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H. G. Dales and A. Millinoton

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# Translation-invariant linear operators

BY H. G. DALES AND A. MILLINGTON
*School of Mathematics, University of Leeds, Leeds LS2 9JT*

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The theory of translation-invariant operators on various spaces of functions (or measures or distributions) is a well-trodden field. The problem is to decide, first, whether or not a linear operator between two function spaces on, say, $\mathbb{R}$ or $\mathbb{R}^+$ which commutes with one or many translations on the two spaces is necessarily continuous, and, second, to give a canonical form for all such continuous operators. In some cases each such operator is zero. The second problem is essentially the 'multiplier problem', and it has been extensively discussed; see [7], for example.

In this paper, we shall give some further results about these two problems. In Section 1, we shall introduce the subject and recall some of the known results. In Section 2, we shall show that, if $E = C_0(\mathbb{R})$ or $C^b(\mathbb{R})$, and if $T: E \to L^1(\mathbb{R})$ is a linear operator which commutes with a single non-trivial shift $S_a$, then necessarily $T = 0$, but that, on the other hand, there is a closed linear subspace $E$ of $C^b(\mathbb{R})$ and a non-zero continuous linear operator $T: E \to L^1(\mathbb{R})$ such that $T$ commutes with each shift $S_a$.

It is well-known that there is a discontinuous linear operator $T: L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ such that $T$ commutes with a single left shift $L_a$. In Section 3, we shall show that there are discontinuous linear operators which commute with all left shift operators.

## 1. *Introduction*

Let $E$ and $F$ be linear spaces. Then $\mathscr{L}(E,F)$ is the space of all linear maps from $E$ into $F$. We write $\mathscr{L}(E)$ for $\mathscr{L}(E,E)$; the identity in $\mathscr{L}(E)$ is $I_E$, and the set of invertible operators in the algebra $\mathscr{L}(E)$ is $\operatorname{Inv}\mathscr{L}(E)$. In the case where $E$ and $F$ are Banach spaces, $\mathscr{B}(E,F)$ denotes the Banach space of all bounded maps in $\mathscr{L}(E,F)$, and we write $\mathscr{B}(E)$ for $\mathscr{B}(E,E)$. The spectrum of $T \in \mathscr{B}(E)$ is $\sigma(T)$.

Let $E$ and $F$ be linear spaces, and let $R \in \mathscr{L}(E)$ and $S \in \mathscr{L}(F)$. A linear map $T: E \to F$ intertwines the pair $(R,S)$ if $TR = ST$. If $E$ and $F$ are Banach spaces and if $R \in \mathscr{B}(E)$ and $S \in \mathscr{B}(F)$, then we ask whether or not a map $T \in \mathscr{L}(E,F)$ which intertwines $(R,S)$ is automatically continuous. More generally, we consider the automatic continuity of a linear map $T$ which intertwines a family of pairs of operators on $E$ and $F$. We also consider the canonical form of continuous operators which intertwine such a family.

To describe the general known results, we require some terminology. Let $E$ be a linear space. Then $R \in \mathscr{L}(E)$ is algebraic if $p(R) = 0$ for some non-zero polynomial $p$. A linear subspace $F$ of $E$ is $\mathbb{C}[R]$-divisible if

$$(zI_E - R)(F) = F \quad (z \in \mathbb{C}).$$

It is easy to see that there is a maximum $\mathbb{C}[R]$-divisible subspace of $E$: this space is the algebraic spectral space of $R$, often denoted by $E_R(\varnothing)$. We shall be concerned with the condition that $E_R(\varnothing) = \{0\}$, that is, the case where $E$ has no non-zero, $\mathbb{C}[R]$-divisible subspaces.

Let $E$ and $F$ be Banach spaces, and let $R\in\mathscr{B}(E)$ and $S\in\mathscr{B}(F)$. Then $z\in\mathbb{C}$ is a critical eigenvalue of $(R,S)$ if $z$ is an eigenvalue of $S$ and if $(zI_E-R)(E)$ has infinite co-dimension in $E$.

An operator $T\in\mathscr{B}(E)$ is super-decomposable ([**8**]) if, for each open cover $\{U,V\}$ of $\mathbb{C}$, there exist $R,S\in\mathscr{B}(E)$ such that $RT = TR$, $R+S = I_E$, $\sigma(T\,|\,\overline{R(E)}) \subset U$, and $\sigma(T\,|\,\overline{S(E)}) \subset V$. For example, suppose that $T\in\mathscr{B}(E)$ is invertible and that

$$(\|T^n\| + \|T^{-n}\|)$$

is bounded. Then it is shown in [**8**] that $T$ is super-decomposable and that $T$ has no non-zero divisible subspace.

The following result is given in [**9**], extending earlier results of Johnson and Sinclair ([**5**], [**6**]).

THEOREM 1·1. *Let $E$ and $F$ be Banach spaces, let $R\in\mathscr{B}(E)$ and $S\in\mathscr{B}(F)$. Consider the following two conditions*:

(*a*) *each linear map which intertwines $(R,S)$ is automatically continuous*;

(*b*) *$(R,S)$ has no critical eigenvalues, and either $R$ is algebraic or $S$ has no non-zero divisible subspaces.*

*Then always* (*a*) *implies* (*b*), *and* (*b*) *implies* (*a*) *in the case where both $R$ and $S$ are super-decomposable.* ▮

(We remark that it would be of interest to establish that (*b*) implies (*a*) in the above theorem under weaker hypotheses than that $R$ and $S$ are super-decomposable. Some partial extensions to the theorem are given in [**9**].)

We apply the general theory to various translation-invariant spaces on $\mathbb{R}$ and $\mathbb{R}^+$.

*First we consider spaces on* $\mathbb{R}$. For $p > 0$, $L^p(\mathbb{R})$ denotes the usual space of functions $f$ such that $|f|^p$ is integrable with respect to Lebesgue measure on $\mathbb{R}$; if $p \geqslant 1$, then $L^p(\mathbb{R})$ is a Banach space. We denote by $C^b(\mathbb{R})$ the space of all bounded, continuous functions on $\mathbb{R}$, and by $C_0(\mathbb{R})$ the space of all continuous functions which vanish at infinity. Throughout the uniform norm over a set $S$ is denoted by $|\cdot|_S$, so that $C^b(\mathbb{R})$ and $C_0(\mathbb{R})$ are Banach spaces with respect to $|\cdot|_{\mathbb{R}}$. Let $E$ be any of these spaces, and, for $a\in\mathbb{R}$, define the shift operator $S_a$ on $E$ by

$$(S_a f)(t) = f(t-a) \quad (t\in\mathbb{R}, f\in E).$$

Then $S_a\in\operatorname{Inv}\mathscr{L}(E)$ and $S_{a+b} = S_a S_b$ $(a,b\in\mathbb{R})$. Clearly $S_a$ is not algebraic. In each case, $\|S_a^n\| = 1$ $(n\in\mathbb{Z})$, and so $S_a$ is a super-decomposable operator on $E$ with no non-zero divisible subspaces. By Theorem 1·1, an operator $T$ between two of these spaces $E$ and $F$ which intertwines $(S_a, S_a)$ for some $a\in\mathbb{R}\backslash\{0\}$ is automatically continuous if and only if the pair $(S_a, S_a)$ has no critical eigenvalue. If $q\in[1,\infty)$, then $S_a$ has no eigenvalue as an operator on $L^q(\mathbb{R})$, and so each intertwining operator mapping into $L^q(\mathbb{R})$ is automatically continuous. In Section 2, we shall show that, for $E = C_0(\mathbb{R})$ or $E = C^b(\mathbb{R})$, each linear operator $T: E\to L^1(\mathbb{R})$ such that $TS_a = S_a T$ for a single $a\in\mathbb{R}\backslash\{0\}$ is necessarily 0. However, there is a closed linear subspace $E$ of $C^b(\mathbb{R})$

which is translation-invariant (i.e. $S_a(E) \subset E$ $(a \in \mathbb{R})$), and $T \in \mathscr{B}(E, L^1(\mathbb{R}))$ such that $TS_a = S_a T$ for all $a \in \mathbb{R}$ and $T$ is non-zero.

Second we consider function spaces on $\mathbb{R}^+$. Here the situation differs according to whether we are working with left or right shifts.

Let $E$ be a function space on $\mathbb{R}^+$, and take $a > 0$. The right shift operator $R_a$ on $E$ is defined by

$$(R_a f)(t) = \begin{cases} f(t-a) & (t \geqslant a) \\ 0 & (0 \leqslant t < a) \end{cases}$$

for $f \in E$. Operators $T$ such that $TR_a = R_a T$ for some $a > 0$ have been much studied. See [**1**, **2**, **10**, **12**], for example. Specifically, let $p, q \in (0, \infty]$, and let $T: L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$ be a linear map such that $TR_a = R_a T$ for a single $a > 0$. Then $T$ is automatically continuous ([**2**]).

The left shift operator $L_a$ is defined for $a > 0$ by

$$(L_a f)(t) = f(t+a) \quad (t \in \mathbb{R}^+)$$

for $f \in E$. It is also defined on $M(\mathbb{R}^+)$, the Banach space of complex-valued, regular Borel measures on $\mathbb{R}^+$, by

$$(L_a \mu)(A) = \mu(A+a)$$

for a Borel set $A \subset \mathbb{R}^+$ and $\mu \in M(\mathbb{R}^+)$.

It is now the case that discontinuous linear operators arise. Let $L^p_{00}(\mathbb{R}^+)$ be the subspace of $L^p(\mathbb{R}^+)$ consisting of the functions of compact support. It is clear that $L^p_{00}(\mathbb{R}^+)$ is a $\mathbb{C}[L_a]$-divisible subspace of $L^p(\mathbb{R}^+)$, and so $E_{L_a}(\varnothing)$ contains $L^p_{00}(\mathbb{R}^+)$. Since $L_a$ is not algebraic, it follows from Theorem 1·1 that, for each fixed $a > 0$, there is a discontinuous linear map $T: L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$ such that $TL_a = L_a T$. For the sake of later comparison we briefly recall the direct argument for this ([**6**, **14**]).

Set $E = L^p(\mathbb{R}^+)$, $F = L^q(\mathbb{R}^+)$, $E_{00} = L^p_{00}(\mathbb{R}^+)$, and $F_{00} = L^q_{00}(\mathbb{R}^+)$. Then $E$ and $F$ are $\mathbb{C}[X]$-modules for the map $(p, f) \mapsto p(L_a)(f)$. Set $G = E_{00} + \mathbb{C}[X].f_0$, where $f_0(t) = \exp(-t^2)$ $(t \in \mathbb{R}^+)$, and let $h_0$ be an arbitrary non-zero element of $F_{00}$. Then the map

$$T_0 : g + p.f_0 \mapsto p.h_0, \quad G \to F_{00},$$

is a well-defined $\mathbb{C}[X]$-module homomorphism. The space $F_{00}$ is a $\mathbb{C}[X]$-divisible module, and so is an injective module because $\mathbb{C}[X]$ is a principal ideal domain. Thus there is a $\mathbb{C}[X]$-module homomorphism $T: E \to F_{00}$ extending $T_0$. Clearly $TL_a = L_a T$, and $T$ is discontinuous because $T | E_{00} = 0$, but $T(f_0) \neq 0$. Thus $T$ is the required map.

However, it is less easy to produce a discontinuous linear operator

$$T: L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$$

such that $TL_a = L_a T$ for *all* $a > 0$. We shall exhibit such an operator $T$ in Section 3.

Finally we consider maps $T: L^1(\mathbb{R}^+) \to L^1(\mathbb{R})$. Suppose that $TL_a = S_a T$ for one fixed $a > 0$. Then $T$ is automatically continuous: this does not follow from Theorem 1·1 because $L_a$ is not a super-decomposable operator on $L^1(\mathbb{R}^+)$, but it does follow from an extension of Theorem 1·1 given in [**9**]. On the other hand, it seems to be an interesting open question whether or not each $T$ such that $TR_a = S_a T$ for some $a > 0$ is automatically continuous.

## 2. *The form of continuous translation-invariant operators*

We first consider the special case of translation-invariant operators $T:E \to L^1(\mathbb{R})$, where $E$ is a closed, translation-invariant linear subspace of $C^b(\mathbb{R})$. As we noted in Section 1, if $TS_a = S_a T$ for one $a \in \mathbb{R}\backslash\{0\}$, then $T$ is automatically continuous. We consider when the only such $T$ is the zero operator. The first result is an easy calculation; it also follows by the argument in Theorem 2·3.

THEOREM 2·1. *Let $T:C_0(\mathbb{R}) \to L^1(\mathbb{R})$ be a linear operator such that $TS_a = S_a T$ for a single $a \in \mathbb{R}\backslash\{0\}$. Then $T = 0$.*

*Proof.* We suppose that $a = 1$. Since $T$ is automatically continuous, we may suppose that $\|T\| \leqslant 1$.

Take $f_0 \in C_0(\mathbb{R})$, and set $g_0 = Tf_0$. Then

$$\left\|\sum_{j=1}^{m} \alpha_j S_{a_j} g_0\right\|_1 \leqslant \left|\sum_{j=1}^{m} \alpha_j S_{a_j} f_0\right|_{\mathbb{R}} \tag{1}$$

for each $m \in \mathbb{N}$, each $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, and each $a_1, \ldots, a_n \in \mathbb{Z}$.

For each $n \in \mathbb{N}$, choose $x_n \in \mathbb{N}$ so that:

(i) $\displaystyle\int_{-x_n}^{x_n} |g_0| \geqslant \left(\frac{n-1}{n}\right) \|g_0\|_1$; (ii) $|f_0(t)| \leqslant \dfrac{1}{n}|f_0|_{\mathbb{R}}$ $(|t| \geqslant x_n)$.

Now choose $\alpha_j = 1$ and $a_j = 2jx_n$ for $j = 0, \ldots, n-1$, and consider the function

$$h_n = \sum_{j=0}^{n-1} S_{2jx_n} g_0 \in L^1(\mathbb{R}).$$

Set

$$I_p = [-x_n + 2px_n, x_n + 2px_n] \quad (p \in \mathbb{Z}).$$

Then, for $p = 0, \ldots, n-1$, we have

$$\begin{aligned}\int_{I_p} |h_n| &\geqslant \int_{I_p} \left(|S_{2px_n} g_0| - \sum_{j \neq p} |S_{2jx_n} g_0|\right) \\ &= \int_{-x_n}^{x_n} |g_0| - \sum_{j \neq p} \int_{I_{p-j}} |g_0| \\ &\geqslant \left(\frac{n-1}{n}\right) \|g_0\|_1 - \frac{1}{n} \|g_0\|_1 = \left(\frac{n-2}{n}\right) \|g_0\|_1.\end{aligned}$$

Thus $\|h_n\|_1 \geqslant (n-2) \|g_0\|_1$ because the intervals $I_0, \ldots, I_{n-1}$ are disjoint.

On the other hand, by (ii),

$$\left|\sum_{j=0}^{n-1} S_{2jx_n} f_0\right|_{\mathbb{R}} \leqslant 2|f_0|_{\mathbb{R}},$$

and so, by (1),

$$(n-2) \|g_0\|_1 \leqslant 2|f_0|_{\mathbb{R}} \quad (n \in \mathbb{N}).$$

Thus $g_0 = 0$, and so $T = 0$. ∎

We now show that the same result holds in the cases where $E = C^b(\mathbb{R})$ or $E = L^p(\mathbb{R})$, where $p > 1$.

Recall that a set $A$ in a Banach space $F$ is weakly sequentially compact if every sequence in $A$ has a subsequence which converges in the weak topology of $F$. Let $E$ and $F$ be Banach spaces, and let $U$ be the closed unit ball in $E$. An operator $T \in \mathscr{B}(E,F)$ is said to be weakly compact if the weak closure of $T(U)$ is compact in the weak topology of $F$.

LEMMA 2·2. *Let $g_0 \in L^1(\mathbb{R})$ be such that $\{S_n g_0 : n \in \mathbb{N}\}$ is weakly sequentially compact. Then $g_0 = 0$.*

*Proof.* There is a sequence $(n_r)$ in $\mathbb{N}$ such that $(S_{n_r} g_0)$ converges weakly in $L^1(\mathbb{R})$, say $S_{n_r} g_0 \to h$ weakly. Let $K$ be a compact subset of $\mathbb{R}$, take $\epsilon > 0$, and take $x_0 > 0$ so that

$$\int_{-x_0}^{x_0} |g_0| > \|g_0\|_1 - \epsilon.$$

Then $\int_K S_{n_r} g_0 \to \int_K h$ as $r \to \infty$. But $[n_r - x_0,\ n_r + x_0] \cap K = \emptyset$ for large $r$, and so $|\int_K S_{n_r} g_0| < \epsilon$ for large $r$. Thus $|\int_K h| < \epsilon$. It follows that $h = 0$.

We now show that $g_0 = 0$. Again take $\epsilon > 0$ and $x_0 > 0$ so that

$$\int_{-x_0}^{x_0} |g_0| > \|g_0\|_1 - \epsilon.$$

By passing to a subsequence of $(n_r)$, we may suppose that the intervals $[n_r - x_0, n_r + x_0]$ are disjoint, and so

$$\sum_{k=1}^{\infty} \int_{-x_0+n_k}^{x_0+n_k} S_{n_r} g_0 \to 0 \quad \text{as } r \to \infty.$$

It follows that

$$\int_{-x_0}^{x_0} |g_0| < 2\epsilon,$$

and so $g_0 = 0$, as required. ▮

THEOREM 2·3. *Let $E$ be $C^b(\mathbb{R})$ or $L^p(\mathbb{R})$, where $p > 1$, and let $T : E \to L^1(\mathbb{R})$ be a linear operator such that $TS_a = S_a T$ for a single $a \in \mathbb{R} \backslash \{0\}$. Then $T = 0$.*

*Proof.* We have noted that $T$ is continuous. We again suppose that $a = 1$.

We first note that each $T \in \mathscr{B}(E, L^1(\mathbb{R}))$ is weakly compact. By [**3**], IV·8·6, the Banach space $L^1(\mathbb{R})$ is weakly complete, and so, by [**3**], VI·7·6, each continuous linear operator from $C^b(\mathbb{R}) = C(\beta\mathbb{R})$ is weakly compact. Certainly each $T \in \mathscr{B}(E, L^1(\mathbb{R}))$ is weakly compact when $E$ is a reflexive Banach space (see [**3**], VI·4·3), and this applies to the spaces $E = L^p(\mathbb{R})$ where $p > 1$.

Now let $f \in E$. Since $\{S_n f : n \in \mathbb{N}\}$ is bounded in $E$ and $T$ is weakly compact,

$$\{TS_n f : n \in \mathbb{N}\} = \{S_n Tf : n \in \mathbb{N}\}$$

is weakly sequentially compact in $L^1(\mathbb{R})$. By Lemma 2·2, $Tf = 0$, and so $T = 0$. ▮

We note a related result ([**7**] theorem 5·2·5). Suppose that $1 < q < p < \infty$. Then each $T \in \mathscr{B}(L^p(\mathbb{R}), L^q(\mathbb{R}))$ such that $TS_a = S_a T$ for every $a \in \mathbb{R}$ is necessarily zero.

In the light of the above two results, the next theorem is perhaps rather surprising.

THEOREM 2·4. *There is a closed, translation-invariant linear subspace $E$ of $C^b(\mathbb{R})$ such that $C_0(\mathbb{R}) \subset E$, and a non-zero, continuous linear operator $T : E \to L^1(\mathbb{R})$ such that $TS_a = S_a T$ for all $a \in \mathbb{R}$.*

It is convenient to proceed via some lemmas.

LEMMA 2·5. *Let $n \in \mathbb{N}$. For each set $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{C}\backslash\{0\}$, there is a subset $\{n_1, \ldots, n_r\}$ of $\{1, \ldots, n\}$ such that*

$$|\alpha_{n_1} + \ldots + \alpha_{n_r}| \geqslant \frac{1}{\pi}(|\alpha_1| + \ldots + |\alpha_n|).$$

*Proof.* This is elementary ([**13**], 6·3). ∎

LEMMA 2·6. *There is a continuous function $f_0 : \mathbb{R}^+ \to [0,1]$ with the property that, for each $n \in \mathbb{N}$, for each $L \in \mathbb{R}$, for each set $\{a_1, \ldots, a_n\}$ of distinct elements of $\mathbb{R}$, and for each set $\{\alpha_1, \ldots, \alpha_n\} \subset \{0, 1\}$, there exists $t_0 \in \mathbb{R}^+$ with $t_0 > L$ such that*

$$f_0(t_0 - a_j) = \alpha_j \quad (j = 1, \ldots, n). \tag{2}$$

*Proof.* Let $((s_k, t_k) : k \in \mathbb{N})$ be an enumeration of the set $\{(p, 1/q) : p, q \in \mathbb{N}\}$.

For each $k \in \mathbb{N}$, define a finite set $\mathscr{F}_k$ of continuous functions as follows. Each $f \in \mathscr{F}_k$ has domain $[0, s_k]$, and range $[0,1]$, $f(0) = f(s_k) = 0$, and $f$ takes the value 0 or 1 on intervals of the form $[(r - \frac{1}{3})t_k, (r + \frac{1}{3})t_k]$, where $r \in \mathbb{N}$ and $(r + \frac{1}{3})t_k < s_k$.

From this collection $\{\mathscr{F}_k : k \in \mathbb{N}\}$ we construct $f_0$ as follows. On the interval $[0, s_1]$, we take $f_0$ to be equal to one of the finitely many elements of $\mathscr{F}_1$, on the interval $[s_1, 2s_1]$, $f_0$ is equal to a translate of one of the other elements of $\mathscr{F}_1$, and so on, until we have all the functions of $\mathscr{F}_1$ 'end-to-end'. We repeat this process successively for $\mathscr{F}_2$, $\mathscr{F}_3, \ldots$, translating functions appropriately so that they are still 'end-to-end'. We have defined a continuous function $f_0$ on $\mathbb{R}^+$ with range $[0,1]$.

Now let $n \in \mathbb{N}$, $L \in \mathbb{R}$, $\{a_1, \ldots, a_n\} \subset \mathbb{R}$, and $\{\alpha_1, \ldots, \alpha_n\} \subset \{0,1\}$ be as specified. Choose an interval of length $s \in \mathbb{N}$ so that $s > \max\{|a_i - a_j| : i,j = 1, \ldots, n\}$ and a 'mesh size' $t$ so that $1/t \in \mathbb{N}$ and

$$t \leqslant \min\{|a_i - a_j| : i,j = 1, \ldots, n,\ i \neq j\}.$$

The ordered pair $(s, t)$ has the form $(s_{k_0}, t_{k_0})$ for some $k_0 \in \mathbb{N}$, and one of the functions $f^* \in \mathscr{F}_{k_0}$ will satisfy

$$f^*(t^* - a_j) = \alpha_j \quad (j = 1, \ldots, n)$$

for some $t^* \in [0, s_{k_0}]$. A suitable translate, say $S_r f^*$, coincides with $f_0$ on an interval of $\mathbb{R}^+$ of length $s_{k_0}$, and so $t_0 = t^* + r$ satisfies (2). ∎

The function $f_0$ of the above lemma, extended to have domain $\mathbb{R}$ by setting it equal to 0 on $\mathbb{R}^-$, is fixed for the remainder of the proof of Theorem 2·4. Clearly the set $\{S_a f_0 : a \in \mathbb{R}\}$ is linearly independent in $C^b(\mathbb{R})$.

LEMMA 2·7. *For each $h \in L^1(\mathbb{R})$ with $\|h\|_1 \leqslant 1$, for each $n \in \mathbb{N}$, for each $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\backslash\{0\}$, and each $a_1, \ldots, a_n \in \mathbb{R}$, we have*

$$\left\| \sum_{j=1}^{n} \alpha_j S_{a_j} h \right\|_1 \leqslant \pi \left| \sum_{j=1}^{n} \alpha_j S_{a_j} f_0 \right|_{\mathbb{R}}.$$

*Proof.* Let $h, n, \alpha_1, \ldots, \alpha_n$, and $a_1, \ldots, a_n$ be as specified; we may suppose that the $a_j$'s are distinct. By Lemma 2·5, there is a subset $S$ of $\{1, \ldots, n\}$ such that

$$|\Sigma\{\alpha_j : j \in S\}| \geqslant \frac{1}{\pi} \sum_{j=1}^{n} |\alpha_j|. \tag{3}$$

By Lemma 2·6, there exists $t_0 \in \mathbb{R}$ such that

$$f_0(t_0 - a_j) = 1 \quad (j \in S) \quad \text{and} \quad f_0(t_0 - a_j) = 0 \quad (j \in \{1, \ldots, n\} \backslash S).$$

We have

$$|\Sigma\{\alpha_j : j \in S\}| = \left|\sum_{j=1}^{n} \alpha_j f_0(t_0 - \alpha_j)\right| \leqslant \left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0\right|_{\mathbb{R}} \tag{4}$$

and

$$\left\|\sum_{j=1}^{n} \alpha_j S_{a_j} h\right\|_1 \leqslant \sum_{j=1}^{n} |\alpha_j| \, \|S_{a_j} h\|_1 \leqslant \sum_{j=1}^{n} |\alpha_j|. \tag{5}$$

The result follows from (3), (4), and (5). ∎

LEMMA 2·8. *For each $g \in C_0(\mathbb{R})$, each $n \in \mathbb{N}$, each $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\backslash\{0\}$, and each distinct set $\{a_1, \ldots, a_n\} \in \mathbb{R}$, we have*

$$\left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0\right|_{\mathbb{R}} \leqslant 2 \left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0 + g\right|_{\mathbb{R}}.$$

*Proof.* For each $\beta_1, \ldots, \beta_m \in \mathbb{C}\backslash\{0\}$, the supremum

$$\sup\{|\beta_1 y_1 + \ldots + \beta_n y_n| : y_1, \ldots, y_n \in [0, 1]\}$$

is attained when each of the $y_j$'s is equal to 0 or 1, and so, by the construction of $f_0$, for each $x_0 > 0$, there exists $t_0 \geqslant x_0$ with

$$\left|\sum_{j=1}^{n} \alpha_j f_0(t_0 - a_j)\right| = \left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0\right|_{\mathbb{R}}.$$

Since $g(t) \to 0$ as $t \to \infty$ and since $|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0|_{\mathbb{R}} > 0$, we may also suppose that

$$|g(t_0)| \leqslant \frac{1}{2}\left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0\right|_{\mathbb{R}}$$

The result follows. ∎

*Proof of Theorem* 2·4. Set

$$X(f_0) = \operatorname{lin}\{S_a f_0 : a \in \mathbb{R}\},$$

the linear span of the functions $S_a f$. Clearly the sum $X(f_0) + C_0(\mathbb{R})$ is direct. Let $E$ be the closure of $X(f_0) \oplus C_0(\mathbb{R})$ in $C^b(\mathbb{R})$, so that $E$ is a closed, translation-invariant linear subspace of $C^b(\mathbb{R})$.

Choose an arbitrary element $h_0 \in L^1(\mathbb{R})$ with $\|h_0\|_1 = 1$, and define

$$T : \sum_{j=1}^{n} \alpha_j S_{a_j} f_0 + g \mapsto \sum_{j=1}^{n} \alpha_j S_{a_j} h_0, \quad X(f_0) \oplus C_0(\mathbb{R}) \to L^1(\mathbb{R}).$$

Then $T$ is a well-defined linear map such that $TS_a = S_a T$ for all $a \in \mathbb{R}$.

We have

$$\begin{aligned} \left\|\sum_{j=1}^{n} \alpha_j S_{a_j} h_0\right\|_1 &\leqslant \pi \left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0\right|_{\mathbb{R}} \quad \text{by Lemma 2·7} \\ &\leqslant 2\pi \left|\sum_{j=1}^{n} \alpha_j S_{a_j} f_0 + g\right|_{\mathbb{R}} \quad \text{by Lemma 2·8,} \end{aligned}$$

and so $T$ is continuous. Since $T(f_0) = h_0$, $T \neq 0$.

The continuous extension of $T$ to $E$ is the required map. ∎

It is easy to see that a linear operator $T: C^b(\mathbb{R}) \to L^1(\mathbb{R})$ is necessarily zero on the almost periodic functions; we could arrange that the Banach space $E$ of Theorem 2·4 also contains these functions, for example.

We now consider the form of a continuous linear operator $T \in \mathscr{B}(L^1(\mathbb{R}^+))$ such that $TL_a = L_a T$ for all $a > 0$. Let $T$ be such an operator.

The duality between $L^1(\mathbb{R}^+)$ and $L^\infty(\mathbb{R}^+)$ is implemented by

$$\langle f, g\rangle = \int_0^\infty f(t)\, g(t)\, dt \quad (f \in L^1(\mathbb{R}^+), g \in L^\infty(\mathbb{R}^+)).$$

Clearly $\langle L_a f, g\rangle = \langle f, R_a g\rangle$ for $f \in L^1(\mathbb{R}^+)$ and $g \in L^\infty(\mathbb{R}^+)$. The duality between $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ is implemented in a similar way.

The adjoint of $T$ is the operator $T' \in \mathscr{B}(L^\infty(\mathbb{R}^+))$ such that

$$\langle f, T'g\rangle = \langle Tf, g\rangle \quad (f \in L^1(\mathbb{R}^+), \quad g \in L^\infty(\mathbb{R}^+));$$

we have $T'R_a = R_a T'$ for all $a > 0$.

Now take $g \in C_{00}(\mathbb{R})$, the linear space of all continuous functions on $\mathbb{R}$ of compact support. We define an element $Ug \in L^\infty(\mathbb{R})$ by the formula

$$Ug = S_{-a} T' S_a g,$$

where $a$ is any real number such that $a + \alpha(g) \geqslant 0$, and $\alpha(g)$ is the infimum of the support of $g$; we are regarding $S_a g$ as an element of $L^\infty(\mathbb{R}^+)$.

LEMMA 2·9. *The map $U: C_{00}(\mathbb{R}) \to L^\infty(\mathbb{R})$ is a well-defined, bounded linear operator with a continuous extension $\tilde{U}: C_0(\mathbb{R}) \to L^\infty(\mathbb{R})$ such that $\tilde{U} S_a = S_a \tilde{U}$ for all $a \in \mathbb{R}$.*

*Proof.* Let $g \in C_{00}(\mathbb{R})$, and take $a_1, a_2 \in \mathbb{R}$ with $a_1 + \alpha(g) \geqslant 0$ and $a_2 + \alpha(g) \geqslant 0$, say $a_1 \leqslant a_2$. Then

$$S_{-a_2} T' S_{a_2} g = S_{-a_2}(T' S_{a_2 - a_1}) S_{a_1} g = S_{-a_1} T' S_{a_1} g,$$

and so $U$ is well-defined. The remainder is clear. ∎

The adjoint of $\tilde{U}$ is the bounded linear operator

$$\tilde{U}': L^\infty(\mathbb{R})' = L^1(\mathbb{R})'' \to C_0(\mathbb{R})' = M(\mathbb{R}),$$

and the restriction of $\tilde{U}'$ to $L^1(\mathbb{R})$ is denoted by $V$.

The duality between $C_0(\mathbb{R}^+)$ and $M(\mathbb{R}^+)$ is implemented by

$$[f, \mu] = \int_{\mathbb{R}^+} f(t)\, d\mu(t) \quad (f \in C_0(\mathbb{R}^+), \mu \in M(\mathbb{R}^+)),$$

and again the duality between $C_0(\mathbb{R})$ and $M(\mathbb{R})$ is implemented similarly. The isometric embedding of $L^1(\mathbb{R}^+)$ in $M(\mathbb{R}^+)$ is $f \mapsto \mu_f$, where

$$[g, \mu_f] = \langle f, g\rangle \quad (g \in C_0(\mathbb{R}^+), f \in L^1(\mathbb{R}^+)).$$

LEMMA 2·10. *The map $V: L^1(\mathbb{R}) \to M(\mathbb{R})$ is a bounded linear operator such that:*

(i) *$VS_a = S_a V$ for all $a \in \mathbb{R}$;*
(ii) *for each $h \in L^1(\mathbb{R}^+)$, $(Vh)|\mathbb{R}^+ = Th$.*

*Proof.* Certainly $V \in \mathscr{B}(L^1(\mathbb{R}), M(\mathbb{R}))$, and it is easy to check that (i) holds.

To establish (ii), we shall show that

$$[g, Th] = [g, Vh] \quad (g \in C_{00}(\mathbb{R}^+), h \in L^1(\mathbb{R}^+)), \tag{6}$$

which is sufficient. Fix $g \in C_{00}(\mathbb{R}^+)$ and $h \in L^1(\mathbb{R}^+)$.

First take a sequence $(g_n)$ in $C_{00}(\mathbb{R})$ defined as follows:

$$g_n(t) = \begin{cases} 0 & (-\infty < t < 1/n) \\ (1+nt)\,g(0) & (-1/n \leqslant t < 0) \\ g(t) & (t \geqslant 0). \end{cases}$$

For $a > 1$, we have

$$\begin{aligned}[g_n, Vh] &= \langle h, Ug_n \rangle = \langle h, S_{-a} T' S_a g_n \rangle \\ &= \langle S_{-a} T S_a h, g_n \rangle = [g_n, S_{-a} T S_a h].\end{aligned}$$

By the dominated convergence theorem,

$$[g_n, S_{-a} T S_a h] \to [g, S_{-a} T S_a h] \quad \text{as } n \to \infty$$

since $g_n \to g$ pointwise on $\mathbb{R}$. Also, by the dominated convergence for complex measures,

$$[g_n, Vh] \to [g, Vh] \quad \text{as } n \to \infty.$$

Thus

$$[g, Vh] = [g, S_{-a} T S_a h]. \tag{7}$$

Now take a second sequence $(\bar{g}_n)$ in $C_{00}(\mathbb{R})$ defined as follows:

$$\bar{g}_n(t) = \begin{cases} 0 & (-\infty < t < 0) \\ ntg(1/n) & (0 \leqslant t < 1/n) \\ g(t) & (t \geqslant 1/n). \end{cases}$$

Just as for $g_n$, we have $[\bar{g}_n, Vh] = [\bar{g}_n, S_{-a} T S_a h]$. Since the measure $S_{-a} T S_a h$ is absolutely continuous with respect to Lebesgue measure,

$$[\bar{g}_n, S_{-a} T S_a h] \to [g, S_{-a} T S_a h] \quad \text{as } n \to \infty.$$

Since $\alpha(\bar{g}_n) \geqslant 0$, we have $[\bar{g}_n, Vh] = [\bar{g}_n, Th] \to [g, Th]$.

Equation (6) now follows from (7). ▮

We remark that the introduction of two sequences $(g_n)$ and $(\bar{g}_n)$ in the above lemma seems to be necessary to obviate the problem that $Vh$ may *a priori* have a point mass at 0.

We can now give the canonical form of $T$. The Banach space $M(\mathbb{R}^+)$ is a Banach algebra with respect to convolution multiplication $*$, and $L^1(\mathbb{R}^+)$ is a closed ideal in $M(\mathbb{R}^+)$: indeed

$$(f * \mu)(t) = \int_{\mathbb{R}^+} f(t-s)\, d\mu(s) \quad (f \in L^1(\mathbb{R}^+), \mu \in M(\mathbb{R}^+)).$$

THEOREM 2·11. *Let $T: L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ be a bounded linear operator such that $TL_a = L_a T$ for each $a > 0$. Then there is a measure $\mu_0 \in M(\mathbb{R})$ such that*

$$Tf = (\mu_0 * f) \mid \mathbb{R}^+ \quad (f \in L^1(\mathbb{R}^+)).$$

*Proof.* Let $V \in \mathscr{B}(L^1(\mathbb{R}), M(\mathbb{R}))$ be as specified in Lemma 2·10. By a slight extension of [**7**, theorem 0·1·1], there is a unique measure $\mu_0 \in M(\mathbb{R})$ such that $Vf = \mu_0 * f$ $(f \in L^1(\mathbb{R}))$ (see also [**4**, theorems 35·5 and 35·9]). The result follows. ∎

## 3. *A discontinuous operator commuting with all left shifts*

The purpose of this section is to prove the following theorem.

Theorem 3·1. *Let $p$ and $q$ be real numbers with $p, q > 0$. Then there is a discontinuous linear operator $T: L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$ such that $TL_a = L_a T$ for each $a \in \mathbb{R}^+$.*

We first obtain a more abstract result, and then apply it to the situation under consideration.

*Definition* 3·2. Let $E$ be a linear space, and let $\mathscr{S}$ be a subset of $\mathscr{L}(E)$. A linear subspace $F$ of $E$ is $\mathscr{S}$-invariant if $S(F) \subset F$ for each $S \in \mathscr{S}$ and is strongly divisible for $\mathscr{S}$ if, further, $S \mid F \in \operatorname{Inv} \mathscr{L}(F)$ for each $S \in \mathscr{S} \backslash \{0\}$.

Proposition 3·3. *Let $E$ be a linear space containing linear subspaces $E_1$ and $E_2$, and let $\mathscr{S}$ be a unital integral domain contained in $\mathscr{L}(E)$. Suppose that:*

(i) *$G_0$ is an $\mathscr{S}$-invariant subspace of $E_1$, and $x_0 \in E_1 \backslash G_0$ is such that*

$$Sx_0 \notin G_0 \quad (S \in \mathscr{S} \backslash \{0\});$$

(ii) *$F$ is a linear subspace of $E_2$ which is strongly divisible for $\mathscr{S}$, and $y_0 \in F$.*

*Then there is a linear map $T: E_1 \to E_2$ such that $TS = ST$ for all $S \in \mathscr{S}$, such that $T \mid G_0 = 0$, and such that $Tx_0 = y_0$.*

*Proof.* Let $\mathscr{G}$ be the family of pairs $(G, T)$, where $G$ is an $\mathscr{S}$-invariant subspace of $E_1$ and $T \in \mathscr{L}(G, F)$ is such that $TS = ST$ for all $S \in \mathscr{S}$. Set $(G_1, T_1) \preccurlyeq (G_2, T_2)$ in $\mathscr{G}$ if $G_1 \subset G_2$ and $T_2 \mid G_1 = T_1$. Clearly $(G_0, 0) \in \mathscr{G}$.

Suppose that $(G, T) \in \mathscr{G}$ and that there exists $u_0 \in E_1 \backslash G$, and set

$$\mathscr{U} = \{S \in \mathscr{S} : Su_0 \in G\}.$$

For each $S \in \mathscr{U} \backslash \{0\}$, $TSu_0 \in F$; set $v_s = (S \mid F)^{-1} TSu_0 \in F$. Now take $S_1, S_2 \in \mathscr{U} \backslash \{0\}$. Then

$$S_1 S_2 v_{s_2} = S_1 T S_2 u_0 = S_2 T S_1 u_0 = S_2 S_1 v_{s_1} = S_1 S_2 v_{s_1},$$

using the facts that $TS_1 = S_1 T$ and $S_1 S_2 = S_2 S_1$. Since $\mathscr{S}$ is an integral domain, we have $S_1 S_2 \in \mathscr{S} \backslash \{0\}$, and so $v_{s_2} = v_{s_1}$. Denote this common value of $v_s$ by $v_0$, so that $Sv_0 = TSu_0$ for all $S \in \mathscr{U}$. In the case where $\mathscr{U} = \{0\}$, choose $v_0$ arbitrarily.

Set $H = G + \{Su_0 : S \in \mathscr{S}\}$. Then $H$ is an $\mathscr{S}$-invariant subspace of $E_1$ with $G \subset H$. Consider the map

$$V: x + Su_0 \mapsto Tx + Sv_0, \quad H \to F.$$

Then $V$ is well-defined, for if $x_1 + S_1 u_0 = x_2 + S_2 u_0$ in $H$, then $S_2 - S_1 \in \mathscr{U}$ and so

$$T(x_1 - x_2) = T(S_2 - S_1)(u_0) = (S_2 - S_1)(v_0).$$

Clearly $V$ is linear and $VS = SV$ for all $S \in \mathscr{S}$, and so $(H, V) \in \mathscr{G}$ with $(G, T) \preccurlyeq (H, V)$ and $u_0 \in H$.

The above argument shows, first, that there exists $(G_1, T_1) \in \mathscr{G}$ with

$$(G_0, 0) \preccurlyeq (G_1, T_1),$$

with $x_0 \in G_1$, and with $T_1 x_0 = y_0$. By Zorn's lemma, there is a maximal element in $\mathscr{G}$, say $(\tilde{G}, \tilde{T})$, with $(G_1, T_1) \preccurlyeq (\tilde{G}, \tilde{T})$. Again by the above argument, necessarily $\tilde{G} = E_1$, and so $\tilde{T}$ is the required map. ▮

*Proof of Theorem* 3·1. We apply the above proposition, taking $E_1 = L^p(\mathbb{R}^+)$, $E_2 = L^q(\mathbb{R}^+)$, $E = E_1 + E_2$, $\mathscr{S} = \operatorname{lin}\{L_a : a \in \mathbb{R}^+\}$, $G_0 = L^p_{00}(\mathbb{R}^+)$, and $x_0$ to be any element of $E_1 \backslash G_0$. Clearly $\mathscr{S}$ is a unital integral domain contained in $\mathscr{L}(E)$, and condition (i) of Proposition 3·3 is satisfied; we shall define a space $F$ satisfying condition (ii) of Proposition 3·3.

The space of infinitely differentiable functions on $\mathbb{R}^+$ is denoted by $C^{(\infty)}(\mathbb{R}^+)$. For $N \in \mathbb{N}$ and $f \in C^{(\infty)}(\mathbb{R}^+)$, set

$$p_{N,n}(f) = \sum_{j=0}^{\infty} \frac{N^j}{j!} |f^{(j)}|_{[n,n+1]} \quad (n \in \mathbb{Z}^+),$$

and consider functions $f$ with $p_{N,n}(f) < \infty$. Clearly each such $f$ extends to be an analytic function on $\{z \in \mathbb{C} : d(z, [n, n+1]) < N\}$, where $d$ denotes the Euclidean metric on $\mathbb{C}$ and we are regarding $[n, n+1]$ as a subset of $\mathbb{C}$.

For $N, k \in \mathbb{N}$ and $f \in C^{(\infty)}(\mathbb{R}^+)$, set

$$\|f\|_{N,k} = \sup\{e^{(n+k)^2} p_{N,n}(f) : n \in \mathbb{Z}^+\},$$

and set

$$F_{N,k} = \{f \in C^{(\infty)}(\mathbb{R}^+) : \|f\|_{N,k} < \infty\},$$

so that $(F_{N,k}, \|\cdot\|_{N,k})$ is a Banach space, and each $f \in F_{N,k}$ extends to be an analytic function on $\{z \in \mathbb{C} : d(z, \mathbb{R}^+) < N\}$.

Now set
$$F = \bigcap\{F_{N,k} : N, k \in \mathbb{N}\},$$
so that $F$ is a linear subspace of $C^{(\infty)}(\mathbb{R}^+)$ and each $f \in F$ is the restriction to $\mathbb{R}^+$ of an entire function, also denoted by $f$.

For each $f \in F$, $|f|_{[n,n+1]} = O(e^{-n^2})$ as $n \to \infty$, and so $F \subset E_2$.

Set $f_0(t) = e^{-t^3}$ for $t \in \mathbb{R}^+$, and take $N$, $k \in \mathbb{N}$. By applying the Cauchy estimates to the entire function $z \mapsto e^{-z^3}$ on $\{z \in \mathbb{C} : d(z, [n, n+1]) \leqslant 2N\}$, we see that there exists $(M_n : n \in \mathbb{Z}^+) \subset \mathbb{R}^+$ such that $M_n = O(e^{-n^3})$ as $n \to \infty$ and

$$\frac{N^j}{j!} |f_0^{(j)}|_{[n,n+1]} \leqslant \frac{M_n}{2^j} \quad (n, j \in \mathbb{Z}^+).$$

We have $p_{N,n}(f_0) = O(e^{-n^3})$ as $n \to \infty$, and so $\|f_0\|_{N,k} < \infty$. Thus $f_0 \in F$, and so $F \neq \{0\}$.

Let $a \in \mathbb{R}^+$, and set $m = [a]$, the integral part of $a$. Take $f \in F$, and set $g = L_a f$ and $h = R_a f$. Fix $N$, $k \in \mathbb{N}$. Since

$$p_{N,n}(g) \leqslant p_{N,n+m}(f) + p_{N,n+m+1}(f) \quad (n \in \mathbb{N})$$

and since $f \in F_{N,k}$, we have $g \in F_{N,k}$. Similarly, since $f \in F_{N,k+m}$, we have $h \in F_{n,k}$. Thus $g$, $h \in F$, and so $L_a(F) = F$. Now take $f \in F$ with $L_a f = 0$. Then $f | [a, \infty) = 0$, and so $f = 0$ because $f$ is the restriction to $\mathbb{R}^+$ of an entire function. Thus $L_a | F$ is injective, and so $L_a | F \in \operatorname{Inv} \mathscr{L}(F)$.

The operator norm in $\mathscr{B}(F_{N,k})$ is also denoted by $\|\cdot\|_{N,k}$. Take $b \in \mathbb{R}^+ \backslash \{0\}$ and $f \in F_{n,k}$. For each $j \in \mathbb{N}$,

$$p_{N,n}(L_{jb}(f)) \leqslant p_{N,n+[jb]}(f) + p_{N,n+[jb]+1}(f) \quad (n \in \mathbb{N}),$$

and so

$$\|L_{jb}(f)\|_{N,k} \leqslant 2\,\|f\|_{N,k}\exp(-[jb]^2).$$

Thus

$$\|L_{jb}\|_{N,k}^{1/j} \to 0 \quad \text{as } j \to \infty.$$

An arbitrary element of $\mathscr{S}\backslash\{0\}$ has the form $L_a(\alpha I_E + L_b T)$, where $a \geqslant 0$, $\alpha \in \mathbb{C}\backslash\{0\}$, $b > 0$, and $T \in \mathscr{S}$. We know that $L_a | F \in \operatorname{Inv} \mathscr{L}(F)$. As an operator on $F_{N,k}$,

$$\|(L_b T)^j\|_{N,k}^{1/j} \leqslant \|L_{jb}\|_{N,k}^{1/j}\|T\|_{N,k} \to 0 \quad \text{as } j \to \infty,$$

and so $(\alpha I_E + L_b T)\,|\,F_{N,k} \in \operatorname{Inv}\mathscr{L}(F_{N,k})$ for each $N, k \in \mathbb{N}$. Thus $(\alpha I_E + L_b T)\,|\,F$ belongs to $\operatorname{Inv}\mathscr{L}(F)$. It follows that $F$ is strongly divisible for $\mathscr{S}$.

We conclude from Proposition 3·3 that there is a linear map $T: E_1 \to E_2$ such that $TL_a = L_a T$ for all $a \in \mathbb{R}^+$, $T\,|\,G_0 = 0$, and $Tx_0 = f_0 \neq 0$, say. Since $G_0$ is dense in $E_1$, the map $T$ is necessarily discontinuous.

This concludes the proof of Theorem 3·1. ∎

It is clear that we may vary the spaces $L^p(\mathbb{R}^+)$ and $L^q(\mathbb{R}^+)$ in Theorem 3·1. For example, either of them can be replaced by $C_0(\mathbb{R}^+)$.

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