ELEMENTARY EVOLUTIONS IN BANACH ALGEBRA

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Abstract. An elementary class of evolutions in unital Banach algebras is obtained by integrating product functions over Guichardet's symmetric measure space on the half-line. These evolutions, along with a useful subclass, are characterised and a Lie–Trotter product formula is proved. The class is rich enough to form the basis for a recent approach to quantum stochastic evolutions.

Introduction

In this note we identify and analyse a simple class of evolutions in unital Banach algebras along with a useful subclass. They have infinitesimal generators, in terms of which they are characterised, and we establish a Lie–Trotter product formula for such evolutions. Our approach is via Guichardet's symmetric measure space ([Gui]) of the Lebesgue space $\mathbb{R}_+$. Apart from the merits of simplicity, one motivation is the fact that the theory forms the basis for a recent approach to quantum stochastic evolutions ([DLT], [DL]) in which quantum stochastic Trotter product formulae are proved (*cf.* [LSi]), characterisations of stochastic cocycles are established (*cf.* [LSk]) and convergence theorems for scaled quantum random walks are proved (*cf.* [Bel]).

After a brief section of preliminaries where notations are fixed, the basic theory occupies Section 2, and the product formula is proved in Section 3.

1. Preliminaries

For a step function f with domain $\mathbb{R}_+ = [0, \infty[$ we write $\operatorname{Disc} f$ for the (possibly empty) complement of the set of points t where f is constant in some neighbourhood of t; for a vector-valued function f on $\mathbb{R}_+$ and subinterval J of $\mathbb{R}_+$, f_J denotes the function on $\mathbb{R}_+$ which agrees with f on J and vanishes outside J. For a Banach space X, $B(X)$ denotes the unital Banach algebra of bounded operators on X. The symbol $\smile$ is used (for both elements of, and subsets of, an algebra) to denote 'commutes with' ([RSz]), $\#$ denotes cardinality, and $\subset\subset$ stands for subset of finite cardinality. For sets A and B, we write $F(A; B)$ rather than B^A, for the set of functions from A to B, and for $f \in F(A; B)$, we denote its range, $f(A)$, by $\operatorname{Ran} f$. Finally, we use the following notation for symplices: for $n \in \mathbb{N}$ and $t \geq r \geq 0$, set

$$\Delta^{(n)}_{[r,t[} := \{\mathbf{a} \in [r, t[^n \colon a_1 < \cdots < a_n\} \text{ and } \Delta^{[n]} := \{\mathbf{a} \in (\mathbb{R}_+)^n : a_1 \leqslant \cdots \leqslant a_n\}.$$

The uniqueness result below will serve us well. In Section 2 we give a very convenient representation of the equation's well-known solution.

Theorem 1.1. *Let $x_0 \in \mathfrak{X}$ and $a \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{A})$ for a right Banach $\mathcal{A}$-module $\mathfrak{X}$.*

(a) *The following integral equation has at most one solution $f \in C\big(\mathbb{R}_+; \mathfrak{X}\big)$:*

$$f(t) = x_0 + \int_0^t \mathrm{d}s\, f(s)a(s) \qquad (t \in \mathbb{R}_+). \tag{1.1}$$

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(b) *Let* $f \in C(\mathbb{R}_+; \mathfrak{X})$*. Then* f *satisfies* (1.1) *if*

$$f(0) = x_0 \quad \text{and} \quad f'(s) = f(s)a(s) \quad (s \in \mathbb{R}_+ \setminus \mathcal{N}),$$

for a Lebesgue-null Borel subset $\mathcal{N}$ *of* $\mathbb{R}_+$ *satisfying* $\operatorname{Haus} f(\mathcal{N}) = 0$*, where* Haus *denotes one-dimensional outer Hausdorff measure.*

(a) is straightforward and classical; for a proof of (b), see [Vol]. The condition $\operatorname{Haus} f(\mathcal{N}) = 0$ is automatic if either $\mathcal{N}$ is countable or $\mathcal{N}$ is Lebesgue-null and f is locally Lipschitz; for us, a will be a step function, so that $\mathcal{N}$ is finite.

2. Evolutions in Banach algebra

In this section we consider norm-continuous evolutions in a unital Banach algebra and analyse two sub-classes. To this end we introduce Guichardet's symmetric measure space of the Lebesgue spaces of subintervals of $\mathbb{R}_+$.

For the rest of the paper $\mathcal{A}$ *is a fixed unital Banach algebra*; its group of invertible elements is denoted $\mathcal{A}^\times$.

Definition. An *evolution* E *in* $\mathcal{A}$ is a family $\big(E_{r,t}\big)_{0\leqslant r\leqslant t}$ in $\mathcal{A}$, or function from $\Delta^{[2]}$ to $\mathcal{A}$, such that

$$E_{r,r} = 1_{\mathcal{A}} \ \text{ and } \ E_{r,s}\,E_{s,t} = E_{r,t} \qquad (0 \leqslant r \leqslant s \leqslant t);$$

The class of evolutions in $\mathcal{A}$ is denoted $\operatorname{Evol}(\mathcal{A})$.

Example. Let $\alpha = (\alpha_t)_{t\geq 0}$ be an E_0-semigroup on a von Neumann algebra $\mathcal{M}$, that is, a one-parameter semigroup of endomorphisms of $\mathcal{M}$ (which is pointwise ultraweakly continuous), and let $V = (V_t)_{t\geq 0}$ be a family of contractions in $\mathcal{M}$ forming an α-cocycle, thus $V_0 = 1$ and $V_{s+t} = V_s\alpha_s(V_t)$ $(s, t \geq 0)$ ([Arv]). Then the family $\big(\alpha_r(V_{t-r})\big)_{0\leqslant r\leqslant t}$ forms an evolution in $\mathcal{M}$.

A family $\big(E_{r,t}\big)_{0\leqslant r\leqslant t}$ in $\mathcal{A}$ is called an *opposite evolution* if instead

$$E_{r,r} = 1_{\mathcal{A}} \ \text{ and } \ E_{s,t}\,E_{r,s} = E_{r,t} \qquad (0 \leqslant r \leqslant s \leqslant t).$$

An evolution is *invertible* if it is $\mathcal{A}^\times$-valued, and *continuous*, respectively *Lipschitz*, if the following maps are continuous, respectively Lipschitz continuous,

$$[r, \infty[\to \mathcal{A},\ s \mapsto E_{r,s} \ \text{ and } \ [0, t] \to \mathcal{A},\ s \mapsto E_{s,t} \qquad (r, t \in \mathbb{R}_+).$$

We denote these classes by $\operatorname{Evol}(\mathcal{A}^\times)$, $\operatorname{Evol}_{\mathrm{c}}(\mathcal{A})$ and $\operatorname{Evol}_{\mathrm{Lc}}(\mathcal{A})$ respectively.

Remarks. For $E \in \operatorname{Evol}(\mathcal{A}^\times)$, $\big((E_{r,t})^{-1}\big)_{0\leqslant r\leqslant t}$ defines an opposite evolution; also E extends to an evolution $(E_{r,t})_{r\leqslant t}$ (where r and t now range over $\mathbb{R}$) by the prescription

$$E_{r,t} := \phi_r^{-1}\phi_t \text{ where } \phi_s := \begin{cases} E_{0,s} & \text{if } s \geq 0 \\ (E_{0,-s})^{-1} & \text{if } s \leqslant 0. \end{cases}$$

Proposition 2.1. *The map* $\{\phi \in F(\mathbb{R}_+; \mathcal{A}^\times) : \phi(0) = 1_{\mathcal{A}}\} \to \operatorname{Evol}(\mathcal{A}^\times)$ *given by* $\phi \mapsto \big(\phi_r^{-1}\phi_t\big)_{0\leqslant r\leqslant t}$ *is bijective, and restricts to a bijection*

$$\{\phi \in C(\mathbb{R}_+; \mathcal{A}^\times) : \phi(0) = 1_{\mathcal{A}}\} \to \operatorname{Evol}_{\mathrm{c}}(\mathcal{A}).$$

Proof. All that needs to be proved is that if $E \in \operatorname{Evol}_{\mathrm{c}}(\mathcal{A})$, then $E_{0,t} \in \mathcal{A}^\times$ for all $t \in \mathbb{R}_+$. Thus let $E \in \operatorname{Evol}_{\mathrm{c}}(\mathcal{A})$ and suppose for a contradiction that $E_{0,s} \notin \mathcal{A}^\times$ for some $s \in \mathbb{R}_+$. Set $t := \inf\{s \in \mathbb{R}_+ : E_{0,s} \notin \mathcal{A}^\times\}$. In view of the facts that the set $\mathcal{A} \setminus \mathcal{A}^\times$ is closed, the map $s \mapsto E_{0,s}$ is right continuous at 0, and $E_{0,0} = 1_{\mathcal{A}} \in \mathcal{A}^\times$, it follows that $E_{0,t} \notin \mathcal{A}^\times$ and $t > 0$. Since $E_{t,t} = 1_{\mathcal{A}} \in \mathcal{A}^\times$, the openness of $\mathcal{A}^\times$ and left continuity of the map $s \mapsto E_{s,t}$ at t imply that, for small enough $h > 0$, the evolution identity $E_{0,t} = E_{0,t-h}\,E_{t-h,t}$ expresses a noninvertible element as a product of invertibles, and we have our contradiction. ☐

Remarks. Thus continuous evolutions are invertible, and invertible evolutions are actually one-parameter objects.

Evolutions generalise one-parameter semigroups, in the sense that every (norm-continuous) one-parameter semigroup $(p_t)_{t\geq 0}$ in $\mathcal{A}$ defines a (continuous) evolution $(p_{t-r})_{0\leqslant r\leqslant t}$. However—in stark contrast to the well-known simple structure of continuous semigroups: $(e^{ta})_{t\geq 0}$ $(a\in\mathcal{A})$ (see *e.g.* [Rud])—continuous evolutions are in general far from being differentiable, as the above proposition shows.

Given a Banach space X, every strongly continuous opposite evolution $(E_{r,t})_{r\leqslant t}$ in $B(X)$ which is exponentially bounded, i.e. where there is $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\|E_{r,t}\|\leqslant Me^{\omega(t-r)}$ $(r\leqslant t)$, the prescription $(T^E_t f)(s):=E_{s-t,s}f(s-t)$ defines a C_0-semigroup on the Banach space $C_0(\mathbb{R};X)$ satisfying $T^E_t M_\varphi = M_{T_t\varphi}T^E_t$ $(\varphi\in C_\kappa(\mathbb{R};X),\ t\in\mathbb{R}_+)$ where T is the right-shift semigroup on $C_0(\mathbb{R})$ and M denotes (scalar) multiplication operator; every such semigroup arises in this way (see [EnN]). An interesting question then is—how might norm continuity of an evolution E be recognised in its semigroup T^E?

Using Guichardet's symmetric measure space, we shall embed the class of evolutions given by semigroups in a much wider class. For $(r,t)\in\Delta^{(2)}$, set

$$\Gamma_{[r,t[}:=\big\{\sigma\subset[r,t[:\#\sigma<\infty\big\}\ \text{ and }\ \Gamma^{(n)}_{[r,t[}:=\{\sigma\subset[r,t[:\#\sigma=n\}\quad(n\in\mathbb{Z}_+),$$

with measurable structure and measure induced from that of Lebesgue measure on each symplex $\Delta^{(n)}_{[r,t[}$, via the bijection

$$\Delta^{(n)}_{[r,t[}\to\Gamma^{(n)}_{[r,t[},\quad \mathsf{s}\mapsto\{s_1,\cdots,s_n\}\qquad(n\in\mathbb{N}),$$

and letting $\varnothing\in\Gamma^{(0)}_{[r,t[}$ be an atom of measure one ([Gui]). Thus $\Gamma^{(n)}_{[r,t[}$ and $\Gamma_{[r,t[}$ have measure $(t-r)^n/n!$ and $\exp(t-r)$ respectively. We use the abbreviations Γ, $\Gamma^{(n)}$, $\Gamma^{\geq 1}$ and $\int \mathrm{d}\sigma$ for $\Gamma_{[0,\infty[}$, $\Gamma^{(n)}_{[0,\infty[}$, $\bigcup_{n\geq 1}\Gamma^{(n)}$ and integration with respect to the symmetric measure on Γ. Each function $\varphi:\mathbb{R}_+\to\mathbb{C}$ determines a function

$$\pi_\varphi:\Gamma\to\mathbb{C},\quad \sigma\mapsto\prod_{s\in\sigma}\varphi(s).$$

Thus $\pi_0=\delta_\varnothing$ and the mapping $\varphi\mapsto\pi_\varphi$ respects measure equivalence classes. For $\varphi\in L^1(\mathbb{R}_+)$, $\pi_\varphi\in L^1(\Gamma)$, $\int\pi_\varphi=\exp\int\varphi$ and $\|\pi_\varphi\|_1=\exp\|\varphi\|_1$. In particular, for nonnegative functions $\varphi,\psi\in L^1(\mathbb{R}_+)$,

$$\|\pi_{\varphi+\psi}\|_1=\|\pi_\varphi\|_1\|\pi_\psi\|_1\ \text{ and }\ \|\pi_{\varphi+\psi}-\pi_\varphi\|_1=\|\pi_\varphi\|_1\big(\|\pi_\psi\|_1-1\big).\tag{2.1}$$

Remark. For $\varphi\in L^2(\mathbb{R}_+)$, let $\varepsilon_\varphi=(1,\varphi,(2!)^{-1/2}\varphi^{\otimes 2},\cdots)$ denote the exponential vector in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. Then the prescription

$$\varepsilon_\varphi\mapsto\pi_\varphi\qquad(\varphi\in L^2(\mathbb{R}_+)),$$

extends to a unitary map $\Gamma(L^2(\mathbb{R}_+))\to L^2(\Gamma)$. For a Hilbert space k, this tensorises to give an isometry from $\Gamma\big(L^2(\mathbb{R}_+;\mathsf{k})\big)=\Gamma\big(L^2(\mathbb{R}_+)\otimes\mathsf{k}\big)$ to $L^2(\Gamma;\Phi_{\mathsf{k}})$, where Φ_{k} denotes the full (unsymmetrised) Fock space over k; its image is

$$\big\{f\in L^2(\Gamma;\Phi_{\mathsf{k}}):\forall_{\sigma\in\Gamma}\ f(\sigma)\in\mathsf{k}^{\otimes\#\sigma}\big\}.$$

For more on Guichardet space analysis, see [$\mathrm{L}_{1,2}$], [Mey] and references therein; a cornerstone is the *integral-sum formula* which we state next—for a proof see [LiP].

Lemma 2.2. *Let $n\in\mathbb{N}$ and $H\in L^1(\Gamma^n;X)$ for a Banach space X. Then*

$$\int\mathrm{d}\alpha_1\cdots\int\mathrm{d}\alpha_n\,H(\alpha_1,\cdots,\alpha_n)=\int\mathrm{d}\sigma\sum H(\sigma_1,\cdots,\sigma_n)$$

where the sum is over all $n^{\#\sigma}$ partitions of σ into n subsets $\sigma_1,\cdots,\sigma_n$.

In particular, for $H \in L^1(\Gamma \times \Gamma; X)$

$$\int d\alpha \int d\beta\, H(\alpha,\beta) = \int d\sigma \sum\nolimits_{\alpha\subset\sigma} H(\alpha, \sigma\setminus\alpha).$$

Note that the integral-sum formula for functions H of the form $(\alpha_1,\cdots,\alpha_n) \mapsto \pi_{\varphi_1}(\alpha_1)\cdots\pi_{\varphi_n}(\alpha_n)\,x$, where $x \in X$ and $\varphi_1,\cdots,\varphi_n \in L^1(\mathbb{R}_+)$, reduces to the simple identity $\big(\prod_{i=1}^n \exp\int\varphi_i\big)x = \big(\exp\int\varphi\big)x$, where $\varphi = \sum_{i=1}^n \varphi_i$.

The composition of $\mathcal{A}$-valued functions on Γ defined by

$$f \circ g : \sigma \mapsto \sum\nolimits_{\alpha\subset\sigma} f(\alpha)g(\sigma\setminus\alpha) \tag{2.2}$$

enjoys the following properties: if $\operatorname{supp} f \subset \Gamma_I$ and $\operatorname{supp} g \subset \Gamma_J$ for disjoint sets I and J, then

$$(f\circ g)(\sigma) = f(\sigma\cap I)g(\sigma\cap J) \text{ for } \sigma\in\Gamma, \tag{2.3}$$

whereas, by the integral-sum formula, if $f, g \in L^1(\Gamma;\mathcal{A})$ then

$$f\circ g \in L^1(\Gamma;\mathcal{A}), \ \int_\Gamma f\circ g = \int_\Gamma f \int_\Gamma g \ \text{ and } \ \|f\circ g\|_1 \leqslant \|f\|_1\|g\|_1. \tag{2.4}$$

Definition. Let $a \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$. Its *associated product functions* π_a and ${}_a\pi$ in $L^1_{\text{loc}}(\Gamma;\mathcal{A})$ are defined by $\pi_a(\varnothing) = {}_a\pi(\varnothing) = 1_\mathcal{A}$ and for $\sigma = \{s_1 < \cdots < s_n\}$, $\pi_a(\sigma) = a(s_1)\cdots a(s_n)$ whereas ${}_a\pi(\sigma) = a(s_n)\cdots a(s_1)$; in short,

$$\pi_a(\sigma) := \overrightarrow{\prod_{s\in\sigma}} a(s) \ \text{ and } \ {}_a\pi(\sigma) := \overleftarrow{\prod_{s\in\sigma}} a(s).$$

For $a \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$ define E^a and ${}^aE$ in $C(\Delta^{[2]};\mathcal{A})$ as follows.

$$E^a_{r,t} := \int_{\Gamma_{[r,t[}} \pi_a = \int \pi_{a_{[r,t[}} \ \text{ and } \ {}^aE_{r,t} := \int_{\Gamma_{[r,t[}} {}_a\pi.$$

Remark. If $a = \varphi(\cdot)1_\mathcal{A}$, for a function $\varphi \in L^1_{\text{loc}}(\mathbb{R}_+)$, then

$$E^a_{r,t} = \int_{\Gamma_{[r,t[}} \pi_\varphi\, 1_\mathcal{A} = e^{\int_r^t \varphi}\, 1_\mathcal{A}.$$

Lemma 2.3. *Let* $c, d, h \in L^1(\mathbb{R}_+;\mathcal{A})$ *and* $a, b \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$.

(a) $\|\pi_c\|_1 \leqslant \exp\|c\|_1$ *and* $\|\pi_{c+h} - \pi_c\|_1 \leqslant \|\pi_c\|_1\big(\|\pi_h\|_1 - 1\big)$.

(b) $\pi_c \circ \pi_d = \pi_{c+d}$ *if* $\operatorname{Ran} d \smile \operatorname{Ran} c$, *whereas* $\pi_c \circ {}_d\pi = \pi_{c+d}$ *provided that* $d(s_1) \smile (c+d)(s_2)$ *when* $s_2 > s_1 \geq 0$.

(c) E^a *is the unique continuous solution of the integral equations* (2.5) *below* (*in turn, for each fixed* r, *and each fixed* t); ${}^bE$ *is likewise for* (2.6).

$$E_{r,t} = 1_\mathcal{A} + \int_r^t ds\, E_{r,s}\, a(s) = 1_\mathcal{A} + \int_r^t ds\, a(s)\, E_{s,t}, \tag{2.5}$$

$$E_{r,t} = 1_\mathcal{A} + \int_r^t ds\, b(s)\, E_{r,s} = 1_\mathcal{A} + \int_r^t ds\, E_{s,t}\, b(s). \tag{2.6}$$

(d) *For* $(r,t), (u,v) \in \Delta^{[2]}$, *setting* $I := [r,t[$ *and* $J := [u,v[$, *the following hold*:

(i) $\|E^b_{r,t} - E^a_{u,v}\| \leqslant \exp\|a_I\|_1\big(\exp\|(b-a)_I\|_1 + \exp\|a_{I\Delta J}\|_1 - 2\big)$.

(ii) $E^a_{r,s}E^b_{s,t} = E^e_{r,t}$ *where* $e := a_{[r,s[} + b_{[s,t[}$.

(iii) $E^a_{r,t}\,{}^bE_{r,t} = E^{a+b}_{r,t}$ *if* $b(s_1) \smile (a+b)(s_2)$ *for* $r < s_1 < s_2 < t$.

(iv) $E^a_{r,t}E^b_{r,t} = E^{e_r}_{r,t}$ *where* $e_r(s) := {}^{(-b)}E_{r,s}a(s)E^b_{r,s} + b(s)$.

(v) $E^{L_w a}_{r,t} = E^a_{r+w,t+w}$ *for* $w \in [-r,\infty[$, *where* $L_w a$ *is given by*

$$(L_w a)(s) = \begin{cases} a(s+w) & \text{if } s+w \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note the following binomial-type identities, for functions $a_1, a_2 : \mathbb{R}_+ \to \mathcal{A}$:

$$\pi_{a_1+a_2}(\sigma) = \sum_{\mathbf{i}\in\{1,2\}^n} a_{i_1}(s_1)\cdots a_{i_n}(s_n) \ \text{ for } \sigma = \{s_1 < \cdots < s_n\} \tag{2.7}$$

$$= \sum_{\alpha\subset\sigma} \pi_{a_1}(\alpha)\pi_{a_2}(\sigma\setminus\alpha) \ \text{ if } \operatorname{Ran} a_1 \smile \operatorname{Ran} a_2. \tag{2.8}$$

(a) The first estimate follows from submultiplicativity of the norm. For the second, note that (2.7) implies that $\big\|\pi_{c+h}(\sigma) - \pi_c(\sigma)\big\| \leqslant \pi_{C+H}(\sigma) - \pi_C(\sigma)$ for $\sigma \in \Gamma$, where $C := \|c(\cdot)\|$ and $H := \|h(\cdot)\|$. Thus, by (2.1),

$$\|\pi_{c+h} - \pi_c\|_1 \leqslant \|\pi_C\|_1\big(\|\pi_H\|_1 - 1\big) = \|\pi_c\|_1\big(\|\pi_h\|_1 - 1\big).$$

(b) The first identity follows from (2.8). The second follows easily from the fact that, under the given commutation assumption,

$$\pi_{c+d}(\{s\}\cup\tau) = c(s)\pi_{c+d}(\tau) + \pi_{c+d}(\tau)d(s),$$

when $s < \tau$ (meaning $s < t$ for all $t \in \tau$).

(c) All four of the required identities follow from the integral-sum formula. For example, for the first one, define $\mathbb{1}(\alpha,\beta)$ to be 1 if $\#\beta = 1$ and $a < b$ for all $a \in \alpha$ and $b \in \beta$, and to be 0 otherwise, then

$$\begin{aligned}\int_r^t \mathrm{d}s\, E^a_{r,s} a(s) &= \int_r^t \mathrm{d}s \int_{\Gamma_{[r,s[}} \mathrm{d}\alpha\, \pi_a(\alpha\cup\{s\}) \\ &= \int \mathrm{d}\alpha \int \mathrm{d}\beta\, \pi_{a_{[r,t[}}(\alpha\cup\beta)\mathbb{1}(\alpha,\beta) \\ &= \int \mathrm{d}\sigma \sum_{\alpha\subset\sigma} \pi_{a_{[r,t[}}(\sigma)\mathbb{1}(\alpha,\sigma\setminus\alpha) = \int_{\Gamma^{\geq 1}} \mathrm{d}\sigma\, \pi_{a_{[r,t[}}(\sigma) = E^a_{r,t} - 1_{\mathcal{A}}.\end{aligned}$$

Uniqueness for the first and last follows from Theorem 1.1; uniqueness for the other two follows from the left module sister version of Theorem 1.1.

(d) (i) follows from Part (a). (ii) follows from (2.4), (2.2) and the identity $\pi_a(\sigma\cap[r,s[)\,\pi_b(\sigma\cap[s,t[) = \pi_e(\sigma)$; with (i) it implies that $E^a \in \mathrm{Evol_c}(\mathcal{A}) \subset \mathrm{Evol}(\mathcal{A}^\times)$. (iii) follows from Part (b) and identity (2.4). In particular, since E^b is invertible, this implies that

$$(E^b_{r,s})^{-1} = {}^{(-b)}E_{r,s} \qquad (r,s) \in \Delta^{[2]}. \tag{2.9}$$

To prove (iv), set E equal to the pointwise product E^aE^b. Integrating by parts using Part (c), the assumed commutation relations, and (2.9), we have

$$E_{r,t} = 1_{\mathcal{A}} + \int_r^t \mathrm{d}s\,\big(E^a_{r,s}a(s)E^b_{r,s} + E^a_{r,s}E^b_{r,s}b(s)\big) = 1_{\mathcal{A}} + \int_r^t \mathrm{d}s\, E_{r,s}e_r(s).$$

Therefore (iv) follows from uniqueness in Part (c). With a simple change of variable, (v) follows from the identity

$$(L_w a)_{[r,t[}(s) = a_{[r+w,t+w[}(s+w) \qquad (s \in \mathbb{R}_+). \qquad \square$$

The summarising proposition below now follows easily.

Proposition 2.4. *Let $a, b \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{A})$, $c \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{A})$ and $(r,t) \in \Delta^{[2]}$. Then*

(a) *$E^a \in \mathrm{Evol_c}(\mathcal{A})$ and $E^c \in \mathrm{Evol_{Lc}}(\mathcal{A})$.*

(b) *$E^a_{r,t}\,{}^bE_{r,t} = E^{a+b}_{r,t}$ if $b(s_1) \smile (a+b)(s_2)$ for $r < s_1 < s_2 < t$, in particular,*

$$(E^a_{r,t})^{-1} = {}^{(-a)}E_{r,t}, \quad \textit{and}$$
$$e^{\int_r^t \varphi} E^a_{r,t} = E^{a+\varphi(\cdot)1_{\mathcal{A}}}_{r,t} \quad \textit{for} \ \ \varphi \in L^1_{\mathrm{loc}}(\mathbb{R}_+).$$

(c) *$E^a_{r,t} = E^{L_r a}_{0,t-r}$ and $E^a_{0,s+u} = E^a_{0,s}E^{L_s a}_{0,u}$, for $s, u \in \mathbb{R}_+$.*

Definition. An evolution of the form E^a where $a \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$ will be called *elementary, with generator* a; we denote this class of evolutions by $\text{Evol}_{\text{e}}(\mathcal{A})$.

The following example is of considerable historical importance (see *e.g.* [EnN]).

Example. Let $a : \mathbb{R}_+ \to B(X)$ be strongly continuous, for a Banach space X. Then, by the Banach-Steinhaus Theorem, a is locally bounded, and by (2.6),

$$^aE_{r,t} = I_X + \int_r^t \mathrm{d}s\, a(s)\, {}^aE_{r,s} \qquad (0 \leqslant r \leqslant t).$$

In particular, for all $x \in X$, the nonautonomous abstract Cauchy problem

$$u'(t) = a(t)u(t) \ (t \geq 0), \quad u(0) = x,$$

has unique "classical" solution ${}^aE_{0,\cdot}\, x \in C^1(\mathbb{R}_+;X)$.

Noting that $\text{Evol}_{\text{e}}(\mathcal{A}) \subset \text{Evol}_{\text{c}}(\mathcal{A})$, we characterise the class of elementary evolutions next.

Theorem 2.5. *Let* $E \in \text{Evol}_{\text{c}}(\mathcal{A})$ *and set* $\phi_t := E_{0,t}$ $(t \in \mathbb{R}_+)$. *Then the following are equivalent*:

(i) *There is a function* $c \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$ *such that*

$$\phi_t - \phi_r = \int_r^t \mathrm{d}s\, c(s) \qquad (0 \leqslant r \leqslant t).$$

(ii) $E \in \text{Evol}_{\text{e}}(\mathcal{A})$.

In this case $c(s) = E_{0,s}\, a(s)$ $(s \in \mathbb{R}_+)$, *where* a *is the generator of* E.

Proof. Multiplying (2.5) on the left by $E_{0,r}$ we see that (ii) implies (i).

Suppose that (i) holds. By Proposition 2.1, $\operatorname{Ran}\phi \subset \mathcal{A}^\times$, and so we may define a function $a \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$ by $a(s) := (\phi_s)^{-1}c(s)$. Since E and E^a are both continuous evolutions, it suffices to show that $\phi_t = E^a_{0,t}$ for all $t \in \mathbb{R}_+$. Now

$$\phi_t = 1_{\mathcal{A}} + \int_0^t \mathrm{d}s\, \phi_s\, a(s) \qquad (t \in \mathbb{R}_+)$$

so, by Part (c) of Lemma 2.3 (uniqueness), it follows that $\phi_t = E^a_{0,t}$ for all $t \in \mathbb{R}_+$, as required. □

Remarks. Evolutions of the above type are a.e.-weakly differentiable in the following sense. By Lebesgue's Differentiation Theorem, for all $\omega \in \mathcal{A}^*$ there is a null set $\mathcal{N}_\omega$ in $\mathbb{R}_+$ such that for all $t \in \mathbb{R}_+ \setminus \mathcal{N}_\omega$,

$$\omega\big(h^{-1}(\phi_{t+h} - \phi_t) - c(t)\big) \to 0 \text{ as } h \to 0.$$

Conversely, it follows from Theorem 1.1 that (ii) holds if there is a Lebesgue-null Borel subset $\mathcal{N}$ of $\mathbb{R}_+$ such that ϕ is differentiable on $\mathbb{R}_+ \setminus \mathcal{N}$, $\phi' \in L^1_{\text{loc}}(\mathbb{R}_+;\mathcal{A})$ and $\operatorname{Haus}\phi(\mathcal{N}) = 0$.

The next result applies to finite-dimensional Banach algebras. A convenient reference for the Radon–Nikodým property is [DiU]; for differentiability of Lipschitz functions, see [LPT].

Corollary 2.6. *Let* $E \in \text{Evol}_{\text{c}}(\mathcal{A})$ *where* $\mathcal{A}$ *has the Radon–Nikodým property, and set* $\phi_t := E_{0,t}$ $(t \in \mathbb{R}_+)$. *Then the following are equivalent*:

(i) $E \in \text{Evol}_{\text{e}}(\mathcal{A})$; *respectively,* $E \in \text{Evol}_{\text{e}}(\mathcal{A})$ *with locally bounded generator.*

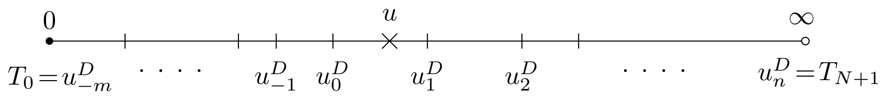
(ii) *There is an absolutely continuous* $\mathcal{A}$*-valued measure* m *on* $\mathbb{R}_+$ *of locally bounded variation such that* $m\big([r,t]\big) = \phi_t - \phi_r$ $(0 \leqslant r \leqslant t)$; *respectively,* ϕ *is locally Lipschitz, so* $E \in \text{Evol}_{\text{Lc}}(\mathcal{A})$.

We next identify a subclass of elementary evolutions which is useful in applications. To this end, and for use in the next section, we adopt the following notation.

Notation. Let $D = \{T_1 < \cdots < T_N\} \subset\subset]0,\infty[$ and set $T_0 := 0$ and $T_{N+1} := \infty$. For $u \in \mathbb{R}_+$, define $m = m(u) \in \mathbb{Z}_+$, $n = n(u) \in \mathbb{N}$ and $\{u_k^D : -m \leqslant k \leqslant n\}$ by

$$\{u_{-m}^D < \cdots < u_n^D\} = \{0\} \cup D \cup \{\infty\} = \{T_0 < T_1 < \cdots < T_{N+1}\}, \quad u \in [u_0^D, u_1^D[,$$

giving the following picture:



Definition. We call E a *piecewise-semigroup evolution* if there are *associated time point and semigroup sets*

$$D = \{T_1 < \cdots < T_N\} \subset\subset]0,\infty[\text{ and } \{P^{(T)} : T \in \{0\} \cup D\} = \{P^{(T_0)}, \cdots, P^{(T_N)}\},$$

where $T_0 := 0$ and each $P^{(T)}$ is a semigroup in $\mathcal{A}$, for which the following holds:

$$E_{r,t} = \begin{cases} P_{t-r}^{(r_0^D)} & \text{if } r_0^D = t_0^D \\ P_{r_1^D - r}^{(r_0^D)} \Big(P_{r_2^D - r_1^D}^{(r_1^D)} \cdots P_{t_0^D - t_{-1}^D}^{(t_{-1}^D)} \Big) P_{t - t_0^D}^{(t_0^D)} & \text{otherwise.} \end{cases} \tag{2.10}$$

Note that, for any such D and $\{P^{(T)}\}$, (2.10) defines an evolution. Let $\mathrm{Evol}_{\mathrm{pws}}(\mathcal{A})$ denote the resulting collection; thus $\mathrm{Evol}_{\mathrm{pws}}(\mathcal{A}) \cap \mathrm{Evol}_{\mathrm{c}}(\mathcal{A}) \subset \mathrm{Evol}_{\mathrm{Lc}}(\mathcal{A})$.

The piecewise-semigroup evolutions are therefore those evolutions which enjoy the *semigroup decomposition property* (2.10). Note that the set D can be empty, and it is only the minimal such set D that is determined by the evolution E. We have the following elementary characterisation.

Proposition 2.7. *Let $E \in \mathrm{Evol}_{\mathrm{c}}(\mathcal{A})$. Then the following are equivalent:*

(i) $E \in \mathrm{Evol}_{\mathrm{pws}}(\mathcal{A})$.

(ii) $E \in \mathrm{Evol}_{\mathrm{e}}(\mathcal{A})$, *with piecewise constant generator.*

In this case, the associated minimal time point and semigroup sets of E are respectively, $\mathrm{Disc}\, a$ *and* $\{(e^{sa(t)})_{s \geq 0} : t \in \{0\} \cup \mathrm{Disc}\, a\}$, *where a is the (right-continuous version of) the generator of E.*

Proof. Suppose that (ii) holds and let a be the generator of E. Let $D = \mathrm{Disc}\, a = \{T_1 < \cdots < T_N\}$, set $T_0 := 0$ and $T_{N+1} := \infty$, and let $(r,t) \in \Delta^{[2]}$. By the evolution property,

$$E_{r,t} = \begin{cases} E_{r,t} & \text{if } r_0^D = t_0^D \\ E_{r,r_1^D} \Big(E_{r_1^D, r_2^D} \cdots E_{t_{-1}^D, t_0^D} \Big) E_{t_0^D, t} & \text{otherwise.} \end{cases} \tag{2.11}$$

Now, for $k = 0, \cdots, N$, a is constant on $[T_k, T_{k+1}[$ so, for $[u,v[\subset [T_k, T_{k+1}[$,

$$E_{u,v} = \int_{\Gamma_{[u,v[}} \mathrm{d}\sigma\, \pi_a(\sigma) = \int_{\Gamma_{[u,v[}} \mathrm{d}\sigma\, a(T_k)^{\#\sigma} = e^{(v-u)a(T_k)} = P_{v-u}^{(T_k)},$$

where $P^{(T)}$ denotes the semigroup generated by $a(T)$. Thus (2.11) becomes (2.10), showing E to be a piecewise-semigroup evolution with associated time and semigroup sets as claimed.

Suppose conversely that (i) holds, and let $D = \{T_1 < \cdots < T_N\}$ and $\{P^{(T)} : T \in \{0\} \cup D\}$ be the associated minimal time point and semigroup sets of E. Since $E \in \mathrm{Evol}_{\mathrm{c}}(\mathcal{A})$, each of these semigroups is norm continuous. Let a be the piecewise constant function $\sum_{k=0}^N a_{[T_k, T_{k+1}[}^{(k)}$ where, for $k = 0, \cdots, N$, $a^{(k)}$ is the generator of $P^{(T_k)}$. Then E^a also satisfies (2.10), and so $E = E^a$. $\square$

Thus the evolutions with piecewise constant generators are the continuous evolutions which enjoy a semigroup decomposition. We characterise a slightly wider class of evolutions next. By *piecewise continuity* for a Banach-space valued function x defined on $\mathbb{R}_+$, we mean that there is a finite subset D of $]0,\infty[$ such that x is continuous on $\mathbb{R}_+ \setminus D$ and the limits $a(0_+)$, $a(s_-)$ and $a(s_+)$ exist, for $s \in D$. For definiteness, we take the unique *right-continuous* (i.e. càdlàg) version of each piecewise continuous function.

Proposition 2.8. *Let $E \in \mathrm{Evol_c}(\mathcal{A})$. Then the following are equivalent:*

(i) *$s \mapsto E_{0,s}$ has piecewise continuous derivative on $\mathbb{R}_+$.*
(ii) *$E \in \mathrm{Evol_e}(\mathcal{A})$ with piecewise continuous generator.*

Proof. By Proposition 2.1, E is invertible. Assume that (i) holds and define $a : \mathbb{R}_+ \to \mathcal{A}$ to be the piecewise continuous function $s \mapsto (E_{0,s})^{-1}\frac{d}{ds}E_{0,s}$. Then $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{A})$ and (ii) holds since $s \mapsto E_{0,s}$ and $s \mapsto E^a_{0,s}$ both satisfy the conditions of Theorem 1.1, Part (b), with $\mathcal{N} := \mathrm{Disc}\, a$.

The converse is clear. □

3. Lie–Trotter product formula

In this section we prove a Trotter product formula and an Euler-type formula, for elementary evolutions. The following notation is convenient for handling Trotter products of evolutions.

Notation. Let $D \subset\subset]0,\infty[$, in other words $D \in \Gamma_{]0,\infty[}$, and let $G \in F(\Delta^{[2]};\mathcal{A})$. Then, in the notation associated with the diagram in Section 2, define *G's D-fold product function* by

$$G^D : \Delta^{[2]} \to \mathcal{A}, \quad G^D_{r,t} = \begin{cases} G_{r^D_1,r^D_2} \cdots G_{t^D_{-1},t^D_0} & \text{if } r^D_1 < t^D_0 \\ 1_{\mathcal{A}} & \text{otherwise.} \end{cases}$$

In particular, if G is an evolution then $G^D_{r,t}$ equals $G_{r^D_1,t^D_0}$ if $[r,t[\,\cap D$ is nonempty, and equals $1_{\mathcal{A}}$ otherwise.

Definition. We say that a sequence $\big(D(n)\big)_{n\ge1}$ in $\Gamma_{]0,\infty[} \setminus \{\varnothing\}$ *converges to* $\mathbb{R}_+$ if

$$\min D(n) \to 0, \quad \max D(n) \to \infty \;\text{ and }\; \mathrm{mesh}\, D(n) \to 0.$$

Similarly, a family $\big(D[h]\big)_{h>0}$ in $\Gamma_{]0,\infty[}$ *converges to* $\mathbb{R}_+$ if, as $h \to 0$,

$$\min D[h] \to 0, \quad \max D[h] \to \infty \;\text{ and }\; \mathrm{mesh}\, D[h] \to 0.$$

Here $\mathrm{mesh}\, D$ is defined to be $\max\{|s-t| : s,t \in D, s \ne t\}$ (or ∞ if $\#D = 1$).

Theorem 3.1. *Let $a_1, a_2 \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{A})$, let $\big(D(n)\big)_{n\ge1}$ be a sequence in $\Gamma_{]0,\infty[} \setminus \{\varnothing\}$ converging to $\mathbb{R}_+$, and let $T \in \mathbb{R}_+$. Then*

$$\sup_{[r,t]\subset[0,T]} \big\| E^{a_1+a_2}_{r,t} - {}^{1,2}E^{D(n)}_{r,t} \big\| \to 0, \quad \textit{where} \quad {}^{1,2}E_{u,v} := E^{a_1}_{u,v}\, E^{a_2}_{u,v}.$$

Proof. Set $a = a_1 + a_2$ and $A = A_1 + A_2$, where $A_i := \|a_i(\cdot)\| \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ $(i = 1, 2)$, and set $\pi := \pi_{a_1} \circ \pi_{a_2}$, for the composition defined in (2.2). Thus $\pi \in L^1_{\mathrm{loc}}(\Gamma;\mathcal{A})$ with

$$\pi(\varnothing) = 1_{\mathcal{A}} \text{ and } \pi(\{s\}) = a(s) \text{ for } s \in \mathbb{R}_+,$$

so the functions π and π_a agree on $\Gamma^{\leqslant 1}$. Also, by (2.4),

$${}^{1,2}E_{u,v} = \int_{\Gamma_{[u,v[}} \mathrm{d}\sigma\, \pi(\sigma) \qquad ((u,v) \in \Delta^{[2]}).$$

By further application of the integral-sum formula—more specifically (2.4), and (2.3),

$$ {}^{1,2}E^{D(n)}_{r,t} = \int_{\Gamma_{[r,t[}} \mathrm{d}\sigma\, \pi^{(n)}(\sigma), \text{ where} $$

$$ \pi^{(n)}(\sigma) := \pi\big(\sigma \cap [r_1^{D(n)}, r_2^{D(n)}[\big) \cdots \pi\big(\sigma \cap [t_{-1}^{D(n)}, t_0^{D(n)}[\big). $$

Now,

$$ \|\pi^{(n)}(\sigma)\| \leqslant \pi_A(\sigma) \qquad (n \in \mathbb{N}, \sigma \in \Gamma). $$

Thus $\pi^{(n)} \in L^1_{\mathrm{loc}}(\Gamma; \mathcal{A})$, $\pi^{(n)}(\varnothing) = 1_{\mathcal{A}} = \pi_a(\varnothing)$ and, for $\sigma \in \Gamma_{]0,\infty[} \setminus \{\varnothing\}$, the equality

$$ \pi^{(n)}(\sigma) = \pi_a(\sigma) $$

holds—as soon as $n \in \mathbb{N}$ is sufficiently large that

$$ \min D(n) < \min \sigma, \ \max D(n) > \max \sigma \ \text{ and } \ \operatorname{mesh} D(n) < \operatorname{mesh} \sigma. $$

The result therefore follows from the Dominated Convergence Theorem:

$$ \sup_{[r,t] \subset [0,T]} \big\| E^a_{r,t} - {}^{1,2}E^{D(n)}_{r,t} \big\| \leqslant \int_{\Gamma_{[0,T[}} \mathrm{d}\sigma\, \big\|\pi_a(\sigma) - \pi^{(n)}(\sigma)\big\| \to 0. $$

□

In order to handle Euler-type products we define, for $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{A})$, the *truncated evolution*:

$$ \widetilde{E}^a : \Delta^{[2]} \to \mathcal{A}, \quad \widetilde{E}^a_{r,t} := \int_{\Gamma_{[r,t[}} \widetilde{\pi}_a \ \text{ where } \ \widetilde{\pi}_a := 1_{\Gamma^{\leqslant 1}} \pi_a. \tag{3.1} $$

Thus $\widetilde{E}^a_{r,t} = 1_{\mathcal{A}} + \int_r^t \mathrm{d}s\, a(s)$.

Theorem 3.2. *Let a_1, a_2, $\big(D(n)\big)_{n \geqslant 1}$ and T be as in Theorem 3.1. Then*

$$ \sup_{[r,t] \subset [0,T]} \big\| E^{a_1+a_2}_{r,t} - {}^{1,2}\widetilde{E}^{D(n)}_{r,t} \big\| \to 0, \quad \textit{where} \quad {}^{1,2}\widetilde{E}_{u,v} := \widetilde{E}^{a_1}_{u,v}\, \widetilde{E}^{a_2}_{u,v}. $$

Proof. A proof is obtained as follows. In the proof of Theorem 3.1 replace π_{a_1}, π_{a_2}, π, ${}^{1,2}E$ and $\pi^{(n)}$ by $\widetilde{\pi}_{a_1}$, $\widetilde{\pi}_{a_2}$, $\widetilde{\pi}$, ${}^{1,2}\widetilde{E}$ and $\widetilde{\pi}^{(n)}$ respectively, where $\widetilde{\pi}$ is defined as π is but with $\widetilde{\pi}_{a_1}$ and $\widetilde{\pi}_{a_2}$ in place of π_{a_1} and π_{a_2}, and $\widetilde{\pi}^{(n)}$ is defined as $\pi^{(n)}$ is, but with $\widetilde{\pi}$ in place of π. In short, drawing on the definitions (3.1), retrace the argument with all π's and E's endowed with tildes. □

Remarks. The above two proofs need little adjustment to deliver the following generalisation. For $a = a_1 + \cdots + a_N$ where $a_1, \cdots, a_N \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{A})$, and $T \in \mathbb{R}_+$,

$$ \sup_{[r,t] \subset [0,T]} \big\| E^a_{r,t} - {}^{(N)}E^{D(n)}_{r,t} \big\| \to 0, \ \text{ where } \ {}^{(N)}E_{u,v} := E^{a_1}_{u,v} \cdots E^{a_N}_{u,v} \qquad \big((u,v) \in \Delta^{[2]}\big), $$

and similarly for the truncations.

The above proofs also yield corresponding results for a continuous-parameter family $\big(D[h]\big)_{h>0}$. In particular, taking a_1 and a_2 constant, respectively $a_2 = 0$ and $a_1 = a$ constant, then gives the following limits

$$ \big(e^{ha_1} e^{ha_2}\big)^{(t_0^{D[h]} - r_1^{D[h]})/h} \to e^{(t-r)(a_1+a_2)} \ \text{ and } \ (1_{\mathcal{A}} + ha)^{(t_0^{D[h]} - r_1^{D[h]})/h} \to e^{(t-r)a} $$

as $h \to 0$; the classical Lie–Trotter product formula ([ReS], Theorem VIII.29) and Euler formula emerge upon taking $r = 0$ and $D[h] = \{nh : 1 \leq n \leq N\}$ where $N = [1/h^2]$:

$$ \big(e^{ha_1} e^{ha_2}\big)^{[t/h]} \to e^{t(a_1+a_2)} \ \text{ and } \ (1_{\mathcal{A}} + ha)^{[t/h]} \to e^{ta}. $$

The close connection between the Trotter product and Euler formulae was richly investigated, at the deeper level of C_0-semigroups, by Chernoff (see [Che]).

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