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COMPLETION OF NORMED ALGEBRAS OF POLYNOMIALS

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Let $\mathcal{P}$ be the algebra of polynomials in one indeterminate x over the complex field $\mathbf{C}$. Suppose $\|\cdot\|$ is a norm on $\mathcal{P}$ such that the coefficient functionals $c_j : \Sigma\, \alpha_i x^i \to \alpha_j$ $(j = 0, 1, 2, \cdots)$ are all continuous with respect to $\|\cdot\|$, and let $K \subset \mathbf{C}$ be the set of characters on $\mathcal{P}$ which are $\|\cdot\|$-continuous. Then K is compact, $\mathbf{C}\backslash K$ is connected, and $0 \in K$. Let A be the completion of $\mathcal{P}$ with respect to $\|\cdot\|$. Then A is a singly generated Banach algebra, with space of characters (homeomorphic with) K. The functionals c_j have unique extensions to bounded linear functionals on A, and the map $a \to \Sigma\, c_i(a) x^i$ $(a \in A)$ is a homomorphism from A onto an algebra of formal power series with coefficients in $\mathbf{C}$. We say that A is an algebra of power series if this homomorphism is one-to-one, that is if $a \in A$ and $a \neq 0$ imply $c_j(a) \neq 0$ for some j.

We are interested in the relationship between the propositions (S): A is semi-simple, and (P): A is an algebra of power series. Loy (1974; Theorem 5) has proved that if $0 \in K^0$ (the interior of K), then (P) implies (S). With the further conditions that K^0 is connected and dense in K, it is easy to see that (S) and (P) are equivalent (Theorem 2). Examples show that without the given restrictions on K, (S) does not imply (P), and without the condition $0 \in K^0$, (P) does not imply (S). The equivalence between (S) and (P) has a generalization to the case of a projective tensor product $B \hat{\otimes} \mathcal{P}$, where B is a commutative Banach algebra with identity and $\mathcal{P}$ is suitably normed (Theorem 5). For a discussion of tensor products of Banach algebras, and in particular of the question of semi-simplicity of $B \hat{\otimes} A$ when B and A are semi-simple, see Gelbaum's paper (1962).

1

EXAMPLES. (a) Let K be a compact set in $\mathbf{C}$ with $0 \in K$ and $\mathbf{C}\backslash K$ connected. If A is the completion of $\mathcal{P}$ with respect to $|\cdot|_K$ (supremum norm over K), then A is the algebra of functions continuous on K and analytic on K^0, and A is semi-simple.

(i) If the coordinate functionals c_0 and c_1 are $|\cdot|_K$-continuous on P, then $0 \in K^0$. This follows from Theorem 3.4.13, Section 2.3, and Corollary 1.6.7 of Browder's book (1969), since c_1 is a point derivation at c_0 on A. Thus if $0 \notin K^0$, A cannot be an algebra of power series in the sense described. On the other hand, if $0 \in K^0$, Cauchy's inequalities show that all the c_j are $|\cdot|_K$-continuous on $\mathcal{P}$.

(ii) Now assume $0 \in K^0$. If K^0 is not dense in K, then there are continuous functions on K, not vanishing identically but vanishing on K^0. Since such a function f is in A and has $c_j(f) = 0$ for all j, A is not an algebra of power series.

(iii) If K^0 is not connected, then A need not be an algebra of power series; for instance if K consists of two disjoint closed discs, A is not an algebra of power series.

On the other hand, it is possible to have K^0 not connected and A an algebra of power series. For example, let K be the "cornucopia", Gamelin (1969; page 152), translated so that 0 is in the interior of the spiral.

(b). The first of the above examples is somewhat unsatisfactory, in that the given completion of $\mathcal{P}$ fails to be an algebra of power series because not all the c_j are continuous. We now give an example of a set K with $0 \in K \backslash K^0$, and a norm $\|\cdot\|$ on $\mathcal{P}$, such that $\|\cdot\|$-continuous characters on $\mathcal{P}$ are just the points of K, all the c_j are $\|\cdot\|$-continuous, and (S) holds but (P) fails for the completion of $\mathcal{P}$ with respect to $\|\cdot\|$.

Let K be a closed disc with positive radius and containing 0 as a boundary point, and let $\{M_k : k = 0, 1, 2, \cdots\}$ be a sequence of positive numbers such that:

(i) $$M_0 = 1 \text{ and } M_k/(M_r M_{k-r}) \geqq \binom{k}{r} \text{ for } r = 0, 1, \cdots, k;$$

(ii) $$(M_k/k!)^{1/k} \to \infty \text{ as } k \to \infty.$$

Let $D^\infty(K)$ denote the algebra of infinitely differentiable functions on K, and define

$$A = \{f \in D^\infty(K) : \|f\| = \sum_{k=0}^{\infty} |f^{(k)}|_K / M_k < \infty\}.$$

Then A is a Banach function algebra on K, Dales and Davie (1973); Theorem 1.6). Clearly $\mathcal{P} \subset A$, and the following lemma implies that the $\|\cdot\|$-completion of $\mathcal{P}$ is A.

LEMMA. *With the above notation, $\mathcal{P}$ is dense in A.*

PROOF. To simplify notation, we suppose temporarily that K is the closed unit disc. Fix $f \in A$. First note that, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\sum_{k=0}^{\infty} \frac{1}{M_k} \sup\{|f^{(k)}(z) - f^{(k)}(w)| : |z - w| < \delta\} < \varepsilon. \tag{1}$$

The n'th Césaro mean of the Taylor series for f is $\sigma_n = f * K_n$, where $\{K_n\}$ is Fejér's kernel. Thus, writing $f(t)$ for $f(e^{it})$,

$$(\sigma_n - f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s)[f(t-s) - f(t)]ds,$$

so that, for any $\delta > 0$,

$$\begin{aligned} \|\sigma_n - f\| &\leqq \sum_{k=0}^{\infty} \frac{1}{M_k} \sup_t \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(s)|\, |f^{(k)}(t-s) - f^{(k)}(t)| ds \\ &\leqq \sum_{k=0}^{\infty} \frac{1}{M_k} \sup_t \sup_{|s| \leqq \delta} |f^{(k)}(t-s) - f^{(k)}(t)| \\ &\quad + 2\|f\| \sup_{|s| \geqq \delta} |K_n(s)|. \end{aligned}$$

Hence, by (1) and the standard properties of $\{K_n\}$, $\sigma_n \to f$ in A. Since $\sigma_n \in \mathcal{P}$, the lemma follows.

We now return to the construction of the example (in particular, we are once again assuming that 0 is a boundary point of K). Clearly, the functionals c_j are all $\|\cdot\|$-continuous on $\mathcal{P}$, and because of (ii), it follows from Theorem 1.9 of Dales and Davie (1973) that each character on A is evaluation at some point of K. Now, an algebra of infinitely differentiable functions on a plane set is *quasi-analytic* if, for each point x in the set and each function f in the algebra,

$$f^{(k)}(x) = 0 \text{ for } k = 0, 1, 2, \cdots \text{ implies } f = 0 \tag{2}$$

(cf. Dales and Davie (1973); Definition 1.10). If f belongs to the algebra A, then $c_k(f) = f^{(k)}(0)/k!$ for $k = 0, 1, 2, \cdots$. Since (2) holds for all $x \in K$ and all $f \in A$ if and only if it holds for $x = 0$ and all $f \in A$, we see that A satisfies (P) if and only if A is quasi-analytic.

Theorem 1 of Korenbljum (1965) states that the class $\mathcal{D}\{M_k\} = \{f \in D(K)$: there is a number C_f such that $|f^{(k)}|_K \leqq C_f M_k$ for $k = 0, 1, 2, \cdots\}$ is quasi-analytic if and only if $\Sigma\, 1/\beta_k = \infty$, where $\beta_k = \inf\{(\sqrt{M_n})^{1/n} : n \geqq k\}$. It follows that, if we take $M_k = (k!)^{\alpha}$ with $\alpha > 1$ (so that (i) and (ii) hold), then A is quasi-analytic if and only if $\alpha \leqq 2$. Therefore, by choosing $M_k = (k!)^{\alpha}$ with $\alpha > 2$, we obtain the required example.

We note incidentally that, if we choose $\{M_k\}$ so that A *is* quasi-analytic, we have an example of a Banach algebra of power series which is semi-simple and also has $0 \notin K^0$, where K is the spectrum of the indeterminate. This appears to answer a question of Loy (1974)—see the sentence immediately preceding Theorem 7.

(c) If the condition $0 \in K^0$ does not hold, then (P) does not imply (S): if $\{\alpha_n\}$ is a sequence of positive numbers with $\alpha_n^{1/n} \to 0$ as $n \to \infty$, then the algebra $K\langle\alpha_n\rangle$ discussed in Rickart (1960; A.2.12) is an appropriate example.

2

THEOREM. *Let $\|\cdot\|$ be a norm on $\mathcal{P}$, and suppose that the set K of $\|\cdot\|$-continuous characters on $\mathcal{P}$ satisfies $0 \in K^0$, K^0 is connected, and K^0 is dense in K. If A is the completion of $\mathcal{P}$ with respect to $\|\cdot\|$, then A is semi-simple if and only if A is an algebra of power series.*

PROOF. It is easy to see that the Gelfand transform $a^\wedge$ of $a \in A$ is analytic at 0, that a Gelfand transform is completely determined by its Taylor series at 0, and that the Taylor coefficients at 0 of $a^\wedge$ are just the numbers $c_j(a)$. The theorem follows from these observations.

The following example gives a norm on $\mathcal{P}$ such that the set of continuous characters satisfies the hypotheses of Theorem 2, and the completion of $\mathcal{P}$ is not semi-simple. Let K be the closed unit disc, and norm $\mathcal{P}$ by $\|p\| = |p|_K + |p'(1)|$. Let A_0 be the disc algebra, and make the Banach space direct sum $A_1 = A_0 \oplus \mathbf{C}$ into a Banach algebra by defining

$$(f, \lambda)(g, \mu) = (fg, f(1)\mu + g(1)\lambda).$$

Then the completion of $\mathcal{P}$ with respect to $\|\cdot\|$ can be identified with the closure in A_1 of $\mathcal{P}_1 = \{(p, p'(1)) \colon p \in \mathcal{P}\}$. Since the linear functional $p \to p'(1)$ is not $|\cdot|_K$-continuous on $\mathcal{P}$, the closure of $\mathcal{P}_1$ contains $(0, 1)$, and therefore is all of A_1. In particular, the $\|\cdot\|$-continuous characters on $\mathcal{P}$ are just the points of K, and the completion of $\mathcal{P}$ is not semi-simple.

This example has also been found by Loy (1974a).

Now let B be a commutative algebra with identity. Since the monomials $\{x^n \colon n = 0, 1, 2, \cdots\}$ are a basis for $\mathcal{P}$, every element of $B \otimes \mathcal{P}$ has a unique representation as a finite sum $\Sigma b_i \otimes x^i (b_i \in B)$, so that there are well-defined linear coefficient mappings $\gamma_j \colon B \otimes \mathcal{P} \to B$, where $\gamma_j(\Sigma b_i \otimes x^i) = b_j$ $(j = 0, 1, 2, \cdots)$. If we identify $\mathcal{P}$ with the subalgebra $1 \otimes \mathcal{P}$ of $B \otimes \mathcal{P}$, then $\gamma_j \mid \mathcal{P} = c_j$ for all j. If B is a Banach algebra, and $\mathcal{P}$ is given a norm which makes $\mathcal{P}$ a normed algebra, then the projective tensor product norm on $B \otimes \mathcal{P}$ is given by

$$\|u\|_p = \inf\{\Sigma\|b_i\| \, \|p_i\| \colon b_i \in B, p_i \in \mathcal{P}, u = \Sigma b_i \otimes p_i\} \text{ for } u \in B \otimes \mathcal{P}$$

and $B \otimes \mathcal{P}$ is a normed algebra with respect to $\|\cdot\|_p$. By saying that the completion $B \hat{\otimes} \mathcal{P}$ of $B \otimes \mathcal{P}$ is an algebra of power series with coefficients in B, we mean that all the γ_i are $\|\cdot\|_p$-continuous, and that their unique continuous extensions to $B \hat{\otimes} \mathcal{P}$ separate the points of $B \hat{\otimes} \mathcal{P}$.

3

LEMMA. *γ_i is continuous on $B \otimes \mathcal{P}$ if and only if c_i is continuous on $\mathcal{P}$.*

PROOF. Since $\|\cdot\|_p$ restricts to the original norm on $\mathcal{P}$, and $\gamma_i \mid \mathcal{P} = c_i$, the necessity is clear. Conversely, suppose c_i is continuous on $\mathcal{P}$. For each linear functional λ on B, there is a well-defined linear mapping $h(\lambda): B \otimes \mathcal{P} \to \mathcal{P}$ defined by $h(\lambda)(\Sigma b_i \otimes p_i) = \Sigma \lambda(b_i) p_i$. Since

$$\| h(\lambda)(\Sigma b_i \otimes p_i \| \leqq \Sigma | \lambda(b_i) | \; \| p_i \|, \lambda$$

continuous implies $(h\lambda)$ continuous and $\| h(\lambda) \| \leqq \| \lambda \|$. Since $\lambda(\gamma_i(u)) = c_i(h(\lambda)(u))$, and $\| \gamma_i(u) \| = \sup\{| \lambda(\gamma_i(u)) | : \| \lambda \| \leqq 1\}$, c_i continuous implies γ_i continuous (actually, $\| \gamma_i \| = \| c_i \|$).

We now assume that the c_i are all continuous on $\mathcal{P}$, and again write A for the completion of $\mathcal{P}$. If Φ is the space of characters on B, and K is the space of continuous characters on $\mathcal{P}$, then the space of characters on $B \hat{\otimes} \mathcal{P}$ is $\Phi \times K$.

4

THEOREM. (i) *If $B \hat{\otimes} \mathcal{P}$ is semi-simple and A is an algebra of power series, then $B \hat{\otimes} \mathcal{P}$ is an algebra of power series with coefficients in B.*

(ii) *If $B \hat{\otimes} \mathcal{P}$ is an algebra of power series with coefficients in B, and if B and A are semi-simple, then $B \hat{\otimes} \mathcal{P}$ is semi-simple.*

PROOF. (i) First, the assumption that A is an algebra of power series implies that all the c_i are continuous on $\mathcal{P}$, so by Lemma 3 the γ_i are continuous on $B \otimes \mathcal{P}$. If $0 \neq u \in B \hat{\otimes} \mathcal{P}$, then there are $\phi \in \Phi$, $\zeta \in K$ such that $u^\wedge(\phi, \zeta) \neq 0$. If $\gamma_i(u) = 0$ for all j, then $\phi(\gamma_i(u)) = 0$, and therefore $c_j(h(\phi)(u)) = 0$ for all j ($h(\phi)$ is defined in the proof of Lemma 3). Since A is an algebra of power series, this implies $h(\phi)(u) = 0$. But $h(\phi)(u)^\wedge(\zeta) = u^\wedge(\phi, \zeta) \neq 0$, a contradiction.

(ii) Suppose $u \in B \hat{\otimes} \mathcal{P}$ and $u^\wedge(\phi, \zeta) = 0$ for all $(\phi, \zeta) \in \Phi \times K$. Then $h(\phi)(u)^\wedge(\zeta) = 0$ for all $(\phi, \zeta) \in \Phi \times K$. Since A is semi-simple, it follows that $h(\phi)(u) = 0$ for all $\phi \in \Phi$. Therefore $\phi(\gamma_i(u)) = c_j(h(\phi)(u)) = 0$ for all $\phi \in \Phi$ and all j. Since B is semi-simple, this implies $\gamma_i(u) = 0$ for all j, and therefore $u = 0$, since $B \hat{\otimes} \mathcal{P}$ is assumed to be an algebra of power series with coefficients in B. Thus $B \hat{\otimes} \mathcal{P}$ is semi-simple, and the proof is complete.

By combining Theorems 2 and 4, we obtain the following generalization of Theorem 2 for the algebra $B \otimes \mathcal{P}$.

5

THEOREM. *Let B be a commutative Banach algebra with identity, and let $\mathcal{P}$ be normed so that the set K of continuous characters on $\mathcal{P}$ satisfies $0 \in K^0$, K^0 is connected, and K^0 is dense in K. Then $B \hat{\otimes} \mathcal{P}$ is semi-simple if and only if B is semi-simple and $B \hat{\otimes} \mathcal{P}$ is an algebra of power series with coefficients in B.*

PROOF. Since $0 \in K^0$, the coefficient functionals c_j are continuous on $\mathcal{P}$, so by Lemma 3, the coefficient mappings γ_j are $\|\cdot\|_p$-continuous on $B \otimes \mathcal{P}$.

If $B \hat{\otimes} \mathcal{P}$ is semi-simple, then A (the completion of $\mathcal{P}$) is semi-simple, since A is the closure of $\mathcal{P}$ in $B \hat{\otimes} \mathcal{P}$. By Theorem 2, A is an algebra of power series, so by Theorem 4(i), $B \hat{\otimes} \mathcal{P}$ is an algebra of power series with coefficients in B.

If $B \hat{\otimes} \mathcal{P}$ is an algebra of power series with coefficients in B, then $c_j = \gamma_j \mid A$ for all j implies that A is an algebra of power series, so by Theorem 2, A is semi-simple. If also B is semi-simple, then Theorem 4(ii) implies that $B \hat{\otimes} \mathcal{P}$ is semi-simple.

To conclude, we indicate two ways in which the above results can be extended. First, let n be a positive integer, replace $\mathcal{P}$ by the algebra $\mathcal{P}_n$ of polynomials in n commuting indeterminates over $\mathbf{C}$, consider the obvious coefficient functionals c_j (and γ_j) indexed by multi-indices $j = (j_1, \cdots, j_n)$, and make the obvious definition of algebra of power series in n indeterminates (over B). Then the set of $\|\cdot\|$-continuous characters on $\mathcal{P}_n$ is a compact, polynomially convex set in $\mathbf{C}^n$, and the results from Theorem 2 to Theorem 5 remain true.

Secondly, let N denote the least cross norm (or injective norm) on $B \otimes \mathcal{P}$:

$$N(\Sigma b_i \otimes p_i) = \sup \{|\Sigma \lambda(b_i)\mu(p_i)| : \lambda \in B_1^*, \mu \in P_1^*\}.$$

(Here we have written E_1^* for the closed unit ball in the dual of a normed space E.) Let ν be any algebra norm on $B \otimes \mathcal{P}$ which is at least as strong as N, and which is equivalent to the given norms on B and $\mathcal{P}$ (identified with $B \otimes 1$ and $1 \otimes \mathcal{P}$ respectively). Then Lemma 3 remains valid if the projective norm is replaced by ν (we are indebted to the referee for that observation and for suggesting this line of extension). Moreover, the space of ν-continuous characters on $B \otimes \mathcal{P}$ is still $\Phi \times K$, and Theorem 4 and 5 hold with $B \hat{\otimes} \mathcal{P}$ replaced by the completion of $B \otimes \mathcal{P}$ with respect to ν; there are no formal changes in the proofs.

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