*Proceedings of the Edinburgh Mathematical Society* (2010) **53**, 97–109 ©
DOI:10.1017/S0013091508000060 Printed in the United Kingdom

# SIMPLICIAL COHOMOLOGY OF AUGMENTATION IDEALS IN $\ell^1(G)$

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(Received 8 January 2008)

*Abstract* Let $G$ be a discrete group. We give a decomposition theorem for the Hochschild cohomology of $\ell^1(G)$ with coefficients in certain $G$-modules. Using this we show that if $G$ is commutative–transitive, the canonical inclusion of bounded cohomology of $G$ into simplicial cohomology of $\ell^1(G)$ is an isomorphism.

*Keywords:* bounded cohomology; simplicial cohomology; Banach algebras; commutative–transitive group

2000 *Mathematics subject classification:* Primary 16E40; 43A20
Secondary 20E45

## 1. Introduction

The bounded cohomology of a (discrete) group $G$ is known to embed as a summand in the simplicial cohomology of the convolution algebra $\ell^1(G)$. Consequently, knowing that the bounded cohomology of $G$ is non-zero, or non-Hausdorff, immediately implies that the simplicial cohomology of $\ell^1(G)$ is non-zero or non-Hausdorff, respectively.

In this paper we observe that for a wide class of discrete groups, including all torsion-free hyperbolic groups, this summand is the only non-zero contribution to simplicial cohomology; more precisely, the aforementioned inclusion of bounded cohomology into simplicial cohomology is an isomorphism. The precise statement is given below as Theorem 3.5. By standard homological arguments (see Lemma 3.1) we may recast our result as saying that the augmentation ideals for these groups are simplicially trivial, in the sense that the 'naive' Hochschild cohomology groups $\mathcal{H}^*(I_0(G), I_0(G)')$ vanish (see Corollary 5.1). Thus, our work is a partial generalization of the results in [**4**] on weak amenability of such ideals.

Our work is also motivated by [**10**], in which a version of our decomposition theorem is given for second-degree cohomology; the conclusion is stronger in [**10**] because the second bounded cohomology of *any* discrete group is known to be a Banach space (no such general result is true for degrees 3 and above).

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## 2. Notation and homological background

Throughout we shall denote the identity map on a Banach space, module or algebra $X$ by $\mathrm{id}_X$ (we will sometimes omit the subscript if there is no risk of ambiguity).

Given a family $(E(x))_{x\in\mathbb{I}}$ of Banach spaces and $p \in [1,\infty]$, we can form the $\ell^p$-direct sum of the $E(x)$ in the obvious way: this will be denoted by

$$\bigoplus_{i\in\mathbb{I}}^{[p]} E(x).$$

Isometric linear isomorphism of two Banach spaces $E$ and $F$ will be denoted by $E \underset{1}{\cong} F$; the dual of a Banach space $E$ will be denoted by $E'$.

Given a Banach algebra $A$, our definition of a Banach $A$-bimodule $M$ is the usual one: we require that the actions of $A$ on $M$ are jointly continuous, but not necessarily that they are contractive. When we write $M'$, we tacitly assume that it is equipped with the canonical $A$-bimodule structure obtained by taking adjoints of the actions of $A$ on $M$.

### 2.1. (Isometric) isomorphism of chain complexes and functors

We assume familiarity with the notions of chain and cochain complexes of Banach spaces and modules. For brevity we adopt the convention that our chain and cochain complexes vanish in degrees less than or equal to $-1$, i.e. are of the form

$$0 \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots \quad \text{or} \quad 0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots .$$

**Definition 2.1.** Let $A$ be a Banach algebra and let $E_*$ and $F_*$ be chain complexes of left Banach $A$-modules. We say that $E_*$ and $F_*$ are *topologically isomorphic as (module) chain complexes* if there exist mutually inverse chain maps $f : E_* \to F_*$ and $g : F_* \to E_*$, with each $f_n$ (and hence each $g_n$) a continuous $A$-module map.

If, moreover, we can arrange that each $f_n$ (and hence each $g_n$) is an isometry, we say that the chain complexes $E_*$ and $F_*$ are *isometrically isomorphic*, and write $E_* \underset{1}{\cong} F_*$.

### 2.2. Hochschild cohomology

We repeat some background material in order to fix our notation. Let $A$ be a Banach algebra and let $M$ be a Banach $A$-bimodule. The Hochschild cochain complex is

$$0 \to \mathcal{C}^0(A,M) \xrightarrow{\delta} \mathcal{C}^1(A,M) \xrightarrow{\delta} \mathcal{C}^2(A,M) \xrightarrow{\delta} \cdots , \tag{2.1}$$

where, for each $n \in \mathbb{Z}_+$, $\mathcal{C}^n(A,M)$ is the Banach space of all bounded $n$-linear maps from $A$ to $M$, and the coboundary operator $\delta : \mathcal{C}^n(A,M) \to \mathcal{C}^{n+1}(A,M)$ is given by

$$(\delta\psi)(a_1,\dots a_{n+1}) := a_1\psi(a_2,\dots,a_{n+1}) + \sum_{j=1}^{n}(-1)^j\psi(a_1,\dots,a_ja_{j+1},\dots,a_{n+1}) \\ + (-1)^{n+1}\psi(a_1,\dots,a_n)a_{n+1}.$$

We denote the kernel of $\delta : \mathcal{C}^n(A, M) \to \mathcal{C}^{n+1}(A, M)$ by $\mathcal{Z}^n(A, M)$ and the range of $\delta : \mathcal{C}^{n-1}(A, M) \to \mathcal{C}^n(A, M)$ by $\mathcal{B}^n(A, M)$. The quotient vector space $\mathcal{Z}^n(A, M)/\mathcal{B}^n(A, M)$ is the *nth cohomology group of* $A$ with *coefficients in* $M$, denoted by $\mathcal{H}^n(A, M)$.

The case where $M = A'$ merits special attention. If $\mathcal{H}^{n+1}(A, A') = 0$ for all $n \geqslant 1$, we say that $A$ is *simplicially trivial*.

For most of this paper $A$ will be the $\ell^1$-convolution algebra of a discrete group $G$. There is a canonical one-dimensional $\ell^1(G)$-module, denoted by $\mathbb{C}_\varepsilon$, corresponding to the augmentation character on $G$: we shall sometimes refer to $\mathcal{H}^n(\ell^1(G), \mathbb{C}_\varepsilon)$ as the *nth bounded cohomology group of* $G$.

Although we do not require much of the machinery of Ext we shall assume familiarity with at least its basic definition and its relation to Hochschild cohomology, as can be found in [**5**, § III.4]. Central to the machinery developed in [**5**] is the notion of an *admissible* resolution or complex. It is convenient (though, as was pointed out to us by the referee, not essential) to have a more precise notion defined. We thank the referee for advice regarding the following terminology, which is modelled on that in [**5**, § III.1].

**Definition 2.2.** Let $0 \leftarrow E_0 \xleftarrow{d_0} E_1 \xleftarrow{d_1} \cdots$ be a chain complex of Banach spaces and continuous linear maps. We say that the complex $E_*$ is 1-*contractible in* $\mathsf{Ban}$ if there exist contractive linear maps $s_j : E_j \to E_{j+1}$, $j \geqslant 0$, such that $d_0 s_0 = \mathrm{id}$ and $s_{j-1}d_{j-1} + d_j s_j = \mathrm{id}$ for all $j \geqslant 1$.

The point of introducing '1-contractibility' explicitly is that it has good stability properties. For instance, we have the following simple observation, whose proof we omit as it is straightforward.

**Lemma 2.3.** *Let $\mathbb{I}$ be an index set and let $p \in [1, \infty]$. Suppose that for each $x \in \mathbb{I}$ we have a* 1-*contractible chain complex*

$$0 \leftarrow E_0(x) \xleftarrow{d_0^x} E_1(x) \xleftarrow{d_1^x} \cdots$$

*in* $\mathsf{Ban}$, *such that for each $n$ we have $\sup_{x \in \mathbb{I}} \|d_n^x\| < \infty$. Then the $\ell^p$-sum*

$$0 \leftarrow \bigoplus_{x \in \mathbb{I}}^{[p]} E_0(x) \xleftarrow{d_0} \bigoplus_{x \in \mathbb{I}}^{[p]} E_1(x) \xleftarrow{d_1} \cdots$$

*is also a* 1-*contractible complex, and is, in particular, exact.*

**Remark 2.4.** Without the 1-contractible condition, the $\ell^p$-sum of a family of contractible chain complexes need not even be exact: we shall return to this point in § 3.

## 3. Augmentation ideals

The original version of the following lemma was stated in the special case of augmentation ideals in discrete group algebras (the author thanks N. Grønbæk for pointing out that a more general result holds).

**Lemma 3.1.** *Let $A$ be a unital Banach algebra which has a character $\varphi : A \to \mathbb{C}$ and let $I = \ker(\varphi)$. Then the following are equivalent:*

(i) *$I$ is simplicially trivial;*

(ii) *$\mathcal{H}^n(A, I') = 0$ for all $n \geqslant 1$;*

(iii) *for each $n \geqslant 1$, the canonical map $\mathcal{H}^n(A, \mathbb{C}_\varphi) \xrightarrow{\varphi^*} \mathcal{H}^n(A, A')$ that is induced by the inclusion $\mathbb{C} \to A'$, $1 \mapsto \varphi$, is a topological isomorphism.*

**Proof.** The implications (i) $\iff$ (ii) are immediate from the observation that $A \cong I^\#$ and the fact (see [**5**, Exercise III.4.10] or [**6**, Chapter 1]) that $\mathcal{H}^n(B^\#, M) \cong \mathcal{H}^n(B, M)$ for any Banach algebra $B$ and Banach $B$-bimodule $M$, where $B^\#$ denotes the forced unitization of $B$.

To obtain the implications (ii) $\iff$ (iii), consider the long exact sequence of cohomology associated with the short exact sequence $0 \to \mathbb{C}_\varphi \to A' \to I' \to 0$, namely

$$\cdots \to \mathcal{H}^n(A, \mathbb{C}_\varphi) \xrightarrow{\varphi^*} \mathcal{H}^n(A, A') \xrightarrow{\rho} \mathcal{H}^n(A, I') \to \mathcal{H}^{n+1}(A, \mathbb{C}_\varphi) \to \cdots$$

We *claim* that the map $\mathcal{H}^0(A, A') \xrightarrow{\rho} \mathcal{H}^0(A, I')$ is surjective. If this is true, then our long exact sequence has the form

$$0 \to \mathcal{H}^1(A, \mathbb{C}_\varphi) \xrightarrow{\varphi^*} \mathcal{H}^1(A, A') \xrightarrow{\rho} \mathcal{H}^1(A, I') \to \mathcal{H}^2(A, \mathbb{C}_\varphi) \to \cdots$$

and the equivalence of (ii) and (iii) now follows from [**5**, Lemma 0.5.9]. Hence, it remains only to justify our claim.

For any $A$-bimodule $X$, $\mathcal{H}^0(A, X)$ is just the centre $Z(X)$ of $X$, so that $\rho : Z(A') \to Z(I')$ is given by the restriction of a trace on $A$ to the ideal $I$. It therefore suffices to show that every element of $Z(I')$ extends to a trace on $A$. But this is easy: if $\psi \in I'$ and $\psi \cdot a = a \cdot \psi$ for all $a \in A$, then the functional $a \mapsto \psi(a - \varphi(a)\mathbf{1}_A)$ gives such a trace, and the proof is complete. ☐

We now specialize to group algebras. Throughout the paper, $G$ will denote a *discrete* group, $\ell^1(G)$ its convolution algebra and $I_0(G)$ the augmentation ideal in $\ell^1(G)$, that is, the kernel of the augmentation character $\varepsilon$ which sends each standard basis vector of $\ell^1(G)$ to 1.

**Definition 3.2.** A group $G$ is said to be *commutative–transitive* if each element of $G \setminus \{\mathbf{1}_G\}$ has an abelian centralizer.

It is not immediately clear that there exist any non-abelian, infinite, commutative–transitive groups: examples can be found in [**8**, Chapter 1] (see in particular the remarks after Proposition 2.19). Let us just mention one family of examples.

**Theorem 3.3 (Gromov [3]; see also [1, Proposition 3.5]).** *Any torsion-free word-hyperbolic group is commutative–transitive.*

The arguments given for this in [**3**] are scattered over several sections and are not easily assembled into a proof. The simplest and clearest account appears in [**1**, Chapter 3]. (I thank K. Goda for drawing these notes to my attention.)

**Remark 3.4.** It is often observed that direct products of hyperbolic groups need not be hyperbolic, the standard example being $F_2 \times F_2$, where $F_2$ denotes the free group on two generators. In the current context it is worth pointing out that clearly $F_2 \times F_2$ is not commutative–transitive (since the centralizer of $(\mathbf{1}, x)$ always contains a copy of $F_2 \times \{\mathbf{1}\}$).

**Theorem 3.5.** *Let $G$ be a commutative–transitive, discrete group. Then, for each $n \geqslant 1$, $\mathcal{H}^n(\ell^1(G), I_0(G)') = 0$.*

The key to the proof is the following well-known idea: when we pass to a conjugation action, $I_0(G)$ decomposes as an $\ell^1$-direct sum of modules of the form $\ell^1(\mathfrak{Cl}_x)$, where $\mathfrak{Cl}_x$ denotes the conjugacy class of $x$. Hence, there is an isomorphism of cochain complexes

$$\mathcal{C}^*(\ell^1(G), I_0(G)') \underset{1}{\cong} \bigoplus_{x \in \mathbb{I}}^{[\infty]} \mathcal{C}^*(\ell^1(G), \ell^1(\mathfrak{Cl}_x)'), \tag{3.1}$$

where $\mathbb{I}$ is a set of representatives for each conjugacy class in $G \setminus \{\mathbf{1}_G\}$. Our theorem will now follow from a computation of the cohomology of the complex on the right-hand side of (3.1).

For each *summand* on the right-hand side of (3.1), the cohomology groups can be reduced to certain bounded cohomology groups: more precisely, it is observed in [**10**] that for each $x$ there are isomorphisms

$$\begin{aligned} \mathcal{H}^*(\ell^1(G), \ell^1(\mathfrak{Cl}_x)') &\cong \operatorname{Ext}^*_{\ell^1(G)}(\ell^1(\mathfrak{Cl}_x), \mathbb{C}) \\ &\cong \operatorname{Ext}^*_{\ell^1(C_x)}(\mathbb{C}, \mathbb{C}) \\ &\cong \mathcal{H}^*(\ell^1(C_x), \mathbb{C}), \end{aligned} \tag{3.2}$$

where $C_x$ denotes the centralizer of $x$. It is implicitly claimed in [**9**, Corollary 3.7] that the cohomology of the cochain complex

$$\bigoplus_{x \in \mathbb{I}}^{[\infty]} \mathcal{C}^*(\ell^1(G), \ell^1(\mathfrak{Cl}_x)')$$

is isomorphic to

$$\bigoplus_{x \in \mathbb{I}}^{[\infty]} \mathcal{H}^*(\ell^1(G), \ell^1(\mathfrak{Cl}_x)').$$

If this were the case, then Theorem 3.5 would follow immediately from (3.2). However, the justification given in [**9**] for this supposed isomorphism is insufficient, because it is *not* in general true that the cohomology of an $\ell^\infty$-sum is the $\ell^\infty$-sum of the cohomology of the summands (see the remarks at the end of § 2).

**Remark 3.6.** In the special case where $G$ is commutative–transitive, each $C_x$ is abelian, hence amenable, and so for each $x$ the cochain complex $\mathcal{C}^*(\ell^1(C_x),\mathbb{C})$ has a contractive linear splitting. Hence, for such $G$, in order to deduce that the cochain complex

$$\bigoplus_{x\in\mathbb{I}}^{[\infty]} \mathcal{C}^*(\ell^1(G),\ell^1(C_x)')$$

splits, it would suffice to prove that the isomorphisms of (3.2) are induced by chain homotopies with norm control independent of $x$. This is implicitly done in [**10**, § 4], but only for second-degree cohomology.

Rather than follow the approach outlined in Remark 3.6, we instead generalize the argument sketched in the final section of [**10**], so that it applies to any left $G$-set $S$ (i.e. we drop their hypothesis that the action is transitive). Since our hypotheses are weaker, we are not able to deduce an isomorphism of cohomology groups as in [**10**]; however, our weaker conclusion suffices to prove Theorem 3.5.

## 4. Disintegration over stabilizers

The promised generalization is as follows.

**Theorem 4.1.** *Let $G$ be a discrete group acting from the left on a set $S$ and let $S=\coprod_{x\in\mathbb{I}}\mathrm{Orb}_x$ be the partition into $G$-orbits. Let $H_x:=\mathrm{Stab}_G(x)$.*

*Regard $\ell^1(S)$ as a Banach $\ell^1(G)$-bimodule with left action given by the $G$-action on $S$ and right action given by the augmentation action $(x,g)\mapsto x$. Then, for each $n$, the Hochschild cohomology group $\mathcal{H}^n(\ell^1(G),\ell^1(S)')$ is topologically isomorphic to the $n$th cohomology group of the complex*

$$0\to\bigoplus_{x\in\mathbb{I}}^{[\infty]}\mathcal{C}^0(\ell^1(H_x),\mathbb{C})\to\bigoplus_{x\in\mathbb{I}}^{[\infty]}\mathcal{C}^1(\ell^1(H_x),\mathbb{C})\to\cdots.$$

**Corollary 4.2.** *Let $G$, $S$ be as above, and assume that each stabilizer subgroup $H_x$ is amenable. Then $\mathcal{H}^n(\ell^1(G),\ell^1(S)')=0$ for all $n\geqslant 1$.*

**Proof of corollary.** Since each $H_x$ is amenable, the cochain complex $\mathcal{C}^*(\ell^1(H_x),\mathbb{C})$ admits a *contractive* linear splitting in degrees 1 and above. Therefore, the chain complex

$$\bigoplus_{x\in\mathbb{I}}^{[\infty]}\mathcal{C}^*(\ell^1(H_x),\mathbb{C})$$

is also split in degrees 1 and above by linear contractions, and is, in particular, exact in degree $n$. Now apply Theorem 4.1. □

**Proof of Theorem 3.5, assuming Corollary 4.2.** By adapting the remarks preceding [**6**, Theorem 2.5], it is straightforward to show that

$$\mathcal{H}^n(\ell^1(G),I_0(G)')\cong\mathcal{H}^n(\ell^1(G),(I_0(G)^\circ)'),$$

where $I_0(G)^\circ$ is the $\ell^1(G)$-bimodule with underlying space $I_0(G)$ but with trivial right action and the conjugation left action.*

Let $S = G \setminus \{\mathbf{1}_G\}$, which may be regarded as a left $G$-set via conjugation action. Then the $\ell^1(G)$-module $\ell^1(G)^\circ$ decomposes into a module-direct sum $\mathbb{C} \oplus \ell^1(S)$, where $\mathbb{C}$ is the point module with trivial action. Composing the truncation map $\ell^1(G) \to \ell^1(S)$ with the inclusion map $I_0(G) \to \ell^1(G)$ gives a linear isomorphism $I_0(G) \to \ell^1(S)$, and this is also a $G$-module map (for the conjugation action). So for this action $I_0(G)^\circ \cong \ell^1(S)$ as $G$-modules, and therefore

$$\mathcal{H}^n(\ell^1(G), I_0(G)') \cong \mathcal{H}^n(\ell^1(G), \ell^1(S)').$$

Write $S$ as the disjoint union $S = \coprod_{x \in \mathbb{I}} \mathfrak{Cl}_x$ of conjugacy classes. The corresponding stabilizer subgroups are precisely the centralizers $C_x$ of each $x \in \mathbb{I}$; since $G$ is assumed to be commutative–transitive and $\mathbf{1}_G \notin S$, each $C_x$ is commutative (hence amenable) and applying Corollary 4.2 completes the proof. $\square$

The proof of Theorem 4.1 is broken into a succession of small lemmas: each is to some extent standard knowledge, but for our purposes we need to make explicit certain uniform bounds and linear splittings for which I can find no precise reference. To do the requisite bookkeeping, we take a functorial viewpoint.

**Notation**

The *projective tensor product* of Banach spaces $E$ and $F$ will be denoted by $E \mathbin{\widehat{\otimes}} F$.

Given a unital Banach algebra $B$, we denote by ${}_B\mathsf{unmod}$ the category whose objects are unit-linked, left Banach $B$-modules and whose morphisms are the $B$-module maps between them. $\mathsf{Ban}$ is the category of Banach spaces and bounded linear maps (equivalently, $\mathsf{Ban} \equiv {}_{\mathbb{C}}\mathsf{unmod}$).

For such a $B$ there are two canonical functors: the 'forgetful functor' $\mathcal{U} : {}_B\mathsf{unmod} \to \mathsf{Ban}$, which sends a module to its underlying Banach space; and the 'free functor' $B \mathbin{\widehat{\otimes}} \cdot : \mathsf{Ban} \to {}_B\mathsf{unmod}$, which sends a Banach space $E$ to the left $B$ module $B \mathbin{\widehat{\otimes}} E$.

If $B$ is a Banach algebra, $M$ is a right Banach $B$-module and $N$ a left Banach $B$-module, we write $M \mathbin{\widehat{\otimes}}_B N$ for the Banach tensor product of $M$ and $N$ over $B$ (see [**5**, § II.3.1] for the definition and basic properties).

**Definition 4.3.** Let $B$, $C$ be unital Banach algebras and let $\mathcal{F}$ and $\mathcal{G}$ be functors ${}_B\mathsf{unmod} \to {}_C\mathsf{unmod}$. We say that $\mathcal{F}$ and $\mathcal{G}$ are *isometrically isomorphic* if there is a natural isomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ such that, for each $M \in {}_B\mathsf{unmod}$, the morphism $\alpha_M : \mathcal{F}(M) \to \mathcal{G}(M)$ is an isometry as a map between Banach $C$-modules.

* In fact, there is a continuous chain isomorphism $\Theta^*$ from $\mathcal{C}^*(\ell^1(G), I_0(G)')$ to $\mathcal{C}^*(\ell^1(G), (I_0(G)^\circ)')$, given by

$$(\Theta^n\psi)(g_1, \dots, g_n) = (g_1 \dots g_n)^{-1} \cdot \psi(g_1, \dots, g_n),$$

where $\psi \in \mathcal{C}^n(\ell^1(G), I_0(G)')$ and $g_1, \dots, g_n \in G$. This formula differs slightly from those in [**6**, § 2], because we wish to reduce to the case of cohomology coefficients with augmentation action on the left, rather than on the right as in [**6**]. The two viewpoints are essentially equivalent but, rather than convert between the two, it is simpler to verify that $\Theta^*$ is a chain map and that each $\Theta^n$ is an isomorphism of Banach spaces.

**Remark 4.4.** Let $B$ be a Banach algebra and let $E_*$ be a chain complex in ${}_B\mathsf{unmod}$. If $\mathcal{F}$ and $\mathcal{G}$ are *isometrically* isomorphic functors from ${}_B\mathsf{unmod}$ to $\mathsf{Ban}$, then the chain complexes $\mathcal{F}(E_*)$ and $\mathcal{G}(E_*)$ are isometrically isomorphic. In particular, if $\mathcal{F}(E_*)$ is 1-contractible then so is $\mathcal{G}(E_*)$.

**Lemma 4.5 (factorization of functors).** *Let $A$ be a unital Banach algebra and let $B \subseteq A$ be a closed subalgebra that contains $\mathbf{1}_A$. Regard $A$ as a right $B$-module via the inclusion homomorphism $B \hookrightarrow A$. Then*

(i) *we have a natural isometric isomorphism of functors*

$$A \underset{B}{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}} \cdot) \underset{1}{\cong} A \mathbin{\widehat{\otimes}} \cdot,$$

*where $B \mathbin{\widehat{\otimes}} \cdot$ and $A \mathbin{\widehat{\otimes}} \cdot$ are the free functors from $\mathsf{Ban}$ (to ${}_B\mathsf{unmod}$ and ${}_A\mathsf{unmod}$ respectively),*

(ii) *we have a natural isometric isomorphism of functors*

$${}_B\operatorname{Hom}(\cdot, \mathbb{C}) \underset{1}{\cong} {}_A\operatorname{Hom}\left(A \underset{B}{\widehat{\otimes}} \cdot, \mathbb{C}\right),$$

*where both sides are functors ${}_B\mathsf{unmod} \to \mathsf{Ban}$.*

The proof is clear (the analogous statements without the qualifier 'isometric' are essentially given in, for instance, [**5**, § II.5.3]).

In what follows, given an indexing set $\mathbb{I}$ (such as discrete group, or coset space) the standard unit basis of $\ell^1(\mathbb{I})$ will be denoted by $(e_x)_{x\in\mathbb{I}}$; thus, $e_x$ is the function which sends $x$ to 1 and all other elements of $\mathbb{I}$ to 0. We shall also abuse notation slightly, to make some of the formulae more legible: if $H$ is a subgroup of $G$ and $M$ and $N$ are, respectively, right and left Banach $\ell^1(H)$-modules, then we shall write $M \underset{H}{\widehat{\otimes}} N$ for the Banach tensor product of $M$ and $N$ over $\ell^1(H)$.

**Lemma 4.6 (a little more than flatness).** *Let $H$ be any subgroup of $G$ and let $G/H = \{gH : g \in G\}$ be the space of left cosets. Then we have a (natural) isometric isomorphism of functors*

$$\mathcal{U}_G(\ell^1(G) \underset{H}{\widehat{\otimes}} \cdot) \underset{1}{\cong} \ell^1(G/H) \mathbin{\widehat{\otimes}} (\mathcal{U}_H \cdot),$$

*where $\mathcal{U}_G$ and $\mathcal{U}_H$ are the forgetful functors to $\mathsf{Ban}$ (from the categories ${}_{\ell^1(G)}\mathsf{unmod}$ and ${}_{\ell^1(H)}\mathsf{unmod}$, respectively).*

**Proof.** Choose a transversal for $G/H$, that is, a function $\tau : G/H \to G$ such that $\tau(\mathcal{J}) \in \mathcal{J}$ for all $\mathcal{J} \in G/H$. (Equivalently, $\tau(\mathcal{J})H = \mathcal{J}$ for all $\mathcal{J}$). This transversal yields a function $\eta : G \to H$ such that

$$g = \tau(gH) \cdot \eta(g) \quad \text{for all } g \in G.$$

Note that $\eta(gh) = \eta(g) \cdot h$ for every $g \in G$ and $h \in H$.

If $E$ is a unit-linked left $\ell^1(H)$-module, define a contractive linear map $\ell^1(G) \widehat{\otimes} E \to \ell^1(G/H) \widehat{\otimes} E$ by $e_g \widehat{\otimes} v \mapsto e_{gH} \otimes (\eta(g) \cdot v)$. This map factors through the quotient map

$$q : \ell^1(G) \widehat{\otimes} E \to \ell^1(G) \underset{H}{\widehat{\otimes}} E,$$

and so induces a linear contraction

$$T_E : \ell^1(G) \underset{H}{\widehat{\otimes}} E \to \ell^1(G/H) \widehat{\otimes} E,$$

where

$$T_E\Big(e_g \underset{H}{\otimes} v\Big) := e_{gH} \otimes (\eta(g) \cdot v).$$

On the other hand, the composite map

$$R_E : \ell^1(G/H) \widehat{\otimes} E \xrightarrow{\tau \widehat{\otimes} \mathrm{id}_E} \ell^1(G) \widehat{\otimes} E \xrightarrow{q} \ell^1(G) \underset{H}{\widehat{\otimes}} E$$

is a linear contraction, defined by the formula

$$R(e_{\mathcal{J}} \otimes v) := e_{\tau(\mathcal{J})} \underset{H}{\otimes} v.$$

$R_E$ is the composition of two maps which are natural in $E$, and hence is itself natural in $E$. Direct checking on elementary tensors shows that $R_E$ and $T_E$ are mutually inverse maps. Hence, $R$ is a natural, isometric isomorphism from $\mathcal{U}_G(\ell^1(G) \underset{H}{\widehat{\otimes}} \cdot)$ to $\ell^1(G/H) \widehat{\otimes} (\mathcal{U}_H \cdot)$ as required. ☐

**Lemma 4.7.** *Let $X$ be a left Banach $\ell^1(G)$-module. Regard it as an $\ell^1(G)$-bimodule $X_\varepsilon$ by defining the right $G$-action on $X$ to be trivial (i.e. augmentation). Then for all $n$ there is a topological isomorphism*

$$\mathcal{H}^n(\ell^1(G), X'_\varepsilon) \cong \mathrm{Ext}^n_{\ell^1(G)}(X, \mathbb{C}).$$

**Proof.** This is a special case of the isomorphisms

$$\mathcal{H}^*(A, \mathcal{L}(E,F)) \cong \mathrm{Ext}^*_{A^e}(A, \mathcal{L}(E,F)) \cong \mathrm{Ext}^*_A(E,F)$$

valid for any unital Banach algebra $A$ and any left Banach $A$-modules $E$ and $F$ (see [**5**, Theorem III.4.12]). ☐

**Lemma 4.8.** *Fix a Banach algebra $A$ and an index set $\mathbb{I}$; for each $x \in \mathbb{I}$ let $0 \leftarrow M_0(x) \leftarrow M_1(x) \leftarrow \cdots$ be a chain complex of contractive left Banach $A$-modules and continuous $A$-module maps.*

*Suppose that, for each $n \in \mathbb{N}$, the family of linear maps $(M_n(x) \to M_{n-1}(x))_{x \in \mathbb{I}}$ is uniformly bounded. Then, for every left Banach $A$-module $N$, there is an isometric isomorphism of chain complexes*

$${}_A\mathrm{Hom}\Bigg(\bigoplus_{x \in \mathbb{I}}^{[1]} M_*(x), N\Bigg) \cong \bigoplus_{x \in \mathbb{I}}^{[\infty]} {}_A\mathrm{Hom}(M_*(x), N).$$

**Outline of the proof.** Let $n \geqslant 0$. Given

$$\psi : \bigoplus_{x}^{[1]} M_n(x) \to N,$$

define $\psi_y : M_n(y) \to N$ to be the map obtained by restricting $\psi$ to the embedded copy of $M_n(y)$. Then $(\psi_y)_{y\in\mathbb{I}}$ is a well-defined element of

$$\bigoplus_{y\in\mathbb{I}}^{[\infty]} {}_A\operatorname{Hom}(M_n(y), N).$$

It is then straightforward to check that the function $\theta^n : \psi \mapsto (\psi_y)_{y\in\mathbb{I}}$ is an isometric linear isomorphism, and that the maps $\theta^n$ assemble to form a chain map. $\square$

**Proof of Theorem 4.1.** First observe that, by Lemma 4.7, there is a topological isomorphism

$$\mathcal{H}^n(\ell^1(G), \ell^1(S)') \cong \operatorname{Ext}^n_{\ell^1(G)}(\ell^1(S), \mathbb{C}).$$

Since Ext may be calculated up to topological isomorphism using any admissible projective resolution of the first variable, it therefore suffices to construct an admissible $\ell^1(G)$-projective resolution $0 \leftarrow \ell^1(S) \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ with the following property:

($*$) the cochain complex $0 \to {}_{\ell^1(G)}\operatorname{Hom}(P_*, \mathbb{C})$ is topologically isomorphic to

$$0 \to \bigoplus_{x\in\mathbb{I}}^{[\infty]} \mathcal{C}^*(\ell^1(H_x), \mathbb{C}).$$

We do this as follows. For each $x \in \mathbb{I}$, let $0 \leftarrow \mathbb{C} \leftarrow P_*(x)$ denote the one-sided bar resolution of $\mathbb{C}$ by left $\ell^1(H_x)$-projective modules, i.e.

$$0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon_x} \ell^1(H_x) \xleftarrow{d_0^x} \ell^1(H_x)^{\widehat{\otimes}\, 2} \xleftarrow{d_1^x} \cdots,$$

where $\varepsilon_x$ is the augmentation character and $d_n^x : \ell^1(H_x)^{\widehat{\otimes}\, n+2} \to \ell^1(H_x)^{\widehat{\otimes}\, n+1}$ is the $\ell^1(H_x)$-module map given by

$$\begin{aligned} d_n^x(e_{h(0)} \otimes \cdots \otimes e_{h(n+1)}) = \sum_{j=0}^{n} (-1)^j e_{h(0)} \otimes \cdots \otimes e_{h(j)h(j+1)} \otimes \cdots \otimes e_{h(n+1)} \\ + (-1)^{n+1} e_{h(0)} \otimes \cdots \otimes e_{h(n)} \end{aligned}$$

for $h(0), h(1), \ldots, h(n+1) \in H_x$. The complex $0 \leftarrow \mathbb{C} \leftarrow P_*(x)$ is 1-contractible in $\mathsf{Ban}$ (one can take as splitting homotopy the sequence $(s_n)$ defined by $s_n : w \mapsto e_{\mathbf{1}} \otimes w$, where $\mathbf{1}$ is the identity element of $G$), and applying $\ell^1(G/H_x) \mathbin{\widehat{\otimes}} \cdot$ to it also yields a 1-contractible complex in $\mathsf{Ban}$. Therefore, by Lemma 4.6 and Remark 4.4, the chain complex

$$0 \leftarrow \ell^1(G/H_x) \xleftarrow{\tilde{\varepsilon}_x} \ell^1(G) \mathbin{\widehat{\otimes}}_{H_x} P_0(x) \xleftarrow{\tilde{d}_0^x} \ell^1(G) \mathbin{\widehat{\otimes}}_{H_x} P_1(x) \xleftarrow{\tilde{d}_1^x} \cdots \tag{4.1}$$

is 1-contractible as a complex in $\mathsf{Ban}$. Here, we have written $\tilde{\varepsilon}_x$ for the $\ell^1(G)$-module map

$$\ell^1(G) \underset{H_x}{\otimes} \varepsilon_x,$$

and $\tilde{d}_n^x$ for the $\ell^1(G)$-module map

$$\ell^1(G) \underset{H_x}{\otimes} d_n^x.$$

For each $n \geqslant 0$ let $P_n$ be the left Banach $\ell^1(G)$-module

$$P_n := \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(G) \underset{H_x}{\widehat{\otimes}} P_n(x);$$

write $\tilde{\varepsilon}$ for the $\ell^1$-sum of all the $\tilde{\varepsilon}_x$, and define $\tilde{d}_n$ similarly for each $n \geqslant 0$. As the $\ell^1$-sum of 1-contractible complexes is 1-contractible (by Lemma 2.3), the complex of Banach $\ell^1(G)$-modules

$$0 \leftarrow \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(G/H_x) \xleftarrow{\tilde{\varepsilon}} P_0 \xleftarrow{\tilde{d}_0} P_1 \xleftarrow{\tilde{d}_1} \cdots \tag{4.2}$$

is 1-contractible as a complex in $\mathsf{Ban}$. There is an isomorphism of $\ell^1(G)$-modules

$$\ell^1(S) = \ell^1\left(\coprod_{x\in\mathbb{I}} \mathrm{Orb}_x\right) \underset{1}{\cong} \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(\mathrm{Orb}_x) \underset{1}{\cong} \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(G/H_x),$$

where in the last step we identified the orbit of $x$ with the coset space $G/H_x$ via the correspondence $g \cdot x \leftrightarrow gH_x$. Hence, $0 \leftarrow \ell^1(S) \leftarrow P_*$ is an admissible complex of Banach $\ell^1(G)$-modules.

Moreover, for each $x \in \mathbb{I}$ and $n \geqslant 0$, Lemma 4.5 provides an isometric isomorphism of left $\ell^1(G)$-modules

$$\ell^1(G) \underset{H_x}{\widehat{\otimes}} P_n(x) \underset{1}{\cong} \ell^1(G) \widehat{\otimes} \ell^1(H_x)^{\widehat{\otimes}\, n},$$

and taking the $\ell^1$-direct sum over all $x$ yields isometric isomorphisms of left $\ell^1(G)$-modules

$$P_n = \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(G) \underset{H_x}{\widehat{\otimes}} P_n(x) \underset{1}{\cong} \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(G) \widehat{\otimes} \ell^1(H_x)^{\widehat{\otimes}\, n} \underset{1}{\cong} \ell^1(G) \widehat{\otimes} \left( \bigoplus_{x\in\mathbb{I}}^{[1]} \ell^1(H_x)^{\widehat{\otimes}\, n} \right),$$

from which we see that each $P_n$ is free, and hence projective, as an $\ell^1(G)$-module.

*Combining the previous two paragraphs we see that $0 \leftarrow \ell^1(S) \leftarrow P_*$ is an admissible resolution of $\ell^1(S)$ by $\ell^1(G)$-projective modules.*

It remains to verify the condition $(*)$. Observe that, for each $x$,

$${}_{\ell^1(H_x)}\mathrm{Hom}(P_*(x), \mathbb{C}) \underset{1}{\cong} \mathcal{C}^*(\ell^1(H_x), \mathbb{C});$$

hence, by Lemma 4.5 we have

$$\mathcal{C}^*(\ell^1(H_x), \mathbb{C}) \underset{1}{\cong}{}_{\ell^1(G)} \operatorname{Hom}\Big(\ell^1(G) \mathop{\hat{\otimes}}_{H_x} P_*(x), \mathbb{C}\Big),$$

and taking the $\ell^\infty$-sum over all $x$ yields

$$\begin{aligned} \bigoplus_{x\in\mathbb{I}}^{[\infty]} \mathcal{C}^*(\ell^1(H_x), \mathbb{C}) &\underset{1}{\cong} \bigoplus_{x\in\mathbb{I}}^{[\infty]} {}_{\ell^1(G)} \operatorname{Hom}\Big(\ell^1(G) \mathop{\hat{\otimes}}_{H_x} P_*(x), \mathbb{C}\Big) \\ &\underset{1}{\cong} {}_{\ell^1(G)} \operatorname{Hom}(P_*, \mathbb{C}), \end{aligned}$$

where for the last isomorphism we applied Lemma 4.8. □

**Remark 4.9.** The proof of Theorem 4.1 does not require the full force of 1-contractibility: given Lemmas 4.5–4.8, it suffices to know only that for each $x$ the one-sided bar resolution of $\mathbb{C}$ as a left $\ell^1(H_x)$-module admits a contracting homotopy (in $\mathsf{Ban}$), whose constituent splitting maps are in each fixed degree $n$ bounded by some constant independent of $x$. (The author thanks the referee for this observation.)

However, it is no easier to show that the bar resolution has such a property of 'uniformly bounded contractibility' than to show that it is 1-contractible in our sense. Since it is the author's belief that the notion of 1-contractibility for chain complexes has good stability properties which turn out to be useful, we have chosen to focus on this property. It would be interesting to know if it fails in situations where a more general notion of 'degreewise uniform contractibility' is successful.

## 5. Corollaries and closing remarks

**Corollary 5.1.** *Let $G$ be a commutative–transitive, discrete group. Then $I_0(G)$ is simplicially trivial.*

**Proof.** This is immediate from combining Lemma 3.1 and Theorem 3.5. □

**Remark 5.2.** Recalling that biflat Banach algebras are simplicially trivial, it is natural to enquire if our result might follow from biflatness of $I_0(G)$. To see that this is not always the case, observe that if $I_0(G)$ is biflat, then $\mathcal{H}^2(I_0(G), \mathbb{C}_{\mathrm{ann}}) = 0$ by [**11**, Theorem 4.13], while it is known that

$$\mathcal{H}^2(I_0(F_2), \mathbb{C}_{\mathrm{ann}}) \cong \mathcal{H}^2(\ell^1(F_2), \mathbb{C}) \neq 0.$$

While it is known that $I_0(G)$ is amenable if and only if $G$ is, there appears to be no analogous characterization of precisely when $I_0(G)$ is biflat.

**Question.** Let $G$ be a discrete group. If $I_0(G)$ is biflat, is $G$ amenable?

**Remark 5.3.** We remarked earlier that $F_2 \times F_2$ is not commutative–transitive. The arguments above show that $I_0(F_2 \times F_2)$ is *not* simplicially trivial, since its second simplicial cohomology will contain a copy of the second bounded cohomology of $C_{(x,\mathbf{1})}$, where $x \in F_2 \setminus \{\mathbf{1}\}$. (To see that $\mathcal{H}^2(\ell^1(C_{(x,\mathbf{1})}), \mathbb{C})$ is non-zero, observe that $C_{(x,\mathbf{1})} \cong C_x \times F_2$ is the direct product of a commutative group with $F_2$, and hence has the same bounded cohomology as $F_2$; by [**6**, Proposition 2.8], $\mathcal{H}^2(\ell^1(F_2), \mathbb{C}) \neq 0$.)

The question of what happens for augmentation ideals in *non-discrete*, locally compact groups is much trickier to solve since measure-theoretic considerations come into play. Johnson and White have shown [**7**] that the augmentation ideal of $\mathrm{PSL}_2(\mathbb{R})$ is not even weakly amenable; in contrast, $\mathrm{PSL}_2(\mathbb{Z})$ is known to be commutative–transitive and so by our results its augmentation ideal is simplicially trivial.

**Acknowledgements.** The results in this paper are taken, with some modifications in the presentation, from Chapter 2 of the author's PhD thesis [**2**], which was written at the University of Newcastle upon Tyne with support from an EPSRC grant.

Particular thanks are due to K. Goda, N. Grønbæk, A. Pourabbas and M. C. White for useful exchanges, and to the referee for comments and suggestions, especially as regards terminology. The author also thanks the University of Manitoba for its support while this paper was written.

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