SESQUILINEAR QUANTUM STOCHASTIC ANALYSIS IN BANACH SPACE

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Abstract. A theory of quantum stochastic processes in Banach space is initiated. The processes considered here consist of Banach space valued sesquilinear maps. We establish an existence and uniqueness theorem for quantum stochastic differential equations in Banach modules, show that solutions in unital Banach algebras yield stochastic cocycles, give sufficient conditions for a stochastic cocycle to satisfy such an equation, and prove a stochastic Lie–Trotter product formula. The theory is used to extend, unify and refine standard quantum stochastic analysis through different choices of Banach space, of which there are three paradigm classes: spaces of bounded Hilbert space operators, operator mapping spaces and duals of operator space coalgebras. Our results provide the basis for a general theory of quantum stochastic processes in operator spaces, of which Lévy processes on compact quantum groups is a special case.

Introduction

The aim of this paper is to initiate a theory of quantum stochastic processes in Banach space. The motivation is twofold: to extend the applicability, and begin to unify, several strands of quantum stochastic analysis. When the results are applied to the paradigm examples discussed below—optimal results are deduced for stochastic Lie–Trotter product formulae, and near-optimal results are obtained for the generation of stochastic cocycles. The Banach space setting presents some obstruction to the development of a 'strong' theory. In a sister paper ([DL$_2$]) we develop quantum stochastic analysis in operator space aided by the superior functorial properties of the operator space projective tensor product compared to that of the Banach space projective tensor product. Broadly speaking, the 'weak' theory is treated here and the 'column' theory there.

The processes considered in this paper are families $(\mathfrak{q}_t)_{t\geqslant 0}$ of sesquilinear maps $\mathcal{E}\times\mathcal{E}\to\mathfrak{X}$ for a Banach space $\mathfrak{X}$ and exponential domain $\mathcal{E}$ in symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$, where k is a Hilbert space which serves as the multiplicity space of the quantum noise. Natural adaptedness and regularity conditions are assumed. The three paradigm examples of $\mathfrak{X}$ are: the space $B(\mathsf{h};\mathsf{h}')$, of bounded operators between Hilbert spaces h and h', and its closed subspaces; the mapping space $CB(\mathsf{V};\mathsf{W})$, of completely bounded maps between operator spaces V and W; and the dual of an operator space coalgebra. The former corresponds to the theory of unitary and contractive operator processes initiated by Hudson and Parthasarathy ([HuP]), the second includes both the theory of quantum stochastic flows on a C^*-algebra founded by Evans and Hudson ([Eva]), and that of completely positive stochastic cocycles on a C^*-algebra initiated by Lindsay and Parthasarathy ([LiP]), and the latter corresponds to the theory of quantum stochastic convolution cocycles, which includes Lévy processes on compact quantum groups in the universal setting

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([LS$_2$]). Expositions of the theory of the first two areas may be found in the monograph [Par], the lecture notes [Mey], and the surveys [Bia] and [L$_1$].

The sesquilinear theory throws light on the paradigm examples mentioned above and we obtain refinements of the standard theory, including that of quantum stochastic differential equations in operator spaces ([LS$_1$]). We also obtain new stochastic Lie–Trotter product formulae for cocycles in all three of the examples, extending the results of [LiS]. Our analysis is founded on some elementary theory of evolutions in unital Banach algebras ([DL$_1$]).

The plan of the paper is as follows. After a section of preliminaries, we review the relevant parts of standard quantum stochastic process theory in Section 2, and the results that we need on evolutions in Section 3. Banach space valued sesquilinear processes are introduced in the fourth section, where sesquilinear multiple quantum Wiener integrals are defined and estimated. In Section 5 the existence and uniqueness theorem is proved for solutions of sesquilinear quantum stochastic differential equations. In Section 6 we show that solutions of such equations are sesquilinear quantum stochastic cocycles and give sufficient conditions for a sesquilinear quantum stochastic cocycle to satisfy an equation of this type. We then apply this to obtain refinements of characterisation theorems in [LS$_1$]. In Section 7 we prove the sesquilinear quantum stochastic Lie–Trotter product formula and deduce corresponding formulae in each of the three paradigm examples.

1. Preliminaries

In this section we establish some general notations and state two propositions which are applied in the paper.

For vector spaces V, V' and W we write $\widehat{V}$ for $\mathbb{C} \oplus V$, $\widehat{v}$ for $\binom{1}{v}$ $(v \in V)$, and $SL(V', V; W)$ for the space of sesquilinear maps $V' \times V \to W$ (or $SL(V;W)$ when $V' = V$), inner products and sesquilinear maps here being linear in their second argument. Basic examples of these are given by $|w\rangle q_T$ for $T \in L(V;V')$, $w \in W$ and inner product spaces V and V', where

$$|w\rangle q_T : V' \times V \to W, \quad (v', v) \mapsto \langle v', Tv\rangle w. \tag{1.1}$$

We also denote by $ASL(V', V; W)$ the collection of maps $\alpha : V' \times V \to W$ which are *affine sesquilinear*, that is, complex affine linear in the second argument and conjugate affine linear in the first (or $ASL(V;W)$ when $V' = V$). For an ordered set A and $n \in \mathbb{N}$, we write

$$A^n_< := \{\mathbf{a} \in A^n : a_1 < \cdots < a_n\} \text{ and } A^n_\leqslant := \{\mathbf{a} \in A^n : a_1 \leqslant \cdots \leqslant a_n\};$$

also, for n-symplices over a subinterval J of $\mathbb{R}_+$ we write

$$\Delta^{(n)}_J := J^n_< \text{ and } \Delta^{[n]}_J := J^n_\leqslant, \tag{1.2}$$

abbreviated to $\Delta^{(n)}$ and $\Delta^{[n]}$ when $J = \mathbb{R}_+$.

For a step function f with domain $\mathbb{R}_+$ we write $\operatorname{Disc} f$ for the (possibly empty) complement of the set of points t where f is constant in some neighbourhood of t; for a vector-valued function f on $\mathbb{R}_+$ and subinterval J of $\mathbb{R}_+$, f_J denotes the function on $\mathbb{R}_+$ which agrees with f on J and vanishes outside J. For Hilbert spaces H and h and vector $e \in \mathsf{h}$, the operator

$$I_{\mathsf{H}} \otimes |e\rangle : \mathsf{H} \to \mathsf{H} \otimes \mathsf{h}, \quad u \mapsto u \otimes e$$

will be denoted by E_e, and its adjoint $I_{\mathsf{H}} \otimes \langle e|$ by E^e, with context dictating the Hilbert space H. Thus $E^e \in B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H})$ and $E^e E_f = \langle e, f\rangle I_{\mathsf{H}}$. Here $\langle e| \in B(\mathsf{h}; \mathbb{C})$ is the adjoint of the operator $|e\rangle \in L(\mathbb{C}; \mathsf{h}) = B(\mathbb{C}; \mathsf{h})$, thus $\langle e| : c \mapsto \langle e, c\rangle$; we set $|\mathsf{h}\rangle := B(\mathbb{C}; \mathsf{h})$ and $\langle \mathsf{h}| := B(\mathsf{h}; \mathbb{C})$. If V is an operator space in $B(\mathsf{H}; \mathsf{H}')$ and

$B = B(\mathsf{h};\mathsf{h}')$, for Hilbert spaces h and h', then the *matrix space tensor product of* V *with* B is the following operator space in $B(\mathsf{H}\otimes\mathsf{h};\mathsf{H}'\otimes\mathsf{h}') = B(\mathsf{H};\mathsf{H}')\overline{\otimes}B$:

$$\mathsf{V}\otimes_{\mathsf{M}} B := \big\{T\in B(\mathsf{H};\mathsf{H}')\overline{\otimes}B : E^{c'}TE_c\in\mathsf{V} \text{ for all } c'\in\mathsf{h}', c\in\mathsf{h}\big\}.$$

Let W be another concrete operator space. If $\phi\in CB(\mathsf{V};\mathsf{W})$ then the map $\phi\underline{\otimes}\,\mathrm{id}_B$ extends uniquely to a map $\phi\otimes_{\mathsf{M}}\mathrm{id}_B\in CB(\mathsf{V}\otimes_{\mathsf{M}}B;\mathsf{W}\otimes_{\mathsf{M}}B)$ ([LiW]). Also, for Hilbert spaces k and k', and map $\psi\in CB_\sigma\big(B;B(\mathsf{k};\mathsf{k}')\big)$, the map $\mathrm{id}_{B(\mathsf{H};\mathsf{H}')}\overline{\otimes}\psi$ restricts to a map in $CB\big(\mathsf{V}\otimes_{\mathsf{M}}B;\mathsf{V}\otimes_{\mathsf{M}}B(\mathsf{k};\mathsf{k}')\big)$, denoted $\mathrm{id}_{\mathsf{V}}\otimes_{\mathsf{M}}\psi$. The following extended composition is very useful. For $\phi_i\in CB\big(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}}B(\mathsf{h}_i;\mathsf{h}_i')\big)$ $(i=1,2)$,

$$\phi_1\bullet\phi_2 := (\phi_1\otimes_{\mathsf{M}}\mathrm{id}_{B(\mathsf{h}_2;\mathsf{h}_2')})\circ\phi_2\in CB\big(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}}B(\mathsf{h};\mathsf{h}')\big). \tag{1.3}$$

Here $\mathsf{h}=\mathsf{h}_1\otimes\mathsf{h}_2$ and $\mathsf{h}'=\mathsf{h}_1'\otimes\mathsf{h}_2'$, so $B(\mathsf{h}_1;\mathsf{h}_1')\otimes_{\mathsf{M}}B(\mathsf{h}_2;\mathsf{h}_2') = B(\mathsf{h};\mathsf{h}')$.

For dense subspaces $\mathcal{D}$ of h and $\mathcal{D}'$ of h', there are natural inclusions

$$\begin{aligned} &\mathsf{V}\otimes_{\mathsf{M}}B\subset L\big(\mathcal{D};\mathsf{V}\otimes_{\mathsf{M}}|\mathsf{h}'\rangle\big)\subset SL(\mathcal{D}',\mathcal{D};\mathsf{V}), \\ &T\mapsto\big(\zeta\mapsto TE_\zeta\big) \ \text{ and } \ R\mapsto\big((\zeta',\zeta)\mapsto E^{\zeta'}R_\zeta\big). \end{aligned} \tag{1.4}$$

Similarly, there are natural inclusions

$$\begin{aligned} CB\big(\mathsf{V};\mathsf{W}\otimes_{\mathsf{M}}B\big) &\subset L\big(\mathcal{D};CB(\mathsf{V};\mathsf{W}\otimes_{\mathsf{M}}|\mathsf{h}'\rangle)\big) \\ &\subset SL\big(\mathcal{D}',\mathcal{D};CB(\mathsf{V};\mathsf{W})\big)\subset SL\big(\mathcal{D}',\mathcal{D};B(\mathsf{V};\mathsf{W})\big). \end{aligned} \tag{1.5}$$

In view of these identifications we are using the subscript notations R_ζ and ϕ_ζ for the images of $\zeta\in\mathcal{D}$ under $R\in L\big(\mathcal{D};\mathsf{V}\otimes_{\mathsf{M}}|\mathsf{h}'\rangle\big)$ and $\phi\in L\big(\mathcal{D};L(\mathsf{V};\mathsf{W}\otimes_{\mathsf{M}}|\mathsf{h}'\rangle)\big)$. Finally we write $\mathcal{O}(\mathcal{D};\mathsf{h}')$ for the linear space of operators from h to h' with domain $\mathcal{D}$, and $\mathcal{O}^\ddagger(\mathcal{D},\mathcal{D}')$ for the subspace of operators T satisfying $\operatorname{Dom}T^*\supset\mathcal{D}'$.

We end this section with two lemmas; the first is elementary linear algebra.

Lemma 1.1. *Let V, V' and W be complex vector spaces. The map $W^{V'\times V}\to W^{\widehat{V}'\times\widehat{V}}$, $\alpha\mapsto\gamma_\alpha$ given by*

$$\gamma_\alpha\Big(\binom{z'}{v'},\binom{z}{v}\Big) = \\ \alpha(v',v)+\overline{z'-1}\,\alpha(0,v)+(z-1)\alpha(v',0)+\overline{z'-1}(z-1)\alpha(0,0),$$

is injective with left inverse given by $\gamma\mapsto\alpha_\gamma$ where $\alpha_\gamma(v',v):=\gamma(\widehat{v'},\widehat{v})$. It restricts to a bijection from $ASL(V',V;W)$ to $SL(\widehat{V'},\widehat{V};W)$.

A useful representation of the well-known solution of the equations in the next lemma is given in Section 3.

Lemma 1.2. *Let $\mathfrak{X}$ be a right Banach $\mathcal{A}$-module, let $x_0\in\mathfrak{X}$ and let a be a step function $\mathbb{R}_+\to\mathcal{A}$ with discontinuity set D. Then the integral equation*

$$f(t) = x_0+\int_0^t \mathrm{d}s\, f(s)a(s) \qquad (t\geqslant 0). \tag{1.6}$$

and the differential equation

$$f(0)=x_0 \ \text{ and } \ f'(s)=f(s)a(s) \ \ (s\in\mathbb{R}_+\setminus D),$$

have the same unique solution in $C\big(\mathbb{R}_+;\mathfrak{X}\big)$.

2. Quantum stochastics

In this section we review some standard quantum stochastic analysis, and establish some notations. Fix now, and for the rest of the paper, a complex Hilbert space k referred to as the *noise dimension space*. For a subinterval J of $\mathbb{R}_+$, let $\mathsf{K}_J := L^2(J; \mathsf{k})$ and, for $f \in \mathsf{K}_J$, write $\widehat{f}$ for the corresponding $\widehat{\mathsf{k}}$-valued function given by $\widehat{f}(s) := \widehat{f(s)}$. Let T be a total subset of k containing 0. The space of T-valued step functions in K_J is denoted $\mathbb{S}_{\mathsf{T},J}$ (we take right-continuous versions). The symmetric Fock space over K_J is denoted $\mathcal{F}_J$; the exponential vectors $\varepsilon(f) := ((n!)^{-1/2} f^{\otimes n})_{n\geqslant 0}$ ($f \in \mathsf{K}_J$) are linearly independent and $\mathcal{E}_{\mathsf{T},J} := \mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{S}_{\mathsf{T},J}\}$ is dense in $\mathcal{F}_J$; when $\mathsf{T} = \mathsf{k}$ or $J = \mathbb{R}_+$, we drop the corresponding subscript; the identity operator on $\mathcal{F}_J$ and vacuum vector $\varepsilon(0)$ in $\mathcal{F}_J$ will be written I_J and Ω_J respectively. The orthogonal decomposition

$$\mathsf{K} = \mathsf{K}_{[0,s[} \oplus \mathsf{K}_{[s,t[} \oplus \mathsf{K}_{[t,\infty[}$$

yields the tensor decompositions

$$\begin{aligned} &\mathcal{F} = \mathcal{F}_{[0,s[} \otimes \mathcal{F}_{[s,t[} \otimes \mathcal{F}_{[t,\infty[}, \quad B(\mathcal{F}) = B_{[0,s[} \overline{\otimes} B_{[s,t[} \overline{\otimes} B_{[t,\infty[}, \quad \text{and} \\ &\mathcal{E}_{\mathsf{T}} = \mathcal{E}_{\mathsf{T},[0,s[} \underline{\otimes} \mathcal{E}_{\mathsf{T},[s,t[} \underline{\otimes} \mathcal{E}_{\mathsf{T},[t,\infty[} \qquad (0 \leqslant s \leqslant t). \end{aligned}$$

Definition. Let h and h' be Hilbert spaces, with dense subspaces $\mathcal{D}$ and $\mathcal{D}'$.

An h-h' *operator quantum stochastic process with exponential domain* $\mathcal{D}\underline{\otimes}\mathcal{E}_{\mathsf{T}}$ is a family of operators $\big(X_t\big)_{t\geqslant 0}$ in $\mathcal{O}(\mathcal{D}\underline{\otimes}\mathcal{E}_{\mathsf{T}}; \mathsf{h}' \otimes \mathcal{F})$ satisfying the following measurability and adaptedness conditions:

(i) $s \mapsto X_s\zeta$ is weakly measurable $\mathbb{R}_+ \to \mathsf{h}' \otimes \mathcal{F}$, for all $\zeta \in \mathcal{D}\underline{\otimes}\mathcal{E}_{\mathsf{T}}$, and
(ii) for all $t \in \mathbb{R}_+$, there is an operator $X_{t)} \in \mathcal{O}\big(\mathcal{D}\underline{\otimes}\mathcal{E}_{\mathsf{T},[0,t[}; \mathsf{h}' \otimes \mathcal{F}_{[0,t[}\big)$ such that $X_t = X_{t)} \underline{\otimes} I'_{[t,\infty[}$ where $I'_{[t,\infty[}$ denotes the restriction of $I_{[t,\infty[}$ to $\mathcal{E}_{\mathsf{T},[t,\infty[}$.

For all $g' \in \mathbb{S}$, $g \in \mathbb{S}_{\mathsf{T}}$, $\varepsilon \in \mathcal{E}_{\mathsf{T}}$ and $t \in \mathbb{R}_+$, set

$$X_t^{g',g} := E^{\varepsilon(g'_{[0,t[})} X_t E_{\varepsilon(g_{[0,t[})} \in \mathcal{O}(\mathcal{D}; \mathsf{h}') \text{ and } X_{t,\varepsilon} = X_t E_\varepsilon \in \mathcal{O}(\mathcal{D}; \mathsf{h}' \otimes \mathcal{F}). \tag{2.1}$$

The process X is *initial space bounded* if $X_t^{g',g}$ is bounded ($t \in \mathbb{R}_+$, $g' \in \mathbb{S}$, $g \in \mathbb{S}_{\mathsf{T}}$); it is *column-bounded* if $X_{t,\varepsilon}$ is bounded ($t \in \mathbb{R}_+$, $\varepsilon \in \mathcal{E}_{\mathsf{T}}$); it is *bounded* if X_t is bounded ($t \in \mathbb{R}_+$), in which case (ii) reads

(ii)′ $\forall_{t\in\mathbb{R}_+}\ \ X_t \subset X_{t)} \otimes I_{[t,\infty[}$ for some operator $X_{t)} \in B\big(\mathsf{h} \otimes \mathcal{F}_{[0,t[}; \mathsf{h}' \otimes \mathcal{F}_{[0,t[}\big)$;

it is *adjointable* if $\mathrm{Dom}(X_t)^* \supset \mathcal{D}'\underline{\otimes}\mathcal{E}_{\mathsf{T}'}$ ($t \in \mathbb{R}_+$) for some dense subspace $\mathcal{D}'$ of h' and total subset T' of k containing 0, in which case $X_t^\dagger := (X_t)^*_{|\mathcal{D}'\underline{\otimes}\mathcal{E}_{\mathsf{T}'}}$ ($t \geqslant 0$) defines an h'-h process $X^\dagger$.

For a column-bounded h-h' process X, and function $g \in \mathbb{S}_{\mathsf{T}}$, we write

$$\begin{aligned} &X^g_{t)} := X_{t)} E_{\varepsilon(g|_{[0,t[})} \in B\big(\mathsf{h}; \mathsf{h}' \otimes \mathcal{F}_{[0,t[}\big) = B(\mathsf{h}; \mathsf{h}') \overline{\otimes} |\mathcal{F}_{[0,t[}\rangle, \quad \text{for } t \geqslant 0, \text{ and} \\ &X^g_{[r,t[} := \big(\mathrm{id}_{B(\mathsf{h};\mathsf{h}')} \overline{\otimes} \tau_{[r,t[}\big)\big(X^{L_r g}_{t-r)}\big) \in B\big(\mathsf{h}; \mathsf{h}' \otimes \mathcal{F}_{[r,t[}\big), \quad \text{for } t \geqslant r \geqslant 0, \end{aligned}$$

where $\tau_{[r,t[}$ denotes the shift $|\mathcal{F}_{[0,t-r[}\rangle \to |\mathcal{F}_{[r,t[}\rangle$, and $(L_t)_{t\geqslant 0}$ denotes the coisometric left shift semigroup on $\mathcal{F}$.

Linear extension of the prescription

$$X_{r,t,\varepsilon(g)} = \Sigma\big(|\varepsilon(g_{[0,r[})\rangle \otimes X^g_{[r,t[} \otimes |\varepsilon(g_{[t,\infty[})\rangle\big), \tag{2.2}$$

in which Σ is the tensor flip

$$\Sigma : |\mathcal{F}_{[0,r[}\rangle \overline{\otimes} B(\mathsf{h}; \mathsf{h}' \otimes \mathcal{F}_{[r,t[}) \overline{\otimes} |\mathcal{F}_{[t,\infty[}\rangle \to B(\mathsf{h}; \mathsf{h}' \otimes \mathcal{F}),$$

then gives a two-parameter family $\big(X_{r,t}\big)_{0\leqslant r\leqslant t}$ in $L\big(\mathcal{E}_{\mathsf{T}}; B(\mathsf{h}; \mathsf{h}' \otimes \mathcal{F})\big)$, which is bi-adapted in an obvious sense.

If X is bounded then

$$X_{r,t} = \sigma_r(X_{t-r}) \quad (t \geqslant r \geqslant 0), \tag{2.3}$$

where $\sigma_r = \mathrm{id}_{B(\mathsf{h};\mathsf{h}')} \overline{\otimes} \sigma_r^{\mathsf{k}}$ for the right shift σ_r^{k} on $B(\mathcal{F})$, thus

$$X_{r,t} \in B(\mathsf{h};\mathsf{h}')\overline{\otimes} I_{[0,r[}\overline{\otimes} B(\mathcal{F}_{[r,t[}) \otimes I_{[t,\infty[}.$$

A bounded h-process X (*i.e.* h-h'-process where $\mathsf{h}' = \mathsf{h}$) is a *quantum stochastic cocycle* if it satisfies

$$X_0 = I_{\mathsf{h}\otimes\mathcal{F}} \ \text{ and } \ X_{s+t} = X_s \sigma_s(X_t) \quad (s,t \geqslant 0). \tag{2.4}$$

By the multiplicativity of the shift, this is equivalent to its associated two-parameter family forming an evolution:

$$X_{r,r} = I_{\mathsf{h}\otimes\mathcal{F}} \ \text{ and } \ X_{r,t} = X_{r,s}X_{s,t} \quad (0 \leqslant r \leqslant s \leq t);$$

it is also equivalent to

$$X_0^{g',g} = I_{\mathsf{h}} \ \text{ and } \ X_{s+t}^{g',g} = X_s^{g',g} X_t^{L_s g', L_s g} \quad (s,t \geqslant 0, g', g \in \mathbb{S}),$$

which makes sense for initial-space bounded processes X. In terms of columns, the cocycle identity is equivalent to

$$X_{0)}^g = I_{\mathsf{h}} \ \text{ and } \ X_{s+t)}^g = \big(X_{s)}^g \otimes I_{[s,s+t[}\big) X_{[s,s+t[}^g, \quad (s,t \geqslant 0, g \in \mathbb{S}),$$

which makes sense for column-bounded processes. The relevance of these is that solutions of quantum stochastic differential equations with bounded coefficients need only be column bounded; however, they are cocycles in the above two senses.

Let V and W be concrete operator spaces and let $B(\mathsf{h};\mathsf{h}')$ be the ambient full operator space of W. A process *in* W is an h-h' operator process X, with exponential domain $\mathsf{h}\underline{\otimes}\mathcal{E}_{\mathsf{T}}$, satisfying $X_t^{g',g} \in \mathsf{W}$ ($t \in \mathbb{R}_+$, $g' \in \mathbb{S}$, $g \in \mathbb{S}_{\mathsf{T}}$).

A *mapping process from* V *to* W is a family $k = (k_t)_{t\geqslant 0}$ in $L\big(\mathsf{V}; \mathcal{O}(\mathsf{h}\underline{\otimes}\mathcal{E}_{\mathsf{T}}; \mathsf{h}' \otimes \mathcal{F})\big)$ such that $(k_t(x))_{t\geqslant 0}$ is a process in W ($x \in \mathsf{V}$); it is *initial-space bounded* (respectively, *initial-space completely bounded*) if $k_t^{g',g} \in B(\mathsf{V};\mathsf{W})$ (respec. $k_t^{g',g} \in CB(\mathsf{V};\mathsf{W})$) for all $t \in \mathbb{R}_+$, $g' \in \mathbb{S}$, $g \in \mathbb{S}_{\mathsf{T}}$, where $k_t^{g',g}(x) := k_t(x)^{g',g}$. It is *column-bounded* (respectively, *cb column bounded*) if $k_{t,\varepsilon} \in B\big(\mathsf{V}; \mathsf{W} \otimes_{\mathsf{M}} |\mathcal{F}\rangle\big)$ (resp. $k_{t,\varepsilon} \in CB\big(\mathsf{V}; \mathsf{W} \otimes_{\mathsf{M}} |\mathcal{F}\rangle\big)$) for all $t \in \mathbb{R}_+$, $\varepsilon \in \mathcal{E}_{\mathsf{T}}$; it is a *completely bounded process* if $k_t \in CB\big(\mathsf{V}; \mathsf{W} \otimes_{\mathsf{M}} B(\mathcal{F})\big)$ ($t \in \mathbb{R}_+$), under the inclusion (1.5); it is *adjointable* if $k_t(\mathsf{V}) \subset \mathcal{O}^{\ddagger}(\mathsf{h}\underline{\otimes}\mathcal{E}_{\mathsf{T}}, \mathsf{h}'\underline{\otimes}\mathcal{E}_{\mathsf{T}'})$, for some total subset T' of k containing 0, so that there is a process $k^\dagger$ from $\mathsf{V}^\dagger$ to $\mathsf{W}^\dagger$ satisfying $k_t^\dagger(x^*) \subset k_t(x)^\dagger$ ($t \in \mathbb{R}_+$, $x \in \mathsf{V}$).

A mapping process k from V to V is a *quantum stochastic cocycle* if,

$$k_0 = \iota_{\mathcal{F}}^{\mathsf{V}} \ \text{ and } \ k_{s+t}^{g',g} = k_s^{g',g} \circ k_t^{L_s g', L_s g} \quad (s,t \in \mathbb{R}_+,\, g' \in \mathbb{S}, g \in \mathbb{S}_{\mathsf{T}}); \tag{2.5}$$

it is *Markov regular* (respectively, *cb Markov regular*) if each function $s \mapsto k_s^{g',g}$ is continuous $\mathbb{R}_+ \to B(\mathsf{V})$ (resp. $\mathbb{R}_+ \to CB(\mathsf{V})$). If k is completely bounded then (2.5) is equivalent to the more recognisable cocycle identity

$$k_{s+t} = \widehat{k}_s \circ (\mathrm{id}_{\mathsf{V}} \otimes_{\mathsf{M}} \tau_{[s,\infty[}^B) \circ k_t \qquad (s,t \in \mathbb{R}_+)$$

where $\widehat{k}_s := k_{s)} \otimes_{\mathsf{M}} \mathrm{id}_{B(\mathcal{F}_{[s,\infty[})}$ for the induced map $k_{s)} : \mathsf{V} \to \mathsf{V} \otimes_{\mathsf{M}} B(\mathcal{F}_{[0,s[})$, and $\tau_{[s,\infty[}^B$ denotes the shift $B(\mathcal{F}) \to B(\mathcal{F}_{[s,\infty[})$.

Denote by $\mathbb{P}_{\mathrm{cbCol}}(\mathsf{V},\mathsf{W} : \mathcal{E}_{\mathsf{T}})$ the set of cb column-bounded quantum stochastic processes k from V to W with exponential domain $\mathcal{E}_{\mathsf{T}}$ and by $\mathbb{QSC}_{\mathrm{cbCol}}(\mathsf{V} : \mathcal{E}_{\mathsf{T}})$ the set of cocycles in $\mathbb{P}_{\mathrm{cbCol}}(\mathsf{V} : \mathcal{E}_{\mathsf{T}}) := \mathbb{P}_{\mathrm{cbCol}}(\mathsf{V},\mathsf{V} : \mathcal{E}_{\mathsf{T}})$.

For $k \in \mathbb{P}_{\mathrm{cbCol}}(\mathsf{V} : \mathcal{E}_{\mathsf{T}})$ and $g \in \mathbb{S}_{\mathsf{T}}$, the notation $k_{t)}^g := k_{t)}(\cdot)E_{\varepsilon(g|_{[0,t[})} \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}_{[0,t[}\rangle\big)$ extends to shifted intervals by setting

$$k_{[r,t[}^g := \big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathsf{M}} \tau_{[r,t[}\big) \circ k_{t-r)}^{L_r g} \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}_{[r,t[}\rangle\big). \tag{2.6}$$

Let $\kappa \in L(\mathsf{V}_0; \mathsf{W})$ and let $\nu \in SL\big(\widehat{\mathsf{D}'}, \widehat{\mathsf{D}}; L(\mathsf{V}_0; \mathsf{V})\big)$ for dense subspaces D' and D of k, and V_0 of V. A process k from V to W is a D'-D *weak solution on* V_0 *of the quantum stochastic differential equation*

$$\mathrm{d}k_t = k_t \circ \mathrm{d}\Lambda_\nu(t) \quad k_0 = \iota^{\mathsf{W}}_{\mathcal{F}} \circ \kappa \tag{2.7}$$

if, for all $x \in \mathsf{V}_0$, $\zeta' \in \mathsf{h}', \zeta \in \mathsf{h}$, $g' \in \mathbb{S}_{\mathsf{D}'}$, $g \in \mathbb{S}_{\mathsf{D}}$ and $t \geqslant 0$,

$$\begin{aligned} \langle \zeta' \otimes \varepsilon(g'), k_t(x)(\zeta \otimes \varepsilon(g)) \rangle &= \langle \zeta', \kappa(x)\zeta \rangle \langle \varepsilon(g'), \varepsilon(g) \rangle \\ &+ \int_0^t \mathrm{d}s \ \langle \zeta' \otimes \varepsilon(g'), k_s\big(\nu(\widehat{g'}(s), \widehat{g}(s))x\big)(\zeta \otimes \varepsilon(g)) \rangle. \end{aligned} \tag{2.8}$$

For $\kappa \in L(\mathsf{V}_0; \mathsf{W})$ and $\phi \in L\big(\mathsf{V}_0; \mathcal{O}(\mathsf{h} \underline{\otimes} \widehat{\mathsf{D}}; \mathsf{h}' \otimes \widehat{\mathsf{k}})\big)$ such that $E^{\widehat{c}}\phi(x)E_{\widehat{d}} \in \mathsf{V}$ for all $x \in \mathsf{V}_0$, $c \in \mathsf{k}$ and $d \in \mathsf{D}$, where $B(\mathsf{h}; \mathsf{h}')$ is the ambient full operator space of V, k is a *strong solution on* V_0 of the quantum stochastic differential equation

$$\mathrm{d}k_t = k_t \circ \mathrm{d}\Lambda_\phi(t) \quad k_0 = \iota^{\mathsf{V}}_{\mathcal{F}} \circ \kappa \tag{2.9}$$

if it is a weak solution of (2.7), where ν is the sesquilinear map associated with ϕ, and, for all $x \in \mathsf{V}_0$, there is a quantum stochastically integrable process X such that, for all $g' \in \mathbb{S}$ and $g \in \mathbb{S}_{\mathsf{D}}$,

$$E^{\varepsilon(g')}\big(E^{g'(s)}X_s E_{g(s)} - k_s\big(\nu(\widehat{g'}(s), \widehat{g}(s))x\big)\big)E_{\varepsilon(g)} = 0 \quad \text{ for a.a. } s.$$

Theorem 2.1 ([LiW])**.** *Let* V *and* W *be concrete operator spaces, let* $\kappa \in CB(\mathsf{V}; \mathsf{W})$ *and let* $\phi \in L\big(\widehat{\mathsf{D}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$ *for a dense subspace* D *of* k*. Then the quantum stochastic differential equation* (2.9) *has a unique weakly regular weak solution. The solution lies in* $\mathbb{P}_{\mathrm{cbCol}}(\mathsf{V}, \mathsf{W} : \mathcal{E}_{\mathsf{D}})$*. Moreover, if* $\mathsf{W} = \mathsf{V}$ *and* $\kappa = \mathrm{id}_{\mathsf{V}}$ *then* $k \in \mathbb{QSC}_{\mathrm{cbCol}}(\mathsf{V} : \mathcal{E}_{\mathsf{D}})$.

Remarks. Weak regularity means: initial space bounded and, for all $T \in \mathbb{R}_+$, $\varepsilon' \in \mathcal{E}$ and $\varepsilon \in \mathcal{E}_{\mathsf{D}}$,

$$\sup \big\{ \|E^{\varepsilon'} k_t(x) E_\varepsilon\| : t \in [0, T], x \in \mathsf{V}, \|x\| \leqslant 1 \big\} < \infty.$$

The unique solution is denoted $k^{\kappa,\phi}$, or k^ϕ when $\mathsf{W} = \mathsf{V}$ and $\kappa = \mathrm{id}_{\mathsf{V}}$; these are related as follows: $k^{\kappa,\phi}_{t,\varepsilon} = \big(\kappa \otimes_{\mathsf{M}} \mathrm{id}_{|\mathcal{F}\rangle}\big) \circ k^\phi_{t,\varepsilon}$ ([LS$_1$]).

3. Evolutions in Banach algebra

In this section we summarise results we need from [DL$_1$]; $\mathcal{A}$ here is a fixed unital Banach algebra, and $\mathcal{A}^\times$ denotes its group of units.

Definition. An *evolution* in $\mathcal{A}$ is a family $\big(F_{r,t}\big)_{0 \leqslant r \leqslant t}$ in $\mathcal{A}$, such that

$$F_{r,r} = 1_{\mathcal{A}} \ \text{ and } \ F_{r,s}\, F_{s,t} = F_{r,t} \qquad (0 \leqslant r \leqslant s \leqslant t).$$

An evolution is *invertible* if it is $\mathcal{A}^\times$-valued, and *continuous* if the following maps are continuous

$$[r, \infty[\to \mathcal{A}, \ s \mapsto F_{r,s} \ \text{ and } \ [0, t] \to \mathcal{A}, \ s \mapsto F_{s,t} \qquad (r, t \in \mathbb{R}_+).$$

These classes are denoted $\mathrm{Evol}(\mathcal{A})$, $\mathrm{Evol}(\mathcal{A}^\times)$ and $\mathrm{Evol}_{\mathrm{c}}(\mathcal{A})$ respectively. We view evolutions as maps $F : \Delta^{[2]} \to \mathcal{A}$.

Remark. Continuous evolutions are invertible:

$$\mathrm{Evol}_{\mathrm{c}}(\mathcal{A}) \subset \mathrm{Evol}(\mathcal{A}^\times),$$

and for $F \in \mathrm{Evol}(\mathcal{A}^\times)$, $F_{r,t} = F^{-1}_{0,r} F_{0,t}$. Thus continuous evolutions are determined by the one parameter family

$$F_t := F_{0,t} \qquad (t \in \mathbb{R}_+). \tag{3.1}$$

For $(r,t) \in \Delta^{(2)}$ and $n \in \mathbb{Z}_+$, set

$$\Gamma_{[r,t[} := \{\sigma \subset [r,t[: \#\sigma < \infty\} \text{ and } \Gamma^{(n)}_{[r,t[} := \{\sigma \subset [r,t[: \#\sigma = n\},$$

with measurable structure and measure induced from that of Lebesgue measure on each symplex $\Delta^{(n)}_{[r,t[}$, as defined in (1.2), via the bijection

$$\Delta^{(n)}_{[r,t[} \to \Gamma^{(n)}_{[r,t[}, \quad \mathbf{s} \mapsto \{s_1, \cdots, s_n\} \qquad (n \in \mathbb{N}),$$

and letting $\varnothing \in \Gamma^{(n)}_{[r,t[}$ be an atom of measure one ([Gui]).

Definition. Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{A})$. Its *associated product function* π_a in $L^1_{\mathrm{loc}}(\Gamma; \mathcal{A})$ is defined by

$$\pi_a(\sigma) := \overrightarrow{\prod_{s\in\sigma}} a(s);$$

its *associated evolution* F^a in $C(\Delta^{[2]}; \mathcal{A})$ is defined by

$$F^a_{r,t} := \int_{\Gamma_{[r,t[}} \pi_a = \int \pi_{a_{[r,t[}}.$$

Proposition 3.1. *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{A})$, and let $(r,t) \in \Delta^{[2]}$. Then the following hold*:

(a) $F^a \in \mathrm{Evol_c}(\mathcal{A})$.

(b) *For $u \in [-r, \infty[$,*

$$F^{L_u a}_{r,t} = F^a_{r+u,t+u}$$

where L_u is the left shift defined by $(L_u a)(s) = a(s+u)$. In particular,

$$F^a_{r,t} = F^{L_r a}_{t-r} \quad \text{and} \quad F^a_{s+u} = F^a_s F^{L_s a}_u \qquad \text{for } s,u \in \mathbb{R}_+.$$

(c)

$$F^a_{r,t} = 1_{\mathcal{A}} + \int_r^t \mathrm{d}s\, F^a_{r,s}\, a(s) = 1_{\mathcal{A}} + \int_r^t \mathrm{d}s\, a(s)\, F^a_{s,t}.$$

In order to characterise the subclass of evolutions that are useful for our analysis, we need some notation.

Notation. Let $D = \{T_1 < \cdots < T_N\} \subset\subset]0, \infty[$ and set $T_0 := 0$ and $T_{N+1} := \infty$. For $u \in \mathbb{R}_+$, letting $k = k(D,u) \in \{0, \cdots, N\}$ be determined by

$$T_k \leqslant u < T_{k+1},$$

we set

$$u^D_j := T_{k+j} \text{ for } j = -k, 1-k, \cdots, N-k. \tag{3.2}$$

Thus for example $u^D_0 = T_k$, the element of $\{0\} \cup D$ immediately to the left of u (or u itself if $u \in \{0\} \cup D$); and $u^D_1 = T_{k+1}$, the element of $D \cup \{\infty\}$ 'immediately' to the right of u.

Definition. We call F a *piecewise-semigroup evolution* if there are *associated time point and semigroup sets*

$$D = \{T_1 < \cdots < T_N\} \subset\subset]0, \infty[\text{ and } \{P^{(T)} : T \in \{0\} \cup D\} = \{P^{(T_0)}, \cdots, P^{(T_N)}\},$$

where $T_0 := 0$ and each $P^{(T_i)}$ is a semigroup in $\mathcal{A}$, for which the following identity holds:

$$F_{r,t} = \begin{cases} P^{(r^D_0)}_{t-r} & \text{if } r^D_0 = t^D_0 \\ P^{(r^D_0)}_{r^D_1 - r}\big(P^{(r^D_1)}_{r^D_2 - r^D_1} \cdots P^{(t^D_{-1})}_{t^D_0 - t^D_{-1}}\big) P^{(t^D_0)}_{t - t^D_0} & \text{otherwise.} \end{cases} \tag{3.3}$$

Let $\mathrm{Evol_{pws}}(\mathcal{A})$ denote the collection of these.

Proposition 3.2. *Let $F \in \mathrm{Evol}_{\mathrm{c}}(\mathcal{A})$. Then the following are equivalent:*

(i) *$F = F^a$ where a is piecewise constant.*

(ii) *$F \in \mathrm{Evol}_{\mathrm{pws}}(\mathcal{A})$.*

In this case (taking the right-continuous version of a), the associated time point and semigroup sets of F are respectively, $\mathrm{Disc}\, a$ *and* $\big\{\big(e^{sa(t)}\big)_{s\geqslant 0} : t \in \{0\} \cup \mathrm{Disc}\, a\big\}$.

Thus the evolutions with piecewise constant generators are the continuous evolutions which enjoy a semigroup decomposition (3.3).

Now let $\mathfrak{X}$ be a right Banach $\mathcal{A}$-module.

Then for $x \in \mathfrak{X}$, $c \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{A})$, and $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $x\, F^c \in C(\Delta^{[2]}; \mathfrak{X})$ and

$$e^{\int_r^t \varphi}\, x\, F^c_{r,t} = x\, F^{c+\varphi(\cdot)1_{\mathcal{A}}}_{r,t} \qquad (0 \leqslant r \leqslant t). \tag{3.4}$$

For Trotter products we adopt the following notations.

Notation. For a finite subset D of $]0,\infty[$ and function $G : \Delta^{[2]} \to \mathcal{A}$, in the notation (3.2), define *G's D-fold product function* by

$$G^D : \Delta^{[2]} \to \mathcal{A}, \quad G^D_{r,t} = \begin{cases} G_{r_1^D, r_2^D} \cdots G_{t^D_{-1}, t^D_0} & \text{if } r_1^D < t_0^D \\ 1_{\mathcal{A}} & \text{otherwise.} \end{cases} \tag{3.5}$$

Definition. A sequence $\big(D(n)\big)_{n\geqslant 1}$ in $\Gamma_{]0,\infty[}$ is said to *converge to* $\mathbb{R}_+$ if, as $n \to \infty$,

$$\min D(n) \to 0, \quad \max D(n) \to \infty \ \text{ and } \ \mathrm{mesh}\, D(n) \to 0. \tag{3.6}$$

Theorem 3.3. *Let $a_1, a_2 \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{A})$, let $\big(D(n)\big)_{n\geqslant 1}$ be a sequence in $\Gamma_{]0,\infty[}$ converging to $\mathbb{R}_+$, and let $T \in \mathbb{R}_+$. Then*

$$\sup_{[r,t]\subset[0,T]} \big\| F^{a_1+a_2}_{r,t} - {}^{1,2}F^{D(n)}_{r,t} \big\| \to 0, \quad \textit{where } \ {}^{1,2}F_{u,v} := F^{a_1}_{u,v}\, F^{a_2}_{u,v}, \qquad \big((u,v) \in \Delta^{[2]}\big).$$

Remark. The theorem remains true if the definition of D-fold product function is modified by replacing $G_{r_1^D, r_2^D} \cdots G_{t^D_{-1}, t^D_0}$ by $H_{r, r_1^D}\big(G_{r_1^D, r_2^D} \cdots G_{t^D_{-1}, t^D_0}\big) K_{t_0^D, t}$ for any continuous functions $H, K : \Delta^{[2]} \to \mathcal{A}$ satisfying $H_{u,u} = K_{u,u} = 1_{\mathcal{A}}$ $(u \in \mathbb{R}_+)$.

4. Sesquilinear processes and Wiener integrals

In this section we consider quantum stochastic processes consisting of Banach space valued sesquilinear maps on Fock space. We define multiple quantum Wiener integrals and establish their basic estimates.

For the rest of the paper we fix a Banach space $\mathfrak{X}$ and a Banach algebra $\mathcal{A}$. Later $\mathfrak{X}$ will be a right Banach $\mathcal{A}$-module, and eventually $\mathcal{A}$ will be assumed to be unital.

Definition. A family of maps $\mathsf{q} = (\mathsf{q}_t)_{t\geqslant 0}$ in $SL(\mathcal{E}; \mathfrak{X})$ is an *$\mathfrak{X}$-valued sesquilinear process, or SL process in $\mathfrak{X}$* if, for all $g', g \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

(i) $\mathsf{q}_t(\varepsilon(g'), \varepsilon(g)) = \mathsf{q}_t(\varepsilon(g'_{[0,t[}), \varepsilon(g_{[0,t[}))\langle \varepsilon(g'_{[t,\infty[}), \varepsilon(g_{[t,\infty[})\rangle$.

It is a *continuous* SL process in $\mathfrak{X}$ if, for all $\varepsilon, \varepsilon' \in \mathcal{E}$,

(ii) $s \mapsto \mathsf{q}_s(\varepsilon', \varepsilon)$ is continuous.

We denote the linear space of SL processes in $\mathfrak{X}$ by $SL\mathbb{P}(\mathfrak{X}, \mathsf{k})$, and the subspace of continuous SL processes by $SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}, \mathsf{k})$. For $\mathsf{q} \in SL\mathbb{P}(\mathfrak{X}, \mathsf{k})$, define

$$\mathsf{q}_t^{g',g} := \mathsf{q}_t(\varepsilon(g'_{[0,t[}), \varepsilon(g_{[0,t[})) \qquad (g', g \in \mathbb{S}_{\mathrm{loc}}, t \in \mathbb{R}_+), \tag{4.1}$$

where $\mathbb{S}_{\mathrm{loc}}$ denotes the space of (right-continuous) step functions, so $\mathbb{S}_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathsf{k})$. Thus $\mathsf{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}, \mathsf{k})$ if and only if $\mathsf{q}^{g',g} \in C(\mathbb{R}_+; \mathfrak{X})$ for all $g', g \in \mathbb{S}_{\mathrm{loc}}$.

For $\mathsf{q} \in SL\mathbb{P}(\mathfrak{X}, \mathsf{k})$, its *time-reversed process* $\mathsf{q}^{\mathrm{R}} \in SL\mathbb{P}(\mathfrak{X}, \mathsf{k})$ is defined by

$$\mathsf{q}^{\mathrm{R}}_t(\varepsilon', \varepsilon) = \mathsf{q}_t(r_t\varepsilon', r_t\varepsilon) \tag{4.2}$$

where r_t is the selfadjoint unitary operator on $\mathcal{F}$ given by $r_t\varepsilon(f) = \varepsilon(h)$ where $h(s)$ equals $f(t-s)$ for $s \in [0,t[$ and equals $f(s)$ for $s \in [t,\infty[$. If $\mathfrak{X}^\dagger$ is the conjugate Banach space of $\mathfrak{X}$ then the *involute* $\mathsf{q}^\dagger \in SL\mathbb{P}(\mathfrak{X}^\dagger, \mathsf{k})$ is defined by

$$\mathsf{q}_t^\dagger(\varepsilon', \varepsilon) = \mathsf{q}_t(\varepsilon, \varepsilon')^\dagger. \tag{4.3}$$

Set

$$S(\mathsf{k}) := \big\{(g', g, t) \in \mathbb{S} \times \mathbb{S} \times \mathbb{R}_+ : \operatorname{supp} g', \operatorname{supp} g \subset [0,t[\big\}.$$

By the linear independence of the exponential vectors, and the definition of adaptedness, the following is easily seen.

Lemma 4.1. *The following map is a bijection*:

$$SL\mathbb{P}(\mathfrak{X}, \mathsf{k}) \to F\big(S(\mathsf{k}); \mathfrak{X}\big), \quad \mathsf{q} \mapsto \phi_\mathsf{q} \quad \textit{where} \quad \phi_\mathsf{q}(g', g, t) := \mathsf{q}_t^{g', g}$$

Here $F\big(S(\mathsf{k}); \mathfrak{X}\big)$ denotes the space of $\mathfrak{X}$-valued functions on the set $S(\mathsf{k})$.

Remarks. The inverse of the above bijection is given by *adapted, sesquilinear extension* of the prescription

$$\phi \mapsto \mathsf{q}^\phi \quad \text{where} \quad \mathsf{q}_t^\phi\big(\varepsilon(g'), \varepsilon(g)\big) = \phi(g', g, t) \text{ for } t \in \mathbb{R}_+ \text{ and } g', g \in \mathbb{S}_{[0,t[}. \tag{4.4}$$

Thus

$$\mathsf{q}_t^\phi\big(\varepsilon(f'), \varepsilon(f)\big) = \exp\langle f'_{[t,\infty[}, f_{[t,\infty[}\rangle \phi\big(f'_{[0,t[}, f_{[0,t[}, t\big).$$

If $\mathfrak{X}$ is a right (or left) Banach module over $\mathcal{A}$, then $SL\mathbb{P}(\mathfrak{X}, \mathsf{k})$ is naturally likewise.

In all that follows, $SL(\mathcal{E}; \mathfrak{X})$ could be replaced by $SL(\mathcal{E}_{\mathsf{T}'}, \mathcal{E}_\mathsf{T}; \mathfrak{X})$, and $S(\mathsf{k})$ by $S(\mathsf{T}', \mathsf{T})$, defined in the obvious way, where T' and T are both total subsets of k containing 0. We shall exploit this fact when applying our results.

Examples. We give a trivial, but useful, example and three paradigm examples.

(a) Let $x \in \mathfrak{X}$. Then, in the notation (1.1), $\mathsf{q}_t := |x\rangle\, q_I$ $(t \in \mathbb{R}_+)$, where $I = I_\mathcal{F}$, defines an SL process in $\mathfrak{X}$. We refer to this as *the constant SL process x*.

(b) Let Z be an initial-space bounded h-h' process, for Hilbert spaces h and h'. Then $\mathsf{q}_t(\varepsilon', \varepsilon) := E^{\varepsilon'} Z_t E_\varepsilon$ defines an SL process in $B(\mathsf{h}; \mathsf{h}')$.

(c) Let k be an initial-space bounded (respectively, completely bounded) mapping process from V to W, for concrete operator spaces V and W. Then $\mathsf{q}_t(\varepsilon', \varepsilon) := E^{\varepsilon'} k_{t,\varepsilon}(\cdot)$ defines an SL process in $B(\mathsf{V}; \mathsf{W})$ (resp. $CB(\mathsf{V}; \mathsf{W})$).

(d) Let l be a cb column bounded quantum stochastic convolution cocycle on C, for an operator space coalgebra C (see [LS$_2$]). Then $\mathsf{q}_t(\varepsilon', \varepsilon) := \omega_{\varepsilon', \varepsilon} \circ l_t$ defines an SL process in the topological dual space C^*.

Remark. In (b), (c) and (d), when the process Z, k, respectively l, is a column-bounded/column-completely bounded Markov-regular quantum stochastic cocycle ([L$_1$]), or satisfies a linear constant-coefficient quantum stochastic differential equation with cb column bounded coefficients and completely bounded initial conditions (as in Theorem 2.1), the corresponding SL process q is continuous.

Multiple quantum Wiener integrals are defined in this setting as follows. For $n \in \mathbb{N}$, $\upsilon_n \in SL(\widehat{\mathsf{k}}^{\otimes n}; \mathfrak{X})$ and $t \geqslant 0$, define a map $\Lambda_t^n(\upsilon_n) \in SL(\mathcal{E}; \mathfrak{X})$ by sesquilinear extension of the prescription

$$\Lambda_t^n(\upsilon_n)(\varepsilon(g'), \varepsilon(g)) := \exp\langle g', g\rangle \int_{\Delta^{[n]}_{[0,t[}} \mathrm{d}\mathbf{s}\ \upsilon_n\big(\widehat{g'}^{\otimes n}(\mathbf{s}), \widehat{g}^{\otimes n}(\mathbf{s})\big) \quad (g', g \in \mathbb{S}),$$

for the convention

$$\widehat{h}^{\otimes n}(\mathbf{s}) := \widehat{h}(s_1) \otimes \cdots \otimes \widehat{h}(s_n), \qquad (\mathbf{s} \in \Delta^{[n]}).$$

The above integral is well-defined, since the integrand is an $\mathfrak{X}$-valued simple function on the simplex. Moreover $\big(\Lambda^n_t(\upsilon_n)\big)_{t\geqslant 0} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X},\mathsf{k})$ in view of the obvious identity

$$\Lambda^n_t(\upsilon_n)\big(\varepsilon(g'_{[0,t[}),\varepsilon(g_{[0,t[})\big) = \exp\langle g'_{[0,t[}, g_{[0,t[}\rangle \int_{\Delta^{[n]}_{[0,t[}} \mathrm{d}\mathbf{s}\ \upsilon_n\big(\widehat{g'}^{\otimes n}(\mathbf{s}), \widehat{g}^{\otimes n}(\mathbf{s})\big).$$

For $\upsilon_0 \in SL(\mathbb{C};\mathfrak{X})$ we define $\Lambda^0_\cdot(\upsilon_0)$ to be the constant SL process $|\upsilon_0(1,1)\rangle\, q_I$. Quantum Wiener integration $\Lambda^n : SL(\widehat{\mathsf{k}}^{\otimes n};\mathfrak{X}) \to SL\mathbb{P}(\mathfrak{X},\mathsf{k})$ is evidently linear and, when $\mathfrak{X}$ is a right (or left) Banach module over $\mathcal{A}$, it is a module map too.

In order to give the basic estimate for these quantum Wiener integrals, define bounding constants for them as follows: $C^{\upsilon_0}_0(g',g) := \|\upsilon_0(1,1)\|$, and for $n\in\mathbb{N}$,

$$C^{\upsilon_n}_n(g',g) := \max\Big\{\big\|\upsilon_n\big(\widehat{c(1)}\otimes\cdots\otimes\widehat{c(n)}, \widehat{d(1)}\otimes\cdots\otimes\widehat{d(n)}\big)\big\| : \\ c(1),\dots,c(n)\in \operatorname{Ran} g',\ d(1),\dots,d(n)\in\operatorname{Ran} g\Big\}; \quad (4.5)$$

abbreviating $C^{\upsilon_1}_1(g',g)$ to $C^{\upsilon_1}(g',g)$.

Lemma 4.2. *Let $n\in\mathbb{Z}_+$, $\upsilon_n \in SL(\widehat{\mathsf{k}}^{\otimes n};\mathfrak{X})$ and $g',g\in\mathbb{S}$. Then, for $t\geqslant 0$,*

$$\big\|\Lambda^n_t(\upsilon_n)(\varepsilon(g'),\varepsilon(g))\big\| \leq |\exp\langle g',g\rangle|\, C^{\upsilon_n}_n(g',g)\frac{t^n}{n!} \quad (t\in\mathbb{R}_+) \qquad (4.6)$$

and, for $n\in\mathbb{N}$ and $t\geqslant r\geqslant 0$,

$$\big\|\Lambda^n_t(\upsilon_n)(\varepsilon(g'),\varepsilon(g)) - \Lambda^n_r(\upsilon_n)(\varepsilon(g'),\varepsilon(g))\big\| \leq (t-r)|\exp\langle g',g\rangle| \frac{t^{n-1}}{(n-1)!} C^{\upsilon_n}_n(g',g).$$

Proof. The first inequality is clear when $n=0$, and for $n\geqslant 1$ it follows from the fact that $\Delta^{[n]}_{[0,t[}$ has n-dimensional volume $t^n/n!$. The second follows from the inequality

$$\big|\Delta^{[n]}_{[0,t[}\setminus\Delta^{[n]}_{[0,r[}\big| = \big|\Delta^{[n]}_{[0,t[}\big| - \big|\Delta^{[n]}_{[0,r[}\big| = \frac{t^n}{n!} - \frac{r^n}{n!} \leqslant \frac{(t-r)}{(n-1)!}t^{n-1}.$$

□

Definition. Let $SL\mathbb{W}(\mathfrak{X},\mathsf{k})$ denote the linear space of SL *Wiener integrands*, that is the space of sequences $\mathcal{U} = (\upsilon_n)_{n\geqslant 0}$, in which $\upsilon_n \in SL(\widehat{\mathsf{k}}^{\otimes n};\mathfrak{X})$ for each $n\in\mathbb{Z}_+$ and

$$\forall_{g',g\in\mathbb{S}}\ \forall_{\alpha\in\mathbb{R}_+}\ \sum_{n\geqslant 0}\frac{\alpha^n}{n!}C^{\upsilon_n}_n(g',g) < \infty. \qquad (4.7)$$

Let $\mathcal{U}\in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$. The *time-reversed integrand* $\mathcal{U}^{\mathrm{R}} \in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$ is defined by

$$\upsilon^{\mathrm{R}}_n(\zeta',\zeta) = \upsilon_n(r_n\zeta', r_n\zeta) \qquad (4.8)$$

where r_n is the selfadjoint unitary on $\widehat{\mathsf{k}}^{\otimes n}$ determined by $r_n\big(\zeta_1\otimes\cdots\otimes\zeta_n\big) = \zeta_n\otimes\cdots\otimes\zeta_1$. If $\mathfrak{X}^\dagger$ is a conjugate Banach space of $\mathfrak{X}$ then $SL\mathbb{W}(\mathfrak{X}^\dagger,\mathsf{k})$ is a conjugate vector space of $SL\mathbb{W}(\mathfrak{X},\mathsf{k})$, with $\mathcal{U}^\dagger \in SL\mathbb{W}(\mathfrak{X}^\dagger,\mathsf{k})$ defined by

$$\upsilon^\dagger_n(\zeta',\zeta) = \upsilon_n(\zeta,\zeta')^\dagger. \qquad (4.9)$$

Remark. By analyticity, if $\mathcal{U}\in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$ then also

$$\forall_{g',g\in\mathbb{S}}\ \forall_{\alpha\in\mathbb{R}_+}\ \sum_{n\geqslant 1}\frac{\alpha^{n-1}}{(n-1)!}C^{\upsilon_n}_n(g',g) < \infty. \qquad (4.10)$$

Proposition 4.3. *Let $\mathcal{U} = (\upsilon_n)_{n\geqslant 0} \in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$. Then*

$$\Lambda_t(\mathcal{U}) := \text{p.w.}\sum_{n\geqslant 0}\Lambda^n_t(\upsilon_n) \qquad (t\geqslant 0)$$

defines an SL process $\Lambda(\mathcal{U})$ in $\mathfrak{X}$ satisfying

$$\big\|\Lambda_t(\mathcal{U})(\varepsilon(g'),\varepsilon(g))\big\| \le |\exp\langle g',g\rangle| \sum_{n\geqslant 0} \frac{t^n}{n!} C_n^{v_n}(g',g)$$

and

$$\big\|\Lambda_t(\mathcal{U})(\varepsilon(g'),\varepsilon(g)) - \Lambda_r(\mathcal{U})(\varepsilon(g'),\varepsilon(g))\big\| \leqslant$$
$$(t-r)|\exp\langle g',g\rangle| \sum_{n\geqslant 1} \frac{t^{n-1}}{(n-1)!} C_n^{v_n}(g',g),$$

for all $g',g\in\mathbb{S}$ and $t\geqslant r\geqslant 0$. Moreover $\Lambda_\cdot(\mathcal{U}^{\mathrm{R}}) = \Lambda_\cdot(\mathcal{U})^{\mathrm{R}}$, and $\Lambda_\cdot(\mathcal{U}^\dagger) = \Lambda_\cdot(\mathcal{U})^\dagger$.

Remarks. The quantum Wiener integral is thereby a linear map

$$\Lambda : SL\mathbb{W}(\mathfrak{X},\mathsf{k}) \to SL\mathbb{P}_{\mathrm{Lip}}(\mathfrak{X},\mathsf{k}),$$

where $SL\mathbb{P}_{\mathrm{Lip}}(\mathfrak{X},\mathsf{k})$ stands for the space of pointwise locally Lipschitz continuous processes:

$$\big\{\mathsf{q}\in SL\mathbb{P}(\mathfrak{X},\mathsf{k}) : \forall_{\varepsilon,\varepsilon'\in\mathcal{E}} \quad s\mapsto \mathsf{q}_s(\varepsilon',\varepsilon) \text{ is locally Lipschitz continuous}\big\}.$$

If $\mathfrak{X}$ is a right (or left) Banach $\mathcal{A}$-module then $SL\mathbb{W}(\mathfrak{X},\mathsf{k})$ is likewise, and Λ is a module map.

Now suppose that $\mathfrak{X}$ is a right Banach $\mathcal{A}$-module; let $\widetilde{\mathcal{A}}$ denote the conditional unitisation of $\mathcal{A}$ that is $\mathcal{A}$ if it is unital, and its unitisation if it is not ([Dal]). For $x\in\mathfrak{X}$ and $\nu\in SL(\widehat{\mathsf{k}};\widetilde{\mathcal{A}})$ define $x\nu^{\otimes} = \big(x\nu^{\underline{\otimes} n}\big)_{n\geqslant 0}$ by $x\nu^{\underline{\otimes} 0} := |x\rangle\, q_I$ and, for $n\in\mathbb{N}$, $x\nu^{\underline{\otimes} n} : \widehat{\mathsf{k}}^{\otimes n}\times\widehat{\mathsf{k}}^{\otimes n}\to\mathfrak{X}$ is the sesquilinearisation of the map

$$\widehat{\mathsf{k}}^n\times\widehat{\mathsf{k}}^n\to\mathfrak{X},\quad (\zeta,\eta)\mapsto x\overrightarrow{\prod_{1\leqslant i\leqslant n}}\nu(\zeta_i,\eta_i).$$

Then

$$C_n^{x\nu^{\underline{\otimes} n}}(g',g) \le \|x\| C^{\nu}(g',g)^n \qquad (n\in\mathbb{Z}_+, g',g\in\mathbb{S}),$$

so $x\nu^{\otimes}\in SL\mathbb{W}(\mathfrak{X},\mathsf{k})$; set ${}^x\mathsf{q}^{\nu} := \Lambda(x\nu^{\otimes})$. Recall the abbreviation $F_t := F_{0,t}$ (3.1) for an evolution F.

Lemma 4.4. *Let $\mathsf{q} = {}^x\mathsf{q}^\nu$ for $x\in\mathfrak{X}$ and $\nu\in SL(\widehat{\mathsf{k}};\widetilde{\mathcal{A}})$, and let $g',g\in\mathbb{S}$ and $t\in\mathbb{R}_+$. Then*

$$\mathsf{q}^{g',g} = x\,F^{\widetilde{a}} \quad \textit{and} \quad \mathsf{q}_t(\varepsilon(g'),\varepsilon(g)) = e^{\langle g',g\rangle}\,x\,F^a_t,$$

where a and a' are the $\widetilde{\mathcal{A}}$-valued step functions given by

$$a(t) := \nu\big(\widehat{g'}(t),\widehat{g}(t)\big), \quad \textit{and} \quad \widetilde{a}(t) := a(t) + \langle g'(t),g(t)\rangle 1_{\widetilde{\mathcal{A}}}.$$

Proof. Set $\varphi(s) = \langle g'(s),g(s)\rangle$ $(s\in\mathbb{R}_+)$. Then

$$\mathsf{q}_t(\varepsilon(g'),\varepsilon(g)) = e^{\langle g',g\rangle}\Big\{x + \sum_{n=1}^{\infty} x\int_{\Delta^{[n]}_{[0,t[}} \mathrm{d}\mathbf{s}\ \overrightarrow{\prod_{1\leqslant i\leqslant n}}\nu(\widehat{g'}(s_i),\widehat{g}(s_i))\Big\} = e^{\langle g',g\rangle}\,x\,F^a_t,$$

and so, by identity (3.4),

$$\mathsf{q}_t^{g',g} = e^{\int_0^t\varphi}\,x\,F^a_t = x\,F^{\widetilde{a}}_t.$$

□

Similarly, if $\mathfrak{Z}$ is a left Banach $\mathcal{B}$-module, $z\in\mathfrak{Z}$ and $\nu\in SL\big(\widehat{\mathsf{k}};\widetilde{\mathcal{B}}\big)$, defining ${}^{\otimes}\nu z = \big({}^{\underline{\otimes} n}\nu z\big)_{n\geqslant 0}$ by ${}^{\underline{\otimes} 0}\nu z := |z\rangle\, q_I$ and, for $n\in\mathbb{N}$, ${}^{\underline{\otimes} n}\nu z$ as the sesquilinear extension of the map

$$\widehat{\mathsf{k}}^n\times\widehat{\mathsf{k}}^n\to\mathfrak{Z},\quad (\zeta,\eta)\mapsto\Big(\overleftarrow{\prod_{1\leqslant i\leqslant n}}\nu(\zeta_i,\eta_i)\Big)\,z,$$

${}^{\otimes}\nu z \in SL\mathbb{W}(\mathfrak{Z}, \mathsf{k})$ and we set ${}^{\nu}\mathfrak{q}^{z} := \Lambda({}^{\otimes}\nu z)$. Thus

$$ {}^{\nu}\mathfrak{q}^{z}_{t}(\varepsilon(g'), \varepsilon(g)) = \exp\langle g', g\rangle \Big(\Big\{ \sum_{n=1}^{\infty} \int_{\Delta^{[n]}_{[0,t[}} \mathrm{d}\mathbf{s} \overleftarrow{\prod_{1 \leqslant i \leqslant n}} \nu(\widehat{g'}(s_i), \widehat{g}(s_i)) \Big\} z + z \Big), $$

and the process $\mathfrak{q} = {}^{\nu}\mathfrak{q}^{z}$ satisfies

$$ \mathfrak{q}^{g',g} = {}^{\widetilde{a}}F.z \quad \text{and} \quad \mathfrak{q}_t(\varepsilon(g'), \varepsilon(g)) = e^{\langle g', g\rangle} \, {}^{a}F_t z. $$

Remarks. From the definitions we have the following relations

$$ ({}^{x}\mathfrak{q}^{\nu})^{\mathrm{op}} = {}^{\mu}\mathfrak{q}^{z} \qquad (x \in \mathfrak{X}, \nu \in SL(\widehat{\mathsf{k}}; \widetilde{\mathcal{A}})) \tag{4.11} $$

in which $\mathfrak{X}^{\mathrm{op}}$ is the left Banach $\mathcal{A}^{\mathrm{op}}$-module opposite to $\mathfrak{X}$, $\mu = \nu^{\mathrm{op}} : (\widehat{e}, \widehat{c}) \mapsto \nu(\widehat{e}, \widehat{c})^{\mathrm{op}} \in \widetilde{\mathcal{A}}^{\mathrm{op}} = \widetilde{\mathcal{A}^{\mathrm{op}}}$ and $z = x^{\mathrm{op}} \in \mathfrak{X}^{\mathrm{op}}$; and

$$ ({}^{x}\mathfrak{q}^{\nu})^{\dagger} = {}^{\mu}\mathfrak{q}^{z} \qquad (x \in \mathfrak{X}, \nu \in SL(\widehat{\mathsf{k}}; \widetilde{\mathcal{A}})) $$

in which $\mathfrak{X}^{\dagger}$ is the left Banach $\mathcal{A}^{\dagger}$-module conjugate to $\mathfrak{X}$, $\mathcal{A}^{\dagger}$ is the Banach algebra conjugate to $\mathcal{A}$, $\mu = \nu^{\dagger}$ and $z = x^{\dagger}$.

Setting $\mathfrak{q}^{\nu} := {}^{1}\mathfrak{q}^{\nu}$ and ${}^{\nu}\mathfrak{q} := {}^{\nu}\mathfrak{q}^{1}$, where $1 = 1_{\widetilde{\mathcal{A}}}$, we have

$$ (\mathfrak{q}^{\nu})^{\mathrm{R}} = {}^{\nu}\mathfrak{q} \qquad (\nu \in SL(\widehat{\mathsf{k}}; \widetilde{\mathcal{A}})). $$

When $\mathfrak{X}$ is a Banach $\mathcal{A}$-bimodule,

$$ {}^{ax}\mathfrak{q}^{\nu} = a\, {}^{x}\mathfrak{q}^{\nu} \quad \text{and} \quad {}^{\nu}\mathfrak{q}^{xa} = {}^{\nu}\mathfrak{q}^{x}\, a, \qquad (x \in \mathfrak{X},\ a \in \mathcal{A}, \nu \in SL(\widehat{\mathsf{k}}; \widetilde{\mathcal{A}})). $$

The following result is an immediate consequence of Proposition 4.3.

Theorem 4.5. *Let $\mathfrak{X}$ be a right Banach $\mathcal{A}$-module and let $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and $x \in \mathfrak{X}$. Then, for all $g', g \in \mathbb{S}$,*

$$ \|{}^{x}\mathfrak{q}^{\nu}_{t}(\varepsilon(g'), \varepsilon(g))\| \leq |\exp\langle g', g\rangle|\, \|x\|\, e^{tC} \quad (t \geqslant 0), $$

and

$$ \|{}^{x}\mathfrak{q}^{\nu}_{t}(\varepsilon(g'), \varepsilon(g)) - \mathfrak{q}^{x,\nu}_{r}(\varepsilon(g'), \varepsilon(g))\| \leq (t - r)\, |\exp\langle g', g\rangle|\, \|x\|\, C e^{tC} \quad (0 \leqslant r \leqslant t), $$

where $C := C^{\nu}(g', g)$. In particular, ${}^{x}\mathfrak{q}^{\nu} \in SL\mathbb{P}_{\mathrm{Lip}}(\mathfrak{X}, \mathsf{k})$.

If $\mathfrak{Z}$ is a left Banach $\mathcal{B}$-module, $\mu \in SL(\widehat{\mathsf{k}}; \mathcal{B})$ and $z \in \mathfrak{Z}$ then ${}^{\nu}\mathfrak{q}^{z}$ satisfies corresponding estimates.

5. Sesquilinear stochastic differential equations

In this section we prove an existence and uniqueness theorem for quantum stochastic differential equations for SL processes in $\mathfrak{X}$. Now $\mathfrak{X}$ *is assumed to be a right Banach $\mathcal{A}$-module.*

Definition. Let $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and $x \in \mathfrak{X}$. Then $\mathfrak{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}, \mathsf{k})$ is a solution of the *left sesquilinear quantum stochastic differential equation*

$$ \mathrm{d}\mathfrak{q}_t = \mathfrak{q}_t \, \mathrm{d}\Lambda_{\nu}(t), \quad \mathfrak{q}_0 = |x\rangle q_I \tag{5.1} $$

if, for all $g', g \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

$$ \mathfrak{q}_t(\varepsilon(g'), \varepsilon(g)) = \langle \varepsilon(g'), \varepsilon(g)\rangle x + \int_0^t \mathrm{d}s\ \mathfrak{q}_s(\varepsilon(g'), \varepsilon(g)) \nu(\widehat{g'}(s), \widehat{g}(s)); \tag{5.2} $$

in other words if, for all $g, g' \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

$$ G_t = e^{\langle g', g\rangle} x + \int_0^t \mathrm{d}s\ G_s\, a(s) \tag{5.3} $$

where the functions G and a are given by

$$ G_t := \mathfrak{q}_t(\varepsilon(g'), \varepsilon(g)), \text{ and } a(t) := \nu(\widehat{g'}(t), \widehat{g}(t)). \tag{5.4} $$

If $\mathfrak{Z}$ is a left Banach $\mathcal{B}$-module then, being a solution of the right sesquilinear quantum stochastic differential equation

$$\mathrm{d}\mathsf{q}_t = \mathrm{d}\Lambda_\nu(t)\mathsf{q}_t, \quad \mathsf{q}_0 = |z\rangle q_I \tag{5.5}$$

is defined analogously, with the order of the images of q_s and ν in (5.2) reversed.

Theorem 5.1. *Let $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and $x \in \mathfrak{X}$. Then ${}^x\mathsf{q}^\nu$ is the unique solution of the left sesquilinear quantum stochastic differential equation* (5.1).

Proof. Fix $g', g \in \mathbb{S}$ and define $G : \mathbb{R}_+ \to \mathfrak{X}$ and $a : \mathbb{R}_+ \to \mathcal{A}$ by (5.4), where $\mathsf{q} = {}^x\mathsf{q}^\nu$. It follows from Lemma 4.4 and Part (c) of Proposition 3.1 that G satisfies the integral equation (5.3). Thus ${}^x\mathsf{q}^\nu$ satisfies (5.1). By the uniqueness part of Lemma 1.2, the integral equation (5.3) has a unique continuous solution. This implies that (5.1) has a unique solution $\mathsf{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}, \mathsf{k})$. □

Remarks. If $\mathsf{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}, \mathsf{k})$, $x \in \mathfrak{X}$ and $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$, then $\mathsf{q}^{\mathrm{op}} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}^{\mathrm{op}}, \mathsf{k})$, $\mathsf{q}^\dagger \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}^\dagger, \mathsf{k})$ and the following are equivalent:

(i) q satisfies (5.1);
(ii) q^{op} satisfies (5.5) for $\mathfrak{Z} = \mathfrak{X}^{\mathrm{op}}$, $\mathcal{B} = \mathcal{A}^{\mathrm{op}}$, $z = x^{\mathrm{op}}$ and $\mu = \nu^{\mathrm{op}}$;
(iii) $\mathsf{q}^\dagger$ satisfies (5.5) for $\mathfrak{Z} = \mathfrak{X}^\dagger$, $\mathcal{B} = \mathcal{A}^\dagger$, $z = x^\dagger$ and $\mu = \nu^\dagger$.

If $\mathsf{q} \in SL\mathbb{P}_{\mathrm{c}}(\widetilde{\mathcal{A}}, \mathsf{k})$ then the following are equivalent:

(i) q satisfies (5.1) with $\mathfrak{X} = \widetilde{\mathcal{A}}$ and $x = 1_{\widetilde{\mathcal{A}}}$;
(ii) q^{R} satisfies (5.5) with $\mathcal{B} = \mathcal{A}$, $\mathfrak{Z} = \widetilde{\mathcal{A}}$, $\mu = \nu$ and $z = 1_{\widetilde{\mathcal{A}}}$.

Corollary 5.2. *Let $\mu \in SL(\widehat{\mathsf{k}}; \mathcal{B})$ and $z \in \mathfrak{Z}$, for a left Banach $\mathcal{B}$-module $\mathfrak{Z}$. Then ${}^\mu\mathsf{q}^z$ is the unique solution of the right sesquilinear quantum stochastic differential equation* (5.5).

We next connect the present theory to standard quantum stochastic differential equations, noting that for operator spaces V and W, $CB(\mathsf{V}; \mathsf{W})$ is a right $CB(\mathsf{V})$-module. Recall the notation $k^{\kappa,\phi}$ for the solution of the QSDE (2.9).

Proposition 5.3. *Let V and W be concrete operator spaces, and let $k = k^{\kappa,\phi}$ where $\kappa \in CB(\mathsf{V}; \mathsf{W})$ and $\phi \in L\big(\widehat{\mathsf{k}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$. Set $\mathfrak{X} = CB(\mathsf{V}; \mathsf{W})$ and $\mathcal{A} = CB(\mathsf{V})$, let $\mathsf{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathfrak{X}, \mathsf{k})$ and $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ be respectively the associated SL process of k and the SL map associated with ϕ:*

$$\mathsf{q}_t(\varepsilon', \varepsilon) := E^{\varepsilon'} k_{t,\varepsilon}(\cdot) \quad (\varepsilon', \varepsilon \in \mathcal{E}, t \in \mathbb{R}_+), \quad \textit{and } \nu(\zeta', \zeta) := E^{\zeta'} \phi_\zeta(\cdot) \quad (\zeta', \zeta \in \widehat{\mathsf{k}}).$$

Then $\mathsf{q} = {}^\kappa\mathsf{q}^\nu$.

Proof. Let $B(\mathsf{h}; \mathsf{h}')$ be the ambient full operator space of V, let $\zeta' \in \mathsf{h}'$, $\zeta \in \mathsf{h}$, $g', g \in \mathbb{S}$ and $t \in \mathbb{R}_+$; set $\varepsilon' = \varepsilon(g')$ and $\varepsilon = \varepsilon(g)$. Applying (2.8),

$$\begin{aligned}&\langle \zeta' \otimes \varepsilon', k_t(x)(\zeta \otimes \varepsilon)\rangle \\ &\qquad = \langle \zeta', \kappa(x)\zeta\rangle \langle \varepsilon', \varepsilon\rangle + \int_0^t \mathrm{d}s\, \big\langle \zeta' \otimes \varepsilon', k_s\big(\nu(\widehat{g'}(s), \widehat{g}(s))x\big)(\zeta \otimes \varepsilon)\big\rangle,\end{aligned}$$

so

$$\langle \zeta', \mathsf{q}_t(\varepsilon', \varepsilon)(x)\zeta\rangle = \langle \zeta', \mathsf{q}_0(\varepsilon', \varepsilon)(x)\zeta\rangle + \int_0^t \mathrm{d}s\, \big\langle \zeta', \mathsf{q}_s(\varepsilon', \varepsilon)\nu(\widehat{g'}(s), \widehat{g}(s))(x)\zeta\big\rangle.$$

Since $s \mapsto \mathsf{q}_s(\varepsilon', \varepsilon) = E^{\varepsilon'} k_{s,\varepsilon}$ is continuous $\mathbb{R}_+ \to CB(\mathsf{V}; \mathsf{W})$ and $s \mapsto \nu(\widehat{g'}(s), \widehat{g}(s))$ is a step function $\mathbb{R}_+ \to CB(\mathsf{V})$, this implies that

$$\mathsf{q}_t(\varepsilon', \varepsilon) = \langle \varepsilon', \varepsilon\rangle\, \kappa + \int_0^t \mathrm{d}s\, \mathsf{q}_s(\varepsilon', \varepsilon) \circ \nu\big(\widehat{g'}(s), \widehat{g}(s)\big) \qquad (t \in \mathbb{R}_+).$$

Therefore $\mathsf{q} = {}^\kappa\mathsf{q}^\nu$, by uniqueness in Theorem 5.1. □

6. Sesquilinear stochastic cocycles

For the rest of the paper $\mathcal{A}$ is assumed to be a unital Banach algebra. We consider stochastic cocycles in the present setting. Examples are provided by solutions of quantum stochastic differential equations, and we give sufficient conditions for a cocycle to be governed by such an equation. The latter entails a new characterisation theorem for standard quantum stochastic cocycles.

For $\mathfrak{q} \in SL\mathbb{P}(\mathcal{A}, \mathsf{k})$ we extend the notation (4.1) to two parameters by setting

$$\mathfrak{q}^{g',g}_{r,t} := \mathfrak{q}^{L_r g', L_r g}_{t-r} \qquad \text{for } (r,t) \in \Delta^{[2]},\ g', g \in \mathbb{S}_{\text{loc}}, \tag{6.1}$$

where $\big(L_r\big)_{r \geqslant 0}$ is the left-shift semigroup on $\mathbb{S}_{\text{loc}}$ given by $(L_r g)(s) := g(s+r)$. Note that $(L_r g)_{[0,t-r[} = L_r(g_{[r,t[})$.

Definition. A process $\mathfrak{q} \in SL\mathbb{P}(\mathcal{A}, \mathsf{k})$ is a *left sesquilinear stochastic cocycle* in $\mathcal{A}$ if it satisfies

$$\mathfrak{q}^{g',g}_0 = 1_{\mathcal{A}} \ \text{ and } \ \mathfrak{q}^{g',g}_{s+t} = \mathfrak{q}^{g',g}_s \, \mathfrak{q}^{L_s g', L_s g}_t \qquad (g', g \in \mathbb{S}_{\text{loc}}, s, t \in \mathbb{R}_+). \tag{6.2}$$

If also $\mathfrak{q} \in SL\mathbb{P}_{\text{c}}(\mathcal{A}, \mathsf{k})$, then $\mathfrak{q}$ is said to be *Markov regular.*

We denote the classes of left SL cocycles and Markov-regular left SL cocycles by $SL\mathbb{SC}(\mathcal{A}, \mathsf{k})$ and $SL\mathbb{SC}_{\text{c}}(\mathcal{A}, \mathsf{k})$ respectively.

Proposition 6.1. *Let $\mathfrak{q} \in SL\mathbb{P}(\mathcal{A}, \mathsf{k})$. Then the following are equivalent*:

(i) $\mathfrak{q} \in SL\mathbb{SC}(\mathcal{A}, \mathsf{k})$.

(ii) *For all $g', g \in \mathbb{S}_{\text{loc}}$, $\big(\mathfrak{q}^{g',g}_{r,t}\big)_{0 \leqslant r \leqslant t}$ defines an evolution in $\mathcal{A}$.*

In this case, for all $g', g \in \mathbb{S}_{\text{loc}}$, $\mathfrak{q}^{g',g} := \big(\mathfrak{q}^{g',g}_{r,t}\big)_{0 \leqslant r \leqslant t} \in \operatorname{Evol}_{\text{pws}}(\mathcal{A})$ with associated time point and semigroup sets $\operatorname{Disc} g' \cup \operatorname{Disc} g$ and $\big\{\mathfrak{q}^{c',c} : (c', c) \in \operatorname{Ran}(g', g)\big\}$ respectively.

Proof. Suppose that $\mathfrak{q} \in SL\mathbb{SC}(\mathcal{A}, \mathsf{k})$, let $g', g \in \mathbb{S}_{\text{loc}}$ and set $D = \operatorname{Disc} g' \cup \operatorname{Disc} g$. Then $\mathfrak{q}^{g',g}_{r,r} = \mathfrak{q}^{L_r g', L_r g}_0 = 1_{\mathcal{A}}$ $(r \geq 0)$ and, in view of the identity $L_s h = L_{s-r}(L_r h)$ $(0 \leq r \leq s,\ h \in \mathbb{S}_{\text{loc}})$,

$$\mathfrak{q}^{g',g}_{r,s} \mathfrak{q}^{g',g}_{s,t} = \mathfrak{q}^{L_r g', L_r g}_{s-r} \mathfrak{q}^{L_{s-r} L_r g', L_{s-r} L_r g}_{t-s} = \mathfrak{q}^{L_r g', L_r g}_{t-r} = \mathfrak{q}^{g',g}_{r,t} \qquad (0 \leqslant r \leqslant s \leqslant t)$$

so $\big(\mathfrak{q}^{g',g}_{r,t}\big)_{0 \leq r \leq t}$ is an evolution. Moreover, for $c', c \in \mathsf{k}$,

$$\mathfrak{q}^{c',c}_0 = 1_{\mathcal{A}} \ \text{ and } \ \mathfrak{q}^{c',c}_{s+t} = \mathfrak{q}^{c',c}_s \, \mathfrak{q}^{L_s c', L_s c}_t = \mathfrak{q}^{c',c}_s \, \mathfrak{q}^{c',c}_t \qquad (s, t \geq 0),$$

so $\big(\mathfrak{q}^{c',c}_t\big)_{t \geq 0}$ is a semigroup. Set

$$P^{(t)} := \mathfrak{q}^{c',c} \ \text{ where } \ (c', c) := \big(g'(t), g(t)\big) \text{ for } t \in \{0\} \cup D,$$

and recall the notation (3.2). If g' and g are constant on an interval $[u, v[$ then $L_u g'$ and $L_u g$ are constant, equal to $g'(u^D_0)$ and $g(u^D_0)$ respectively, on $[0, v-u[$, so

$$\mathfrak{q}^{g',g}_{u,v} = \mathfrak{q}^{L_u g', L_u g}_{v-u} = \mathfrak{q}^{g'(u^D_0), g(u^D_0)}_{v-u} = P^{(u^D_0)}_{v-u}.$$

Let $0 \leq r < t$. If $r^D_0 = t^D_0$ then g' and g are constant on $[r, t[$ so $\mathfrak{q}^{g',g}_{r,t} = P^{(r^D_0)}_{t-r}$; if $r^D_0 < t^D_0$ then, since $\big(\mathfrak{q}^{g',g}_{r,t}\big)_{0 \leq r \leq t}$ is an evolution,

$$\mathfrak{q}^{g',g}_{r,t} = \mathfrak{q}^{g',g}_{r,r^D_1} \big(\mathfrak{q}^{g',g}_{r^D_1, r^D_2} \cdots \mathfrak{q}^{g',g}_{t^D_{-1}, t^D_0}\big) \mathfrak{q}^{g',g}_{t^D_0, t}$$

which equals the RHS of (3.3) since g' and g are constant on each interval of the form $[s^D_k, s^D_{k+1}[$, as well as the intervals $[r, r^D_1[$ and $[t^D_0, t[$. It follows that $\mathfrak{q}^{g',g}$ is a piecewise semigroup evolution, with associated time point and semigroup set as claimed.

Suppose conversely that (ii) holds. Then, for all $g', g \in \mathbb{S}_{\mathrm{loc}}$, $\mathfrak{q}_0^{g',g} = 1_{\mathcal{A}}$ and

$$\mathfrak{q}_{s+t}^{g',g} = \mathfrak{q}_{0,s+t}^{g',g} = \mathfrak{q}_{0,s}^{g',g}\,\mathfrak{q}_{s,s+t}^{g',g} = \mathfrak{q}_s^{g',g}\,\mathfrak{q}_t^{L_s g', L_s g} \qquad (s,t \in \mathbb{R}_+),$$

so (i) holds. □

For a cocycle $\mathfrak{q} \in SL\mathbb{SC}(\mathcal{A}, \mathsf{k})$, we refer to $\{\mathfrak{q}^{c',c} : c', c \in \mathsf{k}\}$ as the family of *associated semigroups of* $\mathfrak{q}$.

Remark. Clearly, $\mathfrak{q}$ is Markov regular if and only if each of its associated semigroups is norm continuous.

Theorem 6.2.

(a) *Let $\mathfrak{q} \in SL\mathbb{SC}_{\mathrm{c}}(\mathcal{A}, \mathsf{k})$, and let $\{\beta_{c',c} : c', c \in \mathsf{k}\}$ be its associated semigroup generators. Then, for all $g', g \in \mathbb{S}_{\mathrm{loc}}$,*

$$\mathfrak{q}^{g',g} = F^{\widetilde{a}} \quad \textit{where} \quad \widetilde{a}(t) = \beta_{g'(t),g(t)} \quad (t \in \mathbb{R}_+).$$

(b) *Let $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$. Then $\mathfrak{q}^\nu \in SL\mathbb{SC}_{\mathrm{c}}(\mathcal{A}, \mathsf{k})$ and its associated semigroup generators are given by*

$$\beta_{c',c} = \nu(\widehat{c'}, \widehat{c}) + \langle c', c\rangle 1_{\mathcal{A}} \qquad (c', c \in \mathsf{k}). \tag{6.3}$$

Proof. (a) This follows from Propositions 6.1 and 3.2.

(b) This follows from Lemma 4.4. □

Remark. By Lemma 1.1 and identity (6.3), the sesquilinear map ν is expressible in terms of the associated semigroup generators $\{\beta_{c',c} : c', c \in \mathsf{k}\}$ of the stochastic cocycle $\mathfrak{q}^\nu$ as follows

$$\nu\left(\begin{pmatrix} z' \\ c' \end{pmatrix}, \begin{pmatrix} z \\ c \end{pmatrix}\right) = \beta_{c',c} - \langle c', c\rangle 1_{\mathcal{A}} + \overline{z'-1}\beta_{0,c} + (z-1)\beta_{c',0} + \overline{z'-1}(z-1)\beta_{0,0}. \tag{6.4}$$

The affine relations enjoyed by the associated semigroup generators read as follows:

$$\beta_{c',c+\lambda d} = \beta_{c',c} + \lambda\beta_{c',d} - \lambda\beta_{c',0}, \quad \text{and} \quad \beta_{c'+\lambda d',c} = \beta_{c',c} + \overline{\lambda}\beta_{d',c} - \overline{\lambda}\beta_{0,c}. \tag{6.5}$$

Sufficient conditions for a cocycle to be governed by a QDSE are given in the next result. We write B_{conj} to denote bounded conjugate-linear.

Theorem 6.3. *Let $\mathfrak{q} \in SL\mathbb{SC}_{\mathrm{c}}(\mathcal{A}, \mathsf{k})$.*

(a) *Suppose that there are separating families of maps $(\varphi_i \in B(\mathcal{A}; \mathfrak{X}_i))_{i \in \mathcal{I}}$ and $(\varphi'_{i'} \in B(\mathcal{A}; \mathfrak{X}'_{i'}))_{i' \in \mathcal{I}'}$ for Banach spaces $\mathfrak{X}_i$ and $\mathfrak{X}'_{i'}$ such that, for all $\varepsilon', \varepsilon \in \mathcal{E}$, $t \in \mathbb{R}_+$, $i \in \mathcal{I}$ and $i' \in \mathcal{I}'$,*

(i) *$\varphi_i \circ \mathfrak{q}_t(\varepsilon', \cdot) \in B(\mathcal{E}; \mathfrak{X}_i)$ and $\varphi'_{i'} \circ \mathfrak{q}_t(\cdot, \varepsilon) \in B_{\mathrm{conj}}(\mathcal{E}; \mathfrak{X}'_{i'})$;*

(ii) *the maps $s \mapsto \varphi_i \circ \mathfrak{q}_s(\varepsilon', \cdot)$ and $s \mapsto \varphi'_{i'} \circ \mathfrak{q}_s(\cdot, \varepsilon)$ are continuous at 0.*

Then $\mathfrak{q} = \mathfrak{q}^\nu$ for a unique map $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$.

(b) *Suppose that* (ii) *is strengthened to the following*:

(ii)′ *$s \mapsto \varphi_i \circ \mathfrak{q}_s(\varepsilon', \cdot)$ and $s \mapsto \varphi'_{i'} \circ \mathfrak{q}_s(\cdot, \varepsilon)$ are Hölder $\frac{1}{2}$ continuous at 0.*

Then, ν enjoys the following weak boundedness properties: *for all $i \in \mathcal{I}$, $i' \in \mathcal{I}'$ and $\zeta, \zeta' \in \widehat{\mathsf{k}}$,*

$$\varphi_i \circ \nu(\zeta', \cdot) \in B(\widehat{\mathsf{k}}; \mathfrak{X}_i) \quad \textit{and} \quad \varphi_{i'} \circ \nu(\cdot, \zeta) \in B_{\mathrm{conj}}(\widehat{\mathsf{k}}; \mathfrak{X}'_{i'}).$$

Proof. (a) Let $\{\beta_{c',c} : c', c \in \mathsf{k}\}$ be the associated semigroup generators of $\mathfrak{q}$. In view of Theorem 6.2, and the remarks that follow it, if there is such a map $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ then it must be given by (6.4). It therefore suffices to show that the map $\nu : \widehat{\mathsf{k}} \times \widehat{\mathsf{k}} \to \mathcal{A}$ *defined by* (6.4) is sesquilinear. By Lemma 1.1 this is equivalent to showing that $\beta_{c',c}$ is complex affine linear in c and conjugate affine linear in c'. Let $t \in \mathbb{R}_+$, $c', c, d \in \mathsf{k}$ and $\lambda \in \mathbb{C}$, set

$$\zeta'_t := \varepsilon(c'_{[0,t[}) \text{ and } \eta_t := \varepsilon\big((1-\lambda)c_{[0,t[} + \lambda d_{[0,t[}\big) - (1-\lambda)\varepsilon(c_{[0,t[}) - \lambda\varepsilon(d_{[0,t[}).$$

Then η_t has no zero or one particle term and so is $O(t)$ as $t \to 0$, and

$$\beta_{c',(1-\lambda)c+\lambda d} - (1-\lambda)\beta_{c',c} - \lambda\beta_{c',d} = \lim_{t\to 0^+} t^{-1}\mathfrak{q}_t(\zeta'_t, \eta_t).$$

As $\eta_t \perp \varepsilon(0)$,

$$\mathfrak{q}_t(\zeta'_t, \eta_t) = (\mathfrak{q}_t - \mathfrak{q}_0)(\zeta'_t, \eta_t) + \langle \zeta'_t - \varepsilon(0), \eta_t\rangle 1_{\mathcal{A}} \qquad (t \in [0,1[).$$

Thus, for all $i \in \mathcal{I}$ and $T \geqslant t > 0$,

$$\big\| t^{-1}(\varphi_i \circ \mathfrak{q}_t)(\zeta'_t, \eta_t)\big\| \leqslant \big\|\varphi_i \circ (\mathfrak{q}_t - \mathfrak{q}_0)(\zeta'_T, \cdot)\big\| t^{-1}\|\eta_t\| + \|\zeta'_t - \varepsilon(0)\| t^{-1}\|\eta_t\|\|\varphi_i\|.$$

Since the family $\big(\varphi_i\big)_{i\in\mathcal{I}}$ is separating, it follows that $\beta_{c',c}$ is complex affine linear in c. By a very similar argument it follows that it is also conjugate affine linear in c', as required.

(b) Now suppose that (ii)$'$ holds and let $i \in \mathcal{I}$. Let $c' \in \mathsf{k}$ and set $\omega := \binom{1}{0} \in \widehat{\mathsf{k}}$ and $C = \big(C_1^2 + C_2^2\big)^{1/2}$, where

$$C_1 := \big\|\varphi_i(\nu(\widehat{c'}, \omega))\big\|_{\mathfrak{X}_i} \text{ and } C_2 := \sup_{t\in]0,1]} t^{-1/2}\big\|\varphi_i \circ (\mathfrak{q}_t - \mathfrak{q}_0)\big(\varepsilon(c'_{[0,1[}), \cdot\big)\big\|_{B(\mathcal{E};\mathfrak{X}_i)}.$$

Then, for $\zeta = \binom{z}{c} \in \widehat{\mathsf{k}}$,

$$\nu(\widehat{c'}, \zeta) = \nu(\widehat{c'}, \widehat{c}) + (z-1)\nu(\widehat{c'}, \omega) = z\nu(\widehat{c'}, \omega) + \big(\nu(\widehat{c'}, \widehat{c}) - \nu(\widehat{c'}, \omega)\big)$$

and (by adaptedness)

$$\varphi_i\big(\nu(\widehat{c'}, \widehat{c}) - \nu(\widehat{c'}, \omega)\big) = \lim_{t\to 0} t^{-1}\varphi_i \circ (\mathfrak{q}_t - \mathfrak{q}_0)\big(\varepsilon(c'_{[0,1[}), \varepsilon(c_{[0,t[}) - \varepsilon(0)\big).$$

Thus, since $t^{-1/2}\|\varepsilon(c_{[0,t[}) - \varepsilon(0)\| \to \|c\|$ as $t \to 0$,

$$\big\|(\varphi_i \circ \nu)(\widehat{c'}, \zeta)\big\| \leqslant C_1|z| + C_2\|c\| \leqslant C\|\zeta\|.$$

It follows that $(\varphi_i \circ \nu)(\zeta', \cdot)$ is bounded for each ζ' of the form $\widehat{c'}$. Therefore, by linearity, $(\varphi_i \circ \nu)(\zeta', \cdot)$ is bounded for all $i \in \mathcal{I}$ and $\zeta' \in \widehat{\mathsf{k}}$. Similarly, $(\varphi'_{i'} \circ \nu)(\cdot, \zeta)$ is bounded for each $i' \in \mathcal{I}'$ and $\zeta \in \widehat{\mathsf{k}}$. $\square$

Corollary 6.4. *Let $\mathfrak{q} \in SL\mathbb{SC}_{\mathrm{c}}(\mathcal{A}, \mathsf{k})$. Suppose that for all $t \in \mathbb{R}_+$ and $\varepsilon', \varepsilon \in \mathcal{E}$,*

(a) *$\mathfrak{q}_t(\varepsilon', \cdot) \in B(\mathcal{E}; \mathcal{A})$ and $\mathfrak{q}_t(\cdot, \varepsilon) \in B_{\mathrm{conj}}(\mathcal{E}; \mathcal{A})$;*
(b) *the resulting maps $s \mapsto \mathfrak{q}_s(\varepsilon', \cdot)$ and $s \mapsto \mathfrak{q}_s(\cdot, \varepsilon)$ are Hölder $\frac{1}{2}$ continuous at 0.*

Then, $\mathfrak{q} = \mathfrak{q}^\nu$ for a unique map $\nu \in BSL(\widehat{\mathsf{k}}; \mathcal{A})$.

Proof. The hypotheses of Theorem 6.3 hold, in their strengthened form (ii)$'$, with $\mathcal{I} = \mathcal{I}'$ being a singleton set, $\mathfrak{X} = \mathfrak{X}' = \mathcal{A}$ and $\varphi = \varphi' = \mathrm{id}_{\mathcal{A}}$. Thus $\mathfrak{q} = \mathfrak{q}^\nu$ for a unique map $\nu \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ and ν is separately continuous. It follows from the Banach-Steinhaus Theorem that ν is jointly continuous, and thus bounded. $\square$

From these results we obtain cocycle characterisations of solutions of quantum stochastic differential equations on operator spaces, refining results in Section 5 of [LS$_1$].

Theorem 6.5. *Let k be an adjointable quantum stochastic cocycle on an operator space V in $B(\mathsf{h}; \mathsf{h}')$ which is Markov regular (respectively, cb-Markov regular).*

(a) *Let k satisfy the following: for all $x \in \mathsf{V}$, $u \in \mathsf{h}$, $u' \in \mathsf{h}'$ and $\varepsilon', \varepsilon \in \mathcal{E}$,*

(i) *the functions $s \mapsto E^{u'}k_s(x)u\varepsilon$ and $s \mapsto E^u k_s^\dagger(x^*)u'\varepsilon'$ are continuous at 0.*

Then there is a map $\nu \in SL(\widehat{\mathsf{k}}; B(\mathsf{V}))$ (*resp.* $\nu \in SL(\widehat{\mathsf{k}}; CB(\mathsf{V}))$) *such that* k *satisfies the weak quantum stochastic differential equation* (2.7).

Suppose further that $\dim \mathsf{k} < \infty$. *Then* ν *is the sesquilinear map associated with a map* $\phi \in B\big(\mathsf{V}; V \otimes_{\mathsf{M}} B(\widehat{\mathsf{k}})\big)$ (*resp.* $\phi \in CB\big(\mathsf{V}; V \otimes_{\mathsf{M}} B(\widehat{\mathsf{k}})\big)$), *and* k *strongly satisfies the quantum stochastic differential equation* (2.9).

(b) *Let* (i) *be strengthened as follows*:

(i)$'$ k *and* $k^\dagger$ *are both pointwise strongly Hölder* $\frac{1}{2}$ *continuous* (*on their exponential domains*),

and let $\nu \in SL(\widehat{\mathsf{k}}; B(\mathsf{V}))$ *be the resulting sesquilinear map. Then, for all* $x \in \mathsf{V}$, $u \in \mathsf{h}$, $u' \in \mathsf{h}'$ *and* $\zeta, \zeta' \in \widehat{\mathsf{k}}$,

$$\nu(\cdot, \zeta)(x)u \in B_{\mathrm{conj}}(\widehat{\mathsf{k}}; \mathsf{h}') \quad \textit{and} \quad \langle u'|\nu(\zeta', \cdot)(x) \in B(\widehat{\mathsf{k}}; \langle \mathsf{h}|).$$

Suppose further that $\dim \mathsf{h}, \dim \mathsf{h}' < \infty$. *Then* ν *is the sesquilinear map associated with a map* $\phi \in L\big(\widehat{\mathsf{k}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$, *and* $k = k^\phi$.

(c) *Let* (i) *be further strengthened as follows*: *for all* $x \in \mathsf{V}$ *and* $\varepsilon', \varepsilon \in \mathcal{E}$,

(i)$''$ $s \mapsto k_{s,\varepsilon}(x)$ *and* $s \mapsto k^\dagger_{s,\varepsilon}(x^*)$ *are Hölder* $\frac{1}{2}$ *continuous* $\mathbb{R}_+ \to \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}\rangle$, *respectively* $\mathbb{R}_+ \to \mathsf{V}^\dagger \otimes_{\mathsf{M}} |\mathcal{F}\rangle$,

and let ν *be the resulting sesquilinear map. Then, for all* $x \in \mathsf{V}$, $\nu(\cdot,\cdot)(x) \in BSL(\widehat{\mathsf{k}}; \mathsf{V})$.

Suppose further that $\dim \mathsf{V} < \infty$. *Then* ν *is the sesquilinear map associated with a map* $\phi \in L\big(\widehat{\mathsf{k}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$, *and* $k = k^\phi$.

Proof. Let $\mathsf{q} \in SL\mathbb{P}_{\mathrm{c}}(\mathcal{A}, \mathsf{k})$ be the corresponding sesquilinear process, with $\mathcal{A} = B(\mathsf{V})$ (resp. $CB(\mathsf{V})$).

(a) Part (a) of Theorem 6.3 applies with $\mathcal{I}' = \mathcal{I} = \mathsf{h}' \times \mathsf{V} \times \mathsf{h}$, $\mathfrak{X}' = \mathfrak{X} = \mathbb{C}$ and $\varphi_{u',x,u} = \varphi'_{u',x,u} : \kappa \mapsto \langle u', \kappa(x)u\rangle$. If $\dim \mathsf{k} < \infty$ then the required map ϕ is defined via the prescription

$$\phi(x)u \otimes \zeta = \sum_\alpha E_{e_\alpha} \nu(e_\alpha, \zeta)(x)u, \tag{6.6}$$

where (e_α) is an arbitrary orthonormal basis of $\widehat{\mathsf{k}}$. The boundedness (resp. complete boundedness) of ϕ is easily verified.

(b) Part (b) of Theorem 6.3 applies, with $\mathcal{I} = \mathsf{V} \times \mathsf{h}$, $\mathfrak{X} = |\mathsf{h}'\rangle$ and $\phi_{x,u} : \kappa \mapsto \kappa(x)|u\rangle$; $\mathcal{I}' = \mathsf{V} \times \mathsf{h}'$, $\mathfrak{X}' = \langle \mathsf{h}|$ and $\phi'_{x,u'} : \kappa \mapsto \langle u'|\kappa(x)$. This gives separate continuity for each map $\nu(\cdot,\cdot)(x) \in BSL(\widehat{\mathsf{k}}; \mathsf{V})$ ($x \in \mathsf{V}$). Their joint continuity again follows from the Banach-Steinhaus Theorem.

If $\dim \mathsf{h}, \dim \mathsf{h}' < \infty$ then there are linear isomorphisms

$$B_{\mathrm{conj}}(\widehat{\mathsf{k}}; \mathsf{h}') \cong \mathsf{h}' \otimes \widehat{\mathsf{k}} \ \text{ and } \ B(\widehat{\mathsf{k}}; \langle \mathsf{h}|) \cong \langle \mathsf{h} \otimes \widehat{\mathsf{k}}|,$$

and the formula (6.6) again defines a linear map ϕ associated with the sesquilinear map ν, moreover by the finite dimensionality of h, $\phi_\zeta(x)$ is bounded for each $x \in \mathsf{V}$ and by the finite dimensionality of V, ϕ_ζ is completely bounded ($\zeta \in \widehat{\mathsf{k}}$). Thus $\phi \in L\big(\widehat{\mathsf{k}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle)\big)$ and, by Theorem 2.1, ϕ generates a quantum stochastic cocycle k^ϕ. Therefore, by uniqueness in Theorem 5.1, $k^\phi = k$.

(c) Part (b) of Theorem 6.3 applies, with $\mathcal{I}' = \mathcal{I} = \mathsf{V}$, $\mathfrak{X}' = \mathfrak{X} = \mathcal{A}$ and both φ'_x and φ_x being evaluation at x. If $\dim \mathsf{V} < \infty$ then there are linear isomorphisms

$$B_{\mathrm{conj}}(\widehat{\mathsf{k}}; \mathsf{V}) \cong CB(\langle \widehat{\mathsf{k}}|; \mathsf{V}) \cong \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}}\rangle, \quad \text{and} \quad B(\widehat{\mathsf{k}}; \mathsf{V}) \cong CB(|\widehat{\mathsf{k}}\rangle; \mathsf{V}) \cong \mathsf{V} \otimes_{\mathsf{M}} \langle \widehat{\mathsf{k}}|,$$

and again (6.6) defines a linear map ϕ associated with ν, and the finite dimensionality of V ensures that ϕ_ζ is completely bounded ($\zeta \in \widehat{\mathsf{k}}$), so the argument of (b) applies. $\square$

Remarks. There is a subtle difference between Parts (b) and (c). As noted in [LS$_1$], finite dimensionality of V does not ensure that V is concretely realisable in a finite dimensional full operator space (see [Pis]).

The full conclusion of Part (c), without the finite-dimensional restriction, which is established in [L$_2$], is recovered and extended to cocycles *in* an operator space, by working in a different category ([DL$_2$]).

In [DL$_2$] we also give a corresponding characterisation of convolution cocycles in an operator space coalgebra.

7. Stochastic Lie–Trotter product formulae

For this section we consider an orthogonal decomposition $\mathsf{k}_1 \oplus \mathsf{k}_2$ of the noise dimension space k, with corresponding tensor decomposition $\mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2$, and prove Lie–Trotter type product formulae for sesquilinear cocycles. This entails product formulae for our three paradigm examples of Markov-regular quantum stochastic cocycle.

Let $\nu_i \in SL(\widehat{\mathsf{k}}_i; \mathcal{A})$ for $i = 1, 2$. The map

$$\mathsf{k} \times \mathsf{k} \to \mathcal{A},\ (c, d) \mapsto \nu_1(\widehat{c^1}, \widehat{d^1}) +\ \nu_2(\widehat{c^2}, \widehat{d^2}) \text{ where } c = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} \text{ and } d = \begin{pmatrix} d^1 \\ d^2 \end{pmatrix},$$

is easily verified to be affine sesquilinear and so, by Lemma 1.1, there is a unique map $\nu_1 \boxplus \nu_2 \in SL(\widehat{\mathsf{k}}; \mathcal{A})$ such that

$$(\nu_1 \boxplus \nu_2)(\widehat{c}, \widehat{d}) = \nu_1(\widehat{c^1}, \widehat{d^1}) + \nu_2(\widehat{c^2}, \widehat{d^2}) \qquad (c, d \in \mathsf{k}). \tag{7.1}$$

The composition $\boxplus$ is the sesquilinear version of the *concatenation product* of quantum stochastic control theory ([GoJ]). The relationship between the generated cocycles $\mathfrak{q}^{\nu_1}$, $\mathfrak{q}^{\nu_2}$ and $\mathfrak{q}^{\nu}$, is given by a stochastic Lie–Trotter product formula. We first establish this formula under more general conditions.

Recall the notation $\mathfrak{q}^{g',g}_{r,t}$ introduced in (6.1), and the notation (3.5) for D-fold product functions.

Definition. Let ${}^i\mathfrak{q} \in SL\mathbb{SC}(\mathcal{A}, \mathsf{k}_i)$, for $i = 1, 2$, and let $D \subset\subset]0, \infty[$. The *stochastic Lie–Trotter product of* ${}^1\mathfrak{q}$ *and* ${}^2\mathfrak{q}$ *determined by* D is the 2-parameter family $\{{}^{1,2}\mathfrak{q}^D_{r,t} : (r, t) \in \Delta^{[2]}\}$ in $SL(\mathcal{E}; \mathcal{A})$, given by bi-adapted sesquilinear extension of the prescription

$${}^{1,2}\mathfrak{q}^D_{r,t}\big(\varepsilon(f_{[r,t[}), \varepsilon(g_{[r,t[})\big) := G^D_{r,t}, \quad \text{where} \quad G_{u,v} := \big({}^1\mathfrak{q}^{f^1,g^1}_{u,v}\big)\big({}^2\mathfrak{q}^{f^2,g^2}_{u,v}\big) \quad ((u, v) \in \Delta^{[2]})$$

for $f = \binom{f^1}{f^2}$ and $g = \binom{g^1}{g^2} \in \mathbb{S}_{\text{loc}}$.

Remark. Thus $\big({}^{1,2}\mathfrak{q}^D_{0,t}\big)_{t \geqslant 0} \in SL\mathbb{P}(\mathcal{A}, \mathsf{k})$, but in general stochastic Trotter products are not cocycles.

Theorem 7.1. *Let* ${}^1\mathfrak{q}$, ${}^2\mathfrak{q}$ *and* $\mathfrak{q}$ *be Markov-regular sesquilinear stochastic cocycles in* $\mathcal{A}$ *with respective noise dimension spaces* k_1, k_2 *and* k, *let* $\big(D(n)\big)_{n\geqslant 1}$ *be a sequence in* $\Gamma_{]0,\infty[}$ *converging to* $\mathbb{R}_+$ *in the sense of* (3.6), *and suppose that the associated semigroup generators of the cocycles are related by*

$$\beta^1_{c^1,d^1} + \beta^2_{c^2,d^2} = \beta_{c,d} \qquad (c, d \in \mathsf{k}). \tag{7.2}$$

Then

$$\sup_{[r,t]\subset[0,T]} \big\|{}^{1,2}\mathfrak{q}^{D(n)}_{r,t}(\varepsilon', \varepsilon) - \mathfrak{q}_{r,t}(\varepsilon', \varepsilon)\big\| \to 0 \quad as \quad n \to 0 \qquad (T \in \mathbb{R}_+,\ \varepsilon', \varepsilon \in \mathcal{E}).$$

Proof. Let $f, g \in \mathbb{S}_{\rm loc}$. By Theorem 6.2,

$$ {}^i\mathsf{q}^{f^i,g^i} = F^{\widetilde{a_i}} \quad \text{where} \quad \widetilde{a_i}(t) := \beta^i_{f^i(t),g^i(t)} \qquad (i = 1, 2,\ t \in \mathbb{R}_+). $$

By assumption, $\widetilde{a_1} + \widetilde{a_2} = \widetilde{a}$ where $\widetilde{a}(t) := \beta_{f(t),g(t)}$ $(t \in \mathbb{R}_+)$. Therefore, applying Theorem 6.2 again, $\mathsf{q}^{f,g} = F^{\widetilde{a}}$. The result therefore follows from the Lie–Trotter product formula of Theorem 3.3, by bi-adapted sesquilinear extension. □

For subsets S_1 of k_1 and S_2 of k_2, we set

$$ S_1 \boxplus S_2 = \Big\{ \begin{pmatrix} c \\ 0 \end{pmatrix} : c \in S_1 \Big\} \cup \Big\{ \begin{pmatrix} 0 \\ c \end{pmatrix} : c \in S_2 \Big\}. $$

Remark. If (7.2) holds only for $c \in \mathsf{T}'$ and $d \in \mathsf{T}$ where $\mathsf{T}' = \mathsf{T}'_1 \boxplus \mathsf{T}'_2$, $\mathsf{T} = \mathsf{T}_1 \boxplus \mathsf{T}_2$ and T'_i and T_i are total subsets of k_i containing 0 $(i = 1, 2)$, then the above proof yields the same conclusion for $\varepsilon' \in \mathcal{E}_{\mathsf{T}'}$ and $\varepsilon \in \mathcal{E}_{\mathsf{T}}$.

Corollary 7.2. *Let ${}^i\mathsf{q} = \mathsf{q}^{\nu_i}$ for $\nu_i \in SL(\widehat{\mathsf{k}_i}; \mathcal{A})$ $(i = 1, 2)$, and let $\big(D(n)\big)_{n\geqslant 1}$ be a sequence in $\Gamma_{]0,\infty[}$ converging to $\mathbb{R}_+$. Then, for all $T \in \mathbb{R}_+$ and $\varepsilon', \varepsilon \in \mathcal{E}$,*

$$ \sup_{[r,t]\subset[0,T]} \big\| {}^{1,2}\mathsf{q}^{D(n)}_{r,t}(\varepsilon', \varepsilon) - \mathsf{q}^{\nu_1 \boxplus \nu_2}_{r,t}(\varepsilon', \varepsilon) \big\| \to 0 \ \text{ as } n \to \infty. $$

Proof. The identity (7.2) for the associated semigroup generators of q^{ν_1}, q^{ν_2} and $\mathsf{q} := \mathsf{q}^{\nu_1 \boxplus \nu_2}$ follows from Part (b) of Theorem 6.2, and therefore the theorem applies. □

Remark. In view of the remark following Theorem 3.3, the above Theorem and Corollary remain true if $\big({}^{1,2}\mathsf{q}^D_{r,t}\big)_{0\leq r\leq t}$ is modified by ${}^{1,2}\mathsf{q}^D_{r,t}\big(\varepsilon(f_{[r,t[}), \varepsilon(g_{[r,t[})\big)$ taking its old value multiplied by $\exp\langle f_J, g_J\rangle$ where $J = [r, r^D_1[\cup [t^D_0, t[$.

We now deduce stochastic Trotter product formulae for mapping, operator and convolution cocycle settings. To this end, we fix a total subset T_i of k_i containing 0, for $i = 1, 2$, and set

$$ \mathsf{T} = \mathsf{T}_1 \boxplus \mathsf{T}_2 \text{ and } \mathsf{D} = \mathsf{D}_1 \oplus \mathsf{D}_2, \text{ where } \mathsf{D}_1 = \operatorname{Lin} \mathsf{T}_1 \text{ and } \mathsf{D}_2 = \operatorname{Lin} \mathsf{T}_2; $$

thus $\mathsf{D} = \operatorname{Lin} \mathsf{T}$, and T is total in k and contains 0.

First let us fix a concrete operator space V. Recall the extended composition described in (1.3), and notions of cb column-bounded processes and cocycles from Section 2, in particular the notation (2.6).

Definition. For $i = 1, 2$, let ${}^ik \in \mathbb{QSC}_{\rm cbCol}(\mathsf{V} : \mathcal{E}_{\mathsf{T}_i})$. First set

$$ {}^{1,2}k^{g,D}_{[r,t[} := \iota^0_{[r,r^D_1[} \bullet \Big({}^{1,2}k^g_{[r^D_1, r^D_2[} \bullet \cdots \bullet {}^{1,2}k^g_{[t^D_{-1}, t^D_0[} \Big) \bullet \iota^0_{[t^D_0, t[} \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}_{[r,t[}\rangle\big), $$

for $D \subset\subset]0, \infty[$, $g \in \mathbb{S}_{\mathsf{T}}$ and $(r, t) \in \Delta^{[2]}$, where

$$ \iota^0_{[u,v[},\ {}^{1,2}k^g_{[u,v[} \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}_{[u,v[}\rangle\big), \text{ with} $$
$$ {}^{1,2}k^g_{[u,v[} := {}^1k^{g^1}_{[u,v[} \bullet {}^2k^{g^2}_{[u,v[} \text{ and } \iota^0_{[u,v[} : x \mapsto x \otimes |\Omega_{[u,v[}\rangle, $$

and we are making the identifications

$$ |\mathcal{F}_{[u,v[}\rangle \otimes_{\mathsf{M}} |\mathcal{F}_{[v,w[}\rangle = |\mathcal{F}_{[u,w[}\rangle \ \text{ and } \ |\mathcal{F}^1_{[u,v[}\rangle \otimes_{\mathsf{M}} |\mathcal{F}^2_{[u,v[}\rangle = |\mathcal{F}_{[u,v[}\rangle \quad (0 \leqslant u \leqslant v \leqslant w). $$

The *stochastic Trotter product of ${}^1k$ and ${}^2k$ determined by $D \subset\subset]0, \infty[$* is the two-parameter family $\big({}^{1,2}k^D_{r,t}\big)_{0\leqslant r\leqslant t}$ in $L\big(\mathcal{E}_{\mathsf{T}}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}\rangle)\big)$ given by bi-adapted linear extension of the prescription $\varepsilon(g_{[r,t[}) \mapsto {}^{1,2}k^{g,D}_{[r,t[}$, as in (2.2). Thus, $\big({}^{1,2}k^D_{0,t}\big)_{t\geqslant 0} \in \mathbb{P}_{\rm cbCol}(\mathsf{V} : \mathcal{E}_{\mathsf{T}})$, but it will not in general be a stochastic cocycle.

If ${}^1k$ and ${}^2k$ are cb cocycles then their stochastic Trotter product determined by D is the family $\big({}^{1,2}k^D_{r,t}\big)_{0\leqslant r\leqslant t}$ in $CB\big(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}} B(\mathcal{F})\big)$ determined by

$${}^{1,2}k^D_{[r,t[} := \iota_{[r,r^D_1[}\bullet\Big({}^{1,2}k_{[r^D_1,r^D_2[}\bullet\cdots\bullet {}^{1,2}k_{[t^D_{-1},t^D_0[}\Big)\bullet\iota_{[t^D_0,t[}\in CB\big(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}} B(\mathcal{F}_{[r,t[})\big),$$

where

$$\iota_{[u,v[},\ {}^{1,2}k_{[u,v[}\in CB\big(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}} B(\mathcal{F}_{[u,v[})\big),\ \text{with}$$
$${}^{1,2}k_{[u,v[} := {}^1k_{[u,v[}\bullet {}^2k_{[u,v[}\ \text{and}\ \iota_{[u,v[}: x\mapsto x\otimes|\Omega_{[u,v[}\rangle\langle\Omega_{[u,v[}|$$

and we are making the identifications

$$B(\mathcal{F}_{[r,s[})\otimes_{\mathsf{M}} B(\mathcal{F}_{[s,t[}) = B(\mathcal{F}_{[r,t[})\quad\text{and}\quad B(\mathcal{F}^1_{[s,t[})\otimes_{\mathsf{M}} B(\mathcal{F}^2_{[s,t[}) = B(\mathcal{F}_{[s,t[}).$$

In this case, bi-adapted extension reads as follows:

$${}^{1,2}k^D_{r,t}(x) = \Sigma\big(I_{[0,r[}\otimes {}^{1,2}k^D_{[r,t[}(x)\otimes I_{[t,\infty[}\big),$$

where Σ is the tensor flip $B(\mathcal{F}_{[0,r[})_{\mathsf{M}}\otimes\mathsf{V}\otimes_{\mathsf{M}} B(\mathcal{F}_{[r,t[})\otimes_{\mathsf{M}} B(\mathcal{F}_{[t,\infty[})\to\mathsf{V}\otimes_{\mathsf{M}} B(\mathcal{F})$; $\big({}^{1,2}k^D_{0,t}\big)_{t\geqslant 0}$ is then a completely bounded process on V.

Theorem 7.3. *Let ${}^ik\in\mathbb{QSC}_{\mathrm{cbCol}}(\mathsf{V}:\mathcal{E}_{\mathsf{T}_i})$ $(i=1,2)$ and $k\in\mathbb{QSC}_{\mathrm{cbCol}}(\mathsf{V}:\mathcal{E}_{\mathsf{T}})$ be cb Markov regular. Suppose that their associated semigroup generators are related by $\phi_{c,d}=\phi^1_{c^1,d^1}+\phi^2_{c^2,d^2}$ $(c\in\mathsf{k},d\in\mathsf{T})$, let $\big(D(n)\big)_{n\geqslant 1}$ be a sequence in $\Gamma_{]0,\infty[}$ converging to $\mathbb{R}_+$, and let $T\in\mathbb{R}_+$. Then*

$$\sup_{[r,t]\subset[0,T]}\big\|E^{\varepsilon'}\big({}^{1,2}k^{D(n)}_{r,t,\varepsilon}-k_{r,t,\varepsilon}\big)(\cdot)\big\|_{\mathrm{cb}}\to 0\quad\textit{as}\quad n\to\infty\qquad(\varepsilon'\in\mathcal{E},\varepsilon\in\mathcal{E}_{\mathsf{T}}).$$

If ${}^1k$, ${}^2k$ and k are completely bounded, with locally bounded cb norms, then convergence holds in the stronger sense:

$$\sup_{[r,t]\subset[0,T]}\big\|\big(\mathrm{id}_{\mathsf{V}}\otimes_{\mathsf{M}}\omega\big)\circ\big({}^{1,2}k^{D(n)}_{r,t}-k_{r,t}\big)\big\|_{\mathrm{cb}}\to 0\quad\textit{as}\quad n\to\infty\qquad(\omega\in B(\mathcal{F})_*).$$

*If V is a C^*-algebra, ${}^1k$ and ${}^2k$ are completely positive and contractive and k is *-homomorphic then*

$$\sup_{[r,t]\subset[0,T]}\big\|\big({}^{1,2}k^{D(n)}_{r,t}-k_{r,t}\big)(\cdot)\xi\big\|\to 0\quad\textit{as}\quad n\to\infty\qquad(\xi\in\mathsf{h}\otimes\mathcal{F}).$$

Proof. The first part follows from Theorem 7.1 by setting $\mathcal{A}=CB(\mathsf{V})$ and letting ${}^i\mathsf{q}$ be the sesquilinear process associated with ${}^ik$ $(i=1,2)$. The second part follows from the first part and the totality of the set $\{\omega_{\varepsilon',\varepsilon}:\varepsilon'\in\mathcal{E},\varepsilon\in\mathcal{E}_{\mathsf{T}}\}$ in $B(\mathcal{F})_*$. The last part follows from the second part, by the operator Schwarz inequality, since each ${}^{1,2}k^D_{r,t}$ is then a composition of completely positive contractions. □

Remark. It follows from the remark after Corollary 7.2 that the above Theorem remains true if, in the definition of ${}^{1,2}k^{g,D}_{[r,t[}$ and ${}^{1,2}k^D_{[r,t[}$ the maps $\iota^0_{[u,v[}$ and $\iota_{[u,v[}$ are replaced by the maps $x\mapsto x\otimes|\varepsilon(g|_{[u,v[})\rangle$ and $x\mapsto x\otimes I_{[u,v[})$ respectively.

For $i=1,2$, let $\phi^i\in L\big(\widehat{\mathsf{D}}_i;CB(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}}|\widehat{\mathsf{k}}_i\rangle)\big)$. Their concatenation product $\phi^1\boxplus\phi^2\in L\big(\widehat{\mathsf{D}};CB(\mathsf{V};\mathsf{V}\otimes_{\mathsf{M}}|\widehat{\mathsf{k}}\rangle)\big)$, is defined by

$$(\phi^1\boxplus\phi^2)_{\widehat{c}}(x) := \begin{pmatrix}\phi^1_{\widehat{c^1}}(x)\\ 0\end{pmatrix}+\Sigma\begin{pmatrix}\phi^2_{\widehat{c^2}}(x)\\ 0\end{pmatrix}\qquad(x\in\mathsf{V},c\in\mathsf{D}),$$

where Σ is the sum-flip $\widehat{\mathsf{k}}_2\oplus\mathsf{k}_1\to\widehat{\mathsf{k}}_1\oplus\mathsf{k}_2=\widehat{\mathsf{k}}$. (This corresponds to (7.1).) Thus

$$E^{\widehat{c}}(\phi^1\boxplus\phi^2)_{\widehat{d}}(\cdot) = E^{\widehat{c^1}}\phi^1_{\widehat{d^1}}(\cdot)+E^{\widehat{c^2}}\phi^2_{\widehat{d^2}}(\cdot)\qquad(c\in\mathsf{k},d\in\mathsf{D}).\tag{7.3}$$

Corollary 7.4. *Let* ${}^ik = k^{\phi^i}$ *for* $\phi^i \in L\big(\widehat{\mathsf{D}_i}; CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} |\widehat{\mathsf{k}_i}\rangle)\big)$ $(i = 1, 2)$*, let* $\big(D(n)\big)_{n \geqslant 1}$ *be a sequence in* $\Gamma_{]0,\infty[}$ *converging to* $\mathbb{R}_+$ *and let* $T \in \mathbb{R}_+$*. Then, for all* $\varepsilon' \in \mathcal{E}$ *and* $\varepsilon \in \mathcal{E}_{\mathsf{D}_1 \oplus \mathsf{D}_2}$*,*

$$\sup_{[r,t] \subset [0,T]} \big\| E^{\varepsilon'}\big({}^{1,2}k^{D(n)}_{r,t,\varepsilon} - k^{\phi^1 \boxplus \phi^2}_{r,t,\varepsilon}\big)(\cdot)\big\|_{\mathrm{cb}} \to 0 \quad as \quad n \to \infty.$$

If k^{ϕ^1}*,* k^{ϕ^2} *and* $k^{\phi^1 \boxplus \phi^2}$ *are completely bounded, with locally bounded cb norms, then, for all* $\omega \in B(\mathcal{F})_*$*,*

$$\sup_{[r,t] \subset [0,T]} \big\| \big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathsf{M}} \omega\big) \circ \big({}^{1,2}k^{D(n)}_{r,t} - k^{\phi^1 \boxplus \phi^2}_{r,t}\big)\big\|_{\mathrm{cb}} \to 0 \quad as \quad n \to \infty.$$

Proof. The stochastic cocycles $k^{\phi^1 \boxplus \phi^2}$, k^{ϕ^1} and k^{ϕ^2} are each cb Markov-regular and cb column-bounded. The identity (7.3) implies that their respective associated semigroup generators are related as required in Theorem 7.3; the result follows. □

Remark. When $\phi^i \in CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} B(\widehat{\mathsf{k}_i}))$ $(i = 1, 2)$, the concatenation product $\phi^1 \boxplus \phi^2$ reads as follows, in terms of block matrices:

$$\begin{bmatrix} \tau_1 & \alpha_1 \\ \chi_1 & \nu_1 \end{bmatrix} \boxplus \begin{bmatrix} \tau_2 & \alpha_2 \\ \chi_2 & \nu_2 \end{bmatrix} := \begin{bmatrix} \tau_1 + \tau_2 & \alpha_1 & \alpha_2 \\ \chi_1 & \nu_1 & 0 \\ \chi_2 & 0 & \nu_2 \end{bmatrix}.$$

Next consider quantum stochastic contraction cocycles on a Hilbert space $\mathfrak{h}$, as in (2.4).

Definition. Let ${}^1V$ and ${}^2V$ be quantum stochastic contraction cocycles on $\mathfrak{h}$ with respective noise dimension spaces k_1 and k_2. Their *stochastic Trotter product determined by* $D \subset\subset]0, \infty[$ is the two-parameter family of contraction operators

$${}^{1,2}V^D_{r,t} := \begin{cases} {}^{1,2}V_{r^D_1, r^D_2} \cdots {}^{1,2}V_{t^D_{-1}, t^D_0} & \text{if } r^D_1 < t^D_0 \\ I_{\mathfrak{h} \otimes \mathcal{F}} & \text{otherwise,} \end{cases}$$

where

$${}^{1,2}V_{u,v} := \big({}^1V_{u,v} \otimes I_{\mathcal{F}^2}\big)\big(\mathrm{id}_{B(\mathfrak{h})} \otimes \Sigma\big)\Big({}^2V_{u,v} \otimes I_{\mathcal{F}^1}\Big) \in B(\mathfrak{h} \otimes \mathcal{F}),$$

in which Σ is the tensor flip $B(\mathcal{F}^2) \overline{\otimes} B(\mathcal{F}^1) \to B(\mathcal{F}^1) \overline{\otimes} B(\mathcal{F}^2) = B(\mathcal{F})$, and, for $i = 1, 2$, ${}^iV_{u,v} := (\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes} \sigma^{\mathsf{k}_i}_u)({}^iV_{v-u}) \in B(\mathfrak{h} \otimes \mathcal{F}^i)$, as in (2.3).

An operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ stochastically generates a Markov-regular quantum stochastic cocycle V^F, and V^F is a contraction cocycle if and only if $F \in C_0(\mathfrak{h}, \mathsf{k})$ where

$$C_0(\mathfrak{h}, \mathsf{k}) := \big\{F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) : r(F) \leqslant 0, \text{ equivalently, } r(F^*) \leqslant 0\big\},$$

where $r(F) := F^* + F + F^* \Delta F$, moreover V^F is isometric, respectively coisometric, if and only if $r(F) = 0$, respectively $r(F^*) = 0$ (see [L$_1$]).

For operators $F_1 \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}_1)$ and $F_2 \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}_2)$, their concatenation product $F_1 \boxplus F_2 \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ is given, in terms of block matrices, by

$$\begin{bmatrix} K_1 & M_1 \\ L_1 & N_1 \end{bmatrix} \boxplus \begin{bmatrix} K_2 & M_2 \\ L_2 & N_2 \end{bmatrix} := \begin{bmatrix} K_1 + K_2 & M_1 & M_2 \\ L_1 & N_1 & 0 \\ L_2 & 0 & N_2 \end{bmatrix}.$$

In view of the easily verified identity $r(F_1 \boxplus F_2) = r(F_1) \boxplus r(F_2)$ and (7.3), $V^{F_1 \boxplus F_2}$ is contractive/isometric/coisometric if V^{F_1} and V^{F_2} both are.

Corollary 7.5 ([LiS])**.** *For $i = 1, 2$, let ${}^{(i)}V = V^{F^i}$ where $F^i \in C_0(\mathfrak{h}, \mathsf{k}_i)$, let $\big(D(n)\big)_{n\geqslant 1}$ be a sequence in $\Gamma_{]0,\infty[}$ converging to $\mathbb{R}_+$, and let $T \in \mathbb{R}_+$. Then, for all $\omega \in B(\mathcal{F})_*$,*

$$\sup_{[r,t]\subset[0,T]} \big\|\big(\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes}\, \omega\big) \circ \big({}^{1,2}V^{D(n)}_{r,t} - V^{F_1 \boxplus F_2}_{r,t}\big)\big\| \to 0 \quad as \quad n \to \infty.$$

If the cocycle $V^{F_1\boxplus F_2}$ is isometric then, for all $\xi \in \mathfrak{h} \otimes \mathcal{F}$,

$$\sup_{[r,t]\subset[0,T]} \big\|\big({}^{1,2}V^{D(n)}_{r,t} - V^{F_1\boxplus F_2}_{r,t}\big)\xi\big\| \to 0 \quad as \quad n \to \infty.$$

Proof. In view of the remark after Theorem 7.3, the first part follows from Corollary 7.4, by means of the completely isometric identifications $B(\mathfrak{h}) = CB(\mathsf{V})$, $B(\mathfrak{h} \otimes \mathcal{F}) = CB(\mathsf{V}; \mathsf{V} \otimes_{\mathsf{M}} B(\mathcal{F}))$ and $|\mathfrak{h} \otimes \mathcal{F}\rangle = \mathsf{V} \otimes_{\mathsf{M}} |\mathcal{F}\rangle$, where V is the column space $|\mathfrak{h}\rangle$. The last part follows since weak operator convergence of a sequence of contractions to an isometry implies strong convergence. ☐

Finally, consider quantum stochastic convolution cocycles on a counital operator space coalgebra C ([LS$_2$]). Denote by $\mathbb{P}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_\mathsf{T})$ the set of cb column-bounded quantum stochastic convolution processes l on C with exponential domain $\mathcal{E}_\mathsf{T}$ and by $\mathbb{QSC}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_\mathsf{T})$ the set of convolution cocycles in $\mathbb{P}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_\mathsf{T})$.

For $l \in \mathbb{P}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_\mathsf{T})$ and $g \in \mathbb{S}_\mathsf{T}$, the notation $l^g_{t)} := l_{t)}(\cdot)|\varepsilon(g_{|[0,t[})\rangle \in CB\big(\mathsf{C}; |\mathcal{F}_{[0,t[}\rangle\big)$ extends to shifted intervals by

$$l^g_{[s,t[} := \tau \circ l^{L_s g}_{t-s)} \in CB\big(\mathsf{C}; |\mathcal{F}_{[s,t[}\rangle\big),$$

where τ here denotes the shift $|\mathcal{F}_{[0,t-s[}\rangle$ to $|\mathcal{F}_{[s,t[}\rangle$.

Definition. Let ${}^il \in \mathbb{QSC}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_{\mathsf{T}_i})$ for $i = 1, 2$, put $\mathsf{T} = \mathsf{T}_1 \boxplus \mathsf{T}_2$ and let $D \subset\subset\,]0, \infty[$. First set

$${}^{1,2}l^{g,D}_{[r,t[} := \begin{cases} \epsilon^0_{[r,r^D_1[} \star \Big({}^{1,2}l^g_{[r^D_1,r^D_2[} \star \cdots \star {}^{1,2}l^g_{[t^D_{-1},t^D_0[}\Big) \star \epsilon^0_{[t^D_0,t[} & \text{if } r^D_1 < t^D_0 \\ \epsilon^0_{[r,t[} & \text{otherwise,} \end{cases}$$

where ϵ is the counit, and the convolution is given, in terms of the coproduct, by $(\phi \star \psi)(x) := (\phi \otimes \psi)(\Delta x)$, and

$$\epsilon^0_{[u,v[},\ {}^{1,2}l^g_{[u,v[} \in CB\big(\mathsf{C}; |\mathcal{F}_{[u,v[}\rangle\big), \text{ with}$$
$${}^{1,2}l^g_{[u,v[} := {}^1l^{g^1}_{[u,v[} \star {}^2l^{g^2}_{[u,v[} \text{ and } \epsilon^0_{[u,v[} := |\Omega_{[u,v[}\rangle \circ \epsilon \qquad (g \in \mathbb{S}_\mathsf{T}).$$

Then the *stochastic Trotter product of ${}^1l$ and ${}^2l$ determined by D* is the two-parameter family $\big({}^{1,2}l^D_{r,t)}\big)_{0\leqslant r\leqslant t}$ in $L\big(\mathcal{E}_\mathsf{T}; CB(\mathsf{C}; |\mathcal{F}\rangle)\big)$ defined by bi-adapted linear extension of the map $\varepsilon(g_{[r,t[}) \mapsto {}^{1,2}l^{g,D}_{[r,t[}$. Again, $\big({}^{1,2}l^D_{0,t)}\big)_{t\geqslant 0} \in \mathbb{P}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_\mathsf{T})$, and if ${}^1l$ and ${}^2l$ are completely bounded then ${}^{1,2}l^D_{r,t} \in CB(\mathsf{C}; B(\mathcal{F}))$ $(0 \leqslant r \leqslant t)$ and $\big({}^{1,2}l^D_{0,t)}\big)_{t\geqslant 0}$ is a cb convolution process on C.

Theorem 7.6. *Let ${}^il \in \mathbb{QSC}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_{\mathsf{T}_i})$ $(i = 1, 2)$ and $l \in \mathbb{QSC}^\star_{\mathrm{cbCol}}(\mathsf{C} : \mathcal{E}_\mathsf{T})$ be cb Markov regular. Suppose that their associated convolution semigroup generators are related by $\varphi_{c,d} = \varphi^1_{c^1,d^1} + \varphi^2_{c^2,d^2}$ $(c, d \in \mathsf{T})$, let $\big(D(n)\big)_{n\geqslant 1}$ be a sequence in $\Gamma_{]0,\infty[}$ converging to $\mathbb{R}_+$, and let $T \in \mathbb{R}_+$. Then*

$$\sup_{[r,t]\subset[0,T]} \big\|\langle\varepsilon'|\big({}^{1,2}l^{D(n)}_{r,t,\varepsilon} - l_{r,t,\varepsilon}\big)(\cdot)\big\| \to 0 \quad as \quad n \to \infty \qquad (\varepsilon' \in \mathcal{E}, \varepsilon \in \mathcal{E}_\mathsf{T}).$$

If ${}^1l$, ${}^2l$ and l are completely bounded, with locally bounded cb norms, then convergence holds in the stronger sense:

$$\sup_{[r,t]\subset[0,T]} \big\|\omega \circ \big({}^{1,2}l^{D(n)}_{r,t} - l_{r,t}\big)\big\| \to 0 \quad as \quad n \to \infty \qquad (\omega \in B(\mathcal{F})_*),$$

and if C *is a* C^**-bialgebra,* l *is *-homomorphic and* ${}^1l$ *and* ${}^2l$ *are completely positive and contractive then*

$$\sup_{[r,t]\subset[0,T]} \big\| \big({}^{1,2}l^{D(n)}_{r,t} - l_{r,t}\big)(\cdot)\xi\big\| \to 0 \quad as \quad n\to\infty \qquad (\xi\in\mathcal{F}).$$

Proof. The first part follows from Theorem 7.1 by setting $\mathcal{A}=\mathsf{C}^*$ with convolution product and letting ${}^i\mathsf{q}$ be the sesquilinear process in $\mathcal{A}$ associated with ${}^il$ $(i=1,2)$. The second and third parts follow in the same way as they do for Theorem 7.3. □

For $i=1,2$, let $\varphi^i \in L\big(\widehat{\mathsf{D}}_i; CB(\mathsf{C};|\widehat{\mathsf{k}}_i\rangle)\big)$. Their concatenation product $\varphi^1 \boxplus \varphi^2 \in L\big(\widehat{\mathsf{k}}; CB(\mathsf{C};|\widehat{\mathsf{k}}\rangle)\big)$ is defined by

$$(\varphi^1\boxplus\varphi^2)_{\widehat{c}} := \begin{pmatrix}\varphi^1_{\widehat{c^1}}(\cdot)\\ 0\end{pmatrix} + \Sigma\begin{pmatrix}\varphi^2_{\widehat{c^2}}(\cdot)\\ 0\end{pmatrix} \qquad (c\in\mathsf{D}),$$

where Σ is the sum-flip $\widehat{\mathsf{k}}_2\oplus\mathsf{k}_1 \to \widehat{\mathsf{k}}_1\oplus\mathsf{k}_2 = \widehat{\mathsf{k}}$.

Corollary 7.7. *Let* ${}^il = l^{\varphi^i}$ *for* $\varphi^i \in L\big(\widehat{\mathsf{D}}_i; CB(\mathsf{C};|\widehat{\mathsf{k}}_i\rangle)\big)$ $(i=1,2)$*, set* $\mathsf{D}=\mathsf{D}_1\oplus\mathsf{D}_2$*, let* $\big(D(n)\big)_{n\geqslant1}$ *be a sequence in* $\Gamma_{]0,\infty[}$ *converging to* $\mathbb{R}_+$*, and let* $T\in\mathbb{R}_+$*. Then, for all* $\varepsilon'\in\mathcal{E}$ *and* $\varepsilon\in\mathcal{E}_{\mathsf{D}}$*,*

$$\sup_{[r,t]\subset[0,T]} \big\|\langle\varepsilon'|\big({}^{1,2}l^{D(n)}_{r,t,\varepsilon} - l^{\varphi^1\boxplus\varphi^2}_{r,t,\varepsilon}\big)(\cdot)\big\| \to 0 \quad as \quad n\to\infty.$$

If l^{φ^1}*,* l^{φ^2} *and* $l^{\varphi^1\boxplus\varphi^2}$ *are completely bounded with locally bounded cb norms then, for all* $\omega\in B(\mathcal{F})_*$*,*

$$\sup_{[r,t]\subset[0,T]} \big\|\omega\circ\big({}^{1,2}l^{D(n)}_{r,t} - l^{\varphi^1\boxplus\varphi^2}_{r,t}\big)\big\| \to 0 \quad as \quad n\to\infty.$$

Proof. The stochastic convolution cocycles $l^{\varphi^1\boxplus\varphi^2}$, l^{φ^1} and l^{φ^2} are each cb Markov-regular and cb column-bounded, moreover

$$\langle\widehat{c}|\,(\varphi^1\boxplus\varphi^2)_{\widehat{d}}(\cdot) = \langle\widehat{c^1}|\,\varphi^1_{\widehat{d^1}}(\cdot) + \langle\widehat{c^2}|\,\varphi^2_{\widehat{d^2}}(\cdot) \qquad (c\in\mathsf{k}, d\in\mathsf{D}),$$

which implies that their respective associated convolution semigroup generators are related as required in Theorem 7.6; the result follows. □

Remark. When $\varphi^i\in CB\big(\mathsf{C};B(\widehat{\mathsf{k}}_i)\big)$ $(i=1,2)$, the concatenation product $\varphi^1\boxplus\varphi^2$ reads as follows, in terms of block matrices:

$$\begin{bmatrix}\gamma_1 & \zeta_1\\ \eta_1 & \nu_1\end{bmatrix} \boxplus \begin{bmatrix}\gamma_2 & \zeta_2\\ \eta_2 & \nu_2\end{bmatrix} := \begin{bmatrix}\gamma_1+\gamma_2 & \zeta_1 & \zeta_2\\ \eta_1 & \nu_1 & 0\\ \eta_2 & 0 & \nu_2\end{bmatrix}.$$

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