{u}{W} \right\rceil R^f_{*i} + uR^u_{*i} + (z - u)L_i \right\} \tag{4.27}$$

gives a lower bound on shortage costs at location i. The form of the expression appropriate for the case $W = \infty$ is $\rho_i(z) := \min \left\{ R^f_{*i} + zR^u_{*i}, zL_i \right\}$. Combining the above elements yields the following result:

Proposition 2 *A lower bound on the network costs incurred over a review period of length T and with replenishment levels* $\mathbf{S}$ *is given by*

$$LB(\mathbf{S}, T) = v_{tot}\Big\{S_{tot}, 0, T; hold\Big\} + \sum_{i=1}^{N} \sum_{j=S_i+1}^{\infty} P(D_{iT} = j)\rho_i(j - S_i). \tag{4.28}$$

5 Experimentation

To test the performance of the new policy an extensive simulation study has been carried out. We first explore how different heuristic approaches, including the new hybrid policy, perform compared to optimal for small problems. Given the complexity of the decision problem, the analysis is restricted to a single item in a network with three locations. Alongside the hybrid policy (H) developed in Section 4 we tested the performance of the myopic policies *no pooling* (NP) in which no transshipments occur and *complete pooling* (CP) in which transshipments to meet shortages are designed on a minimum immediate cost basis. We also study a standard reactive policy (R) which was adapted from Archibald et al. (2010). All policies were applied under the same conditions using common random numbers. For the optimal policy, the cost rate was determined via dynamic programming. Table 2 summarises the optimality gaps obtained for the above policies and highlights how policy H improves significantly upon R closing the gap to optimal considerably.

In addition of the evaluation of the hybrid policy H via comparisons to optimal in small networks we study its performance in larger networks with 10 and 50 locations and two distinct item types. In Tables 3-7 the cost rate performances of the policies mentioned above are compared in larger networks along with that of an artificial policy (Hpar) which runs the decision rule H for each item type separately before aggregating costs. Comparing H to Hpar shows the improvement achieved by modelling item types together and allowing the coordinated proactive transshipments of multiple item types at each decision epoch. In Table 8 the cost rates incurred by NP, CP, R and H are compared with the lower bound established in Section 4.5 for problems with 10 locations which are replenished simultaneously. Subsequent studies aim to assess the benefits achieved through the generality of our demand modelling (Table 9) and to characterise competing transshipment heuristics in terms of the size, frequency and timing of transshipments (Figure 1).

In all of the numerical studies reported in the section we shall take the unit of time to be one day and shall assume that stock is replenished on a weekly basis ($T_i = 7$, $\forall i$). Successive replenishments at location i occur at $\kappa_i + 7m$, $m \in \mathbb{N}$, for some offset $\kappa_i \in [0, 7)$, $1 \leq i \leq N$. We also assume a weekly demand pattern. We write λ_i for the mean number of customer arrivals at location i per week and φ_{ik} for the long run proportion of customers who arrive during phase (day) $k \in \{1, \ldots, 7\}$ of the week. Hence the customer arrival rate at i during phase k is $\lambda_i \varphi_{ik}$. These choices are informed by the motivating application concerning the sale of car parts. Note that the parameters λ_i and φ_{ik} are chosen constant here, but our approach easily accommodates varying these for successive replenishment cycles to model any trend in demand. In what follows we shall use D-Pat as an abbreviation for the pattern of weekly demands $\boldsymbol{\lambda}$ in the network and P-Pat for the associated phase patterns $\boldsymbol{\varphi}$.

Parameter		Values
Weekly demand	D-Pat 1	$\lambda_{\bar{1}} = 20, \lambda_{\bar{2}} = 20, \lambda_{\bar{3}} = 20$
	D-Pat 2	$\lambda_{\bar{1}} = 25, \lambda_{\bar{2}} = 20, \lambda_{\bar{3}} = 15$
	D-Pat 3	$\lambda_{\bar{1}} = 30, \lambda_{\bar{2}} = 20, \lambda_{\bar{3}} = 10$
Phase pattern	P-Pat 0	$\varphi_k = \frac{1}{7}\ \forall k$ (Constant/Stationary)
	P-Pat 1	$\varphi = (0.100, 0.250, 0.250, 0.100, 0.100, 0.100, 0.100)$
	P-Pat 2	$\varphi = (0.050, 0.375, 0.375, 0.050, 0.050, 0.050, 0.050)$
	P-Pat 3	$\varphi = (0.150, 0.350, 0.200, 0.075, 0.075, 0.075, 0.075)$

Table 1: Overview of demand and phase patterns used

In our numerical studies we assign each location to one of three similarly sized groups. Locations within group g have a common customer arrival rate $\lambda_{\bar{g}}$. We further always assume a common phase pattern φ_k, $1 \leq k \leq 7$, across all locations. Table 1 contains details of the D-Pat and P-Pat we use in the study. We further take $f_{ixd} = 0.8(1-0.8)^{d-1}$, $d \geq 1$, as our model for type-x demand per customer at location i, with an associated mean of $0.8^{-1} = 1.25$. With the exception of the simultaneous replenishment setting of Table 8, the offsets κ_i determining the times of location replenishments are drawn independently and uniformly from the interval $[0, 7)$. Transshipment costs are characterised by the triple $(R^{\text{fix}}, R^{\text{dist}}, R^{u})$. The fixed element of the cost of a transshipment from j to i is given by $R^{f}_{ji} = R^{\text{fix}} + \xi_{ji} R^{\text{dist}}$, while the per unit cost is $R^{u}_{jix} = R^{u}$ for all choices of j, i and x. The factor ξ_{ji} is the normalised distance between locations j and i. Since throughout our experimentation, we found that the lost sales and backordered sales models produced comparable results, we include results only for the former. For the most part, we assume that holding and lost sales cost rates do not vary with location and item type. When this is the case we also take the holding cost rate to be the unit in which all costs are measured. Hence we have $h_{ix} = 1$ and $L_{ix} = L$ for all choices of i, x. For such cases, we assume from the discussion leading to (4.25) that replenishment levels take the form $S_{ix} = 1.25\lambda_i + \alpha\sqrt{1.25\lambda_i}$, where the parameter α is either set equal to 1 or 1.5 or is optimised in the manner described in subsection 4.4. In order to demonstrate that our results are not dependent on assumptions of homogeneity of inventory and transshipment costs across item types we include a set of results in Table 5 where this is not the case. The appropriate form of (4.25) then becomes $S_{ix} = 1.25\lambda_i + \alpha_x \sqrt{1.25\lambda_i}$. In all of the experiments 50 simulation repetitions were performed with each running for 200 replenishment periods (weeks).

Parameter		Suboptimality gap for policy (%)			
		NP	CP	R	H
L	20	10.09	4.98	4.40	0.14
	60	76.72	15.00	13.47	0.54
	100	133.63	12.43	12.18	0.79
$(R^{\text{fix}}, R^{\text{dist}}, R^{u})$	(10, 40, 0)	64.28	12.46	11.52	0.51
	(10, 40, 1)	60.13	11.21	10.16	0.41
	(5, 20, 1)	96.04	8.75	8.36	0.55
worst case		126.80	15.84	15.48	1.14

Table 2: Suboptimality gap results for a three location network using $\alpha = 1$

Table 2 summarises results obtained for different heuristic transshipment policies expressed as the deviation (percentage excess) from the optimal cost rate. These are all three location problems with replenishment levels set by taking $\alpha = 1$. Experiments were carried out for all combinations of the demand and phase patterns in Table 1 and three levels of both lost sales penalties and transshipment costs. This yields 108 problem configurations in all. We present average figures for the results obtained for different cost levels as well as the worst case. Please note that the hybrid heuristic H closes the greater part of the suboptimality gap left by other heuristics.

The 10 location experiments whose results are given in Tables 3 and 4 were conducted on 10 randomly generated maps. The experiments were as described above and the relevant model parameters are given in the tables. We include results for just one phase/demand pattern since we found varying P-Pat and D-Pat had little impact on the relative performance of the heuristic policies. The tables give values of the cost per week incurred under different policies and for a variety of problem contexts also record the percentage cost reduction achieved by H in comparison to other policies. Table 3 considers contexts in which limited storage space dictates low replenishment levels ($\alpha = 1$) while in Table 4, the value of α has been chosen to achieve a minimum cost rate for each policy. This optimal value lies in the range [1.3,1.8] for CP, [1.3,1.6] for R and [1.1,1.3] for H, with larger optimising α obtained when lost sales penalties and/or transshipments costs are high. For policy NP, optimal values of α were obtained from (4.21). We can infer that the new hybrid policy allows for considerably lower levels of safety stock compared to other policies thus keeping holding costs low. This is especially important for inventory systems where holding costs constitute a major part of the operating costs. We can also see that the hybrid policy is more robust towards higher shortage costs which is also very important for industries where high penalties apply for unmet customer demand. For high levels of shortage costs, it is notable that for non-simultaneous replenishments, as is the case here, the myopic policy CP in many cases outperforms policy R. This is due to the fact that the purely reactive quasi-myopic approach overestimates future shortage costs at locations where the remaining time until the next replenishment is long and thus produces inferior decisions. This deficiency is completely removed by the hybrid approach.

Parameter		Cost per period using policy					Improvement H over (%)			
$(R^{\text{fix}}, R^{\text{dist}}, R^u)$	L	NP	CP	R	Hpar	H	NP	CP	R	Hpar
	20	543.22	482.19	480.59	411.59	397.82	36.55	21.21	20.80	3.46
(10, 40, 0)	60	975.92	495.74	516.04	420.06	403.63	141.79	22.82	27.85	4.07
	100	1408.63	495.74	521.01	422.92	405.06	247.76	22.39	28.63	4.41
	20	543.22	491.67	487.97	429.06	417.77	30.03	17.69	16.80	2.70
(10, 40, 1)	60	975.92	510.90	527.26	439.29	425.92	129.13	19.95	23.79	3.14
	100	1408.63	510.90	532.19	442.56	428.00	229.12	19.37	24.34	3.40
	20	543.22	422.37	434.45	389.82	385.13	41.05	9.67	12.81	1.22
(5, 20, 1)	60	975.92	422.38	437.51	393.40	387.43	151.90	9.02	12.93	1.54
	100	1408.63	422.38	438.95	394.67	388.36	262.71	8.76	13.03	1.63

Table 3: Lost sales results for 10 locations using $\alpha = 1$ (D-Pat 3, P-Pat 2)

Parameter		Cost per period using policy					Improvement H over (%)			
$(R^{\text{fix}}, R^{\text{dist}}, R^u)$	L	NP	CP	R	Hpar	H	NP	CP	R	Hpar
	20	465.90	440.98	445.51	405.85	396.26	17.57	11.28	12.43	2.42
(10, 40, 0)	60	529.36	443.72	454.11	411.70	400.48	32.18	10.80	13.39	2.80
	100	561.26	443.72	455.53	413.52	401.83	39.68	10.43	13.37	2.91
	20	465.90	444.10	448.24	417.97	412.28	13.01	7.72	8.72	1.38
(10, 40, 1)	60	529.36	447.79	457.51	423.19	417.78	26.71	7.18	9.51	1.30
	100	561.26	447.79	458.72	425.42	419.40	33.82	6.77	9.37	1.43
	20	465.90	412.11	423.19	389.82	385.13	20.97	7.00	9.88	1.22
(5, 20, 1)	60	529.36	412.11	424.67	393.40	387.43	36.63	6.37	9.61	1.54
	100	561.26	412.11	425.53	394.67	388.36	44.52	6.12	9.57	1.63

Table 4: Lost sales results for 10 locations using respective optimal values of α (D-Pat 3, P-Pat 2)

The studies in Tables 3 and 4 (and those elsewhere in this subsection) suppose that inventory and transshipment costs are constant over item types. In Table 5 find results from a set of experiments in which we have introduced item cost heterogeneity and set $h_{i1} = 0.5$, $h_{i2} = 1.5$, $L_{i1} = L$, $L_{i2} = 2L, R^u_{ji1} = R^u$, $R^u_{ji2} = 3R^u$. Other aspects of the studies are unchanged from those reported in Tables 3 and 4. The reader will note that this introduction of item cost heterogeneity has not materially effected the nature of the results.

Parameter		Cost per period using policy					Improvement H over (%)			
$(R^{\text{fix}}, R^{\text{dist}}, R^{u})$	L	NP	CP	R	Hpar	H	NP	CP	R	Hpar
	20	640.40	483.15	488.85	409.20	395.03	62.12	22.31	23.75	3.59
(10, 40, 0)	60	1277.62	490.93	514.97	416.84	400.06	219.35	22.71	28.72	4.19
	100	1914.83	490.93	519.73	419.37	401.37	377.07	22.31	29.49	4.49
	20	640.40	509.30	508.42	443.54	433.48	47.73	17.49	17.29	2.32
(10, 40, 1)	60	1277.62	520.27	536.67	453.97	442.86	188.49	17.48	21.18	2.51
	100	1914.83	520.27	541.35	457.54	445.70	329.62	16.73	21.46	2.66
	20	640.40	431.75	440.91	401.83	398.62	60.66	8.31	10.61	0.81
(5, 20, 1)	60	1277.62	431.75	444.25	406.94	403.49	216.64	7.01	10.10	0.86
	100	1914.83	431.75	445.67	408.66	405.29	372.46	6.53	9.96	0.83

Table 5: Lost sales results for 10 locations with heterogeneous item types using $\alpha = 1$ (D-Pat 3, P-Pat 2)

In order to evaluate how the benefits of the hybrid policy scale with the size of the network, experiments were conducted using a network with 50 locations. Here geographical data on 50 branches of a car parts dealer were used. Tables 6 and 7 report a set of results equivalent to those for 10 locations in Tables 3 and 4. In the determination of replenishment levels the parameter α was both set to be 1 (Table 6) and optimised (Table 7). The larger number of locations means that the chance of a suitable sending location when a shortage occurs is enhanced. Hence it is true for all transshipment policies that safety stock levels, as reflected by the optimal α values computed for Table 7 were significantly reduced. Optimal α are now in the range [1.0,1.4] for CP, [1.0,1.3] for R and [0.6,1.0] for H. We can see that with regard to choosing α optimally the benefit of H observed earlier is increased. The importance of transshipments per se is seen in the dominance of all transshipment policies over NP.

Parameter		Cost per period for policy					Improvement H over (%)			
$(R^{\text{fix}}, R^{\text{dist}}, R^{u})$	L	NP	CP	R	Hpar	H	NP	CP	R	Hpar
	20	2758.67	2205.63	2183.64	1921.43	1879.52	46.78	17.35	16.18	2.23
(10, 40, 0)	60	4914.75	2205.92	2195.93	1945.19	1899.59	158.73	16.13	15.60	2.40
	100	7070.83	2205.92	2206.71	1957.60	1908.49	270.49	15.58	15.63	2.57
	20	2758.67	2270.44	2236.99	2013.65	1982.72	39.14	14.51	12.82	1.56
(10, 40, 1)	60	4914.75	2270.84	2250.22	2044.90	2015.43	143.86	12.67	11.65	1.46
	100	7070.83	2270.84	2260.89	2061.33	2030.00	248.32	11.86	11.37	1.54
	20	2758.67	2000.74	1996.39	1898.55	1890.06	45.96	5.86	5.63	0.45
(5, 20, 1)	60	4914.75	2000.74	2007.04	1922.02	1914.10	156.77	4.53	4.86	0.41
	100	7070.83	2000.74	2013.43	1931.62	1923.36	267.63	4.02	4.68	0.43

Table 6: Lost sales results for a 50 location network using $\alpha = 1$ (D-Pat 3, P-Pat 2)

Parameter		Cost per period for policy					Improvement H over (%)			
$(R^{\text{fix}}, R^{\text{dist}}, R^{u})$	L	NP	CP	R	Hpar	H	NP	CP	R	Hpar
(10, 40, 0)	20	2374.59	2144.02	2143.82	1914.62	1853.31	28.13	15.69	15.68	3.31
	60	2688.90	2144.07	2146.67	1945.19	1881.31	42.93	13.97	14.11	3.40
	100	2842.91	2144.07	2150.34	1957.60	1894.10	50.09	13.20	13.53	3.35
(10, 40, 1)	20	2374.59	2178.19	2174.29	2013.65	1982.72	19.76	9.86	9.66	1.56
	60	2688.90	2178.27	2177.42	2044.90	2015.43	33.42	8.08	8.04	1.46
	100	2842.91	2178.27	2181.04	2061.33	2030.00	40.04	7.30	7.44	1.54
(5, 20, 1)	20	2374.59	2000.74	1996.39	1872.23	1853.04	28.15	7.97	7.74	1.04
	60	2688.90	2000.74	2007.04	1900.39	1883.84	42.73	6.21	6.54	0.88
	100	2842.91	2000.74	2013.43	1911.29	1894.15	50.09	5.63	6.30	0.90

Table 7: Lost sales results for a 50 location network using respective optimal values of α (D-Pat 3, P-Pat 2)

It is clear from the results obtained in Tables 2-7 that the hybrid policy improves significantly upon the competing heuristics. For networks of size larger than three locations the full potential of applying the hybrid approach remains unknown as an optimal solution cannot be determined for use as a comparator. Section 4.5 introduced an approach which provides a lower bound for the cost per period achievable under any policy. For this setup an assumption of simultaneous replenishment of all locations is required. The results presented in Table 8 use the same underlying parameters as before with the exception that the offset κ_i from the weekly repeating replenishment pattern is set to zero for all i. To allow a common lower bound for all policies a fixed value of α is used. This was set at level $\alpha = 1.5$ to achieve a reasonably strongly performing set of replenishment levels for all the policies. We can see that the hybrid policy performs very well which indicates that the quasi-myopic approach indeed yields a good approximation to solving the sequential decision problem. The deviation from the lower bound ranges from roughly 1 to 2.5% for the hybrid policy. As was the case in Table 2, Table 8 again makes clear that the hybrid heuristic H closes the major part of the suboptimality gap left by the competing heuristics in these larger problems. Further, upon close inspection the reader should observe that the lower bound developed in Section 4.5 applies to all approaches of stock rebalancing between (simultaneous) replenishments, not simply those triggered by shortages of the kind considered here. Hence for the problems in Table 8, heuristic H is competitive with a wide range of possible approaches including those which take a different approach to proactive transshipment and/or which allow simultaneous transshipments from more than a single location.

Parameter		Deviation from lower bound (%)			
$(R^{\text{fix}}, R^{\text{dist}}, R^{u})$	L	NP	CP	R	H
(10, 40, 0)	20	18.24	19.18	11.11	1.32
	60	11.89	21.25	12.77	1.52
	100	16.99	21.25	12.85	1.73
(10, 40, 1)	20	18.24	19.76	10.65	2.07
	60	11.89	22.39	12.66	2.38
	100	16.99	22.39	12.74	2.68
(5, 20, 1)	20	18.24	11.73	6.56	1.42
	60	11.89	11.73	6.61	1.82
	100	16.99	11.73	6.67	2.03

Table 8: Performance analysis for 10 locations using the derived lower bound and $\alpha = 1.5$ (D-Pat 3, P-Pat3)

		$\alpha = 1$			$\alpha = 1.5$		
Parameter		Cost		Gain	Cost		Gain
$(R^{\text{fix}}, R^{\text{dist}}, R^{u})$	L	Ave	H	(%)	Ave	H	(%)
(10, 40, 0)	20	177.7	173.0	2.66	165.6	162.7	1.75
	60	188.0	182.5	2.88	168.9	165.2	2.16
	100	193.7	188.7	2.57	170.1	166.6	2.06
(10, 40, 1)	20	190.2	185.1	2.69	173.4	170.0	1.96
	60	201.6	195.6	3.01	177.5	173.2	2.41
	100	207.6	202.1	2.65	179.0	174.9	2.27
(5, 20, 1)	20	162.2	158.7	2.14	159.1	156.5	1.59
	60	169.0	166.0	1.77	161.1	158.8	1.45
	100	173.7	171.1	1.48	161.9	159.9	1.28

Table 9: Non-homogeneous benefit analysis for 10 locations in a single item network (D-Pat 3, P-Pat 3)

To assess the contribution made to the results by our incorporation of non-homogeneous demand, for each of phase patterns 1-3 we designed a hybrid heuristic (Ave) on the basis of a false assumption of homogeneous demand. In Table 9, find cost rates which compare H with Ave over a set of cases similar to those used in Tables 3 and 4, but for a single item model and with replenishment levels set by taking $\alpha = 1$ and $\alpha = 1.5$. From our entire set of results we note that a cost rate benefit of up to 3% can be achieved by correctly incorporating demand seasonality in the model. In an unreported study available from the authors, they demonstrate the superiority of H over competing heuristics even in the case of pure Poisson demand standard in the literature hitherto.

We finally analyse the nature of different policies by evaluating statistics collected for the set of experiments reported in Table 3. The left hand plot of Figure 1 shows how often transshipments were made under the policies CP, R and H and how close the receiving location was to its next replenishment expressed in phases. We can see that under the hybrid policy, transshipments are less frequent than for policies CP and R. Particularly striking is the extent to which H mitigates the spike in the frequency of transshipments which occur at the end of a location's review period in comparison to CP and R. This is reflected in the cost benefit analysis as fixed costs for transshipments increase. We can also see that due to its myopic nature, policy CP has an increased transshipment frequency compared to R. Studying the right hand plot of Figure 1, we can understand the huge potential the hybrid policy offers. Here the average shipment size across different item types is illustrated. While in the majority of cases CP and R ship only one item to meet an occurring shortage, policy H makes significantly larger transshipments. This not only enables fewer transshipments in the future due to a reduced chance of stockouts it also makes efficient use of the capacity of vehicles and exploits the dominance of fixed over variable costs.

6 Conclusion

The hybrid policy improves significantly upon a reactive policy and other heuristics when a substantial part of the cost of transshipments is fixed. This is particularly relevant for inventory networks which are spread over a wide geographic area where the cost of transshipping will be predominantly determined by distance and time travelled rather than the amount transported. The main improvement lies in the fact that fewer transshipments of larger size are made thus making efficient use of the resources involved. Not only will reducing the frequency of transshipments reduce costs, it also reflects a more strategic approach to stock rebalancing and will reduce the extent to which stock is shuffled repeatedly between locations. We have provided evidence that our hybrid heuristic not only improves upon previous proposals but also comes close to optimal.

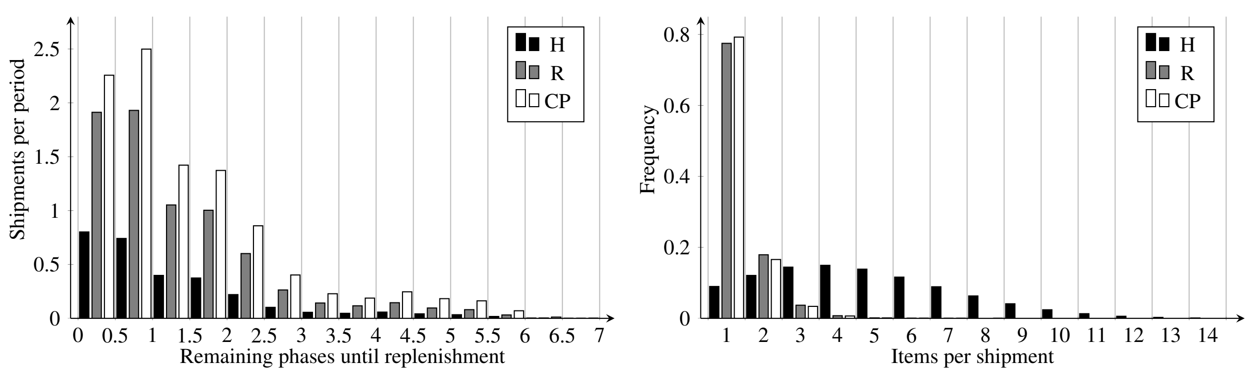

Figure 1: Timing and size of transshipments for different policies

In addition to the considerable cost savings in operating an inventory system, our approach also enables a much greener business operation as the capacity of transport vehicles is used more efficiently and fewer journeys are needed. Allowing compound non-homogeneous demand provides further performance improvements and greater precision in the policy's application. The approach enables a very general setting allowing multiple item types where demand is drawn from a general multivariate distribution. Further, a more flexible way to model shortage costs is offered. This increased generality allows the hybrid policy to exploit the benefits offered by economies of scale in a wide range of practical settings.

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