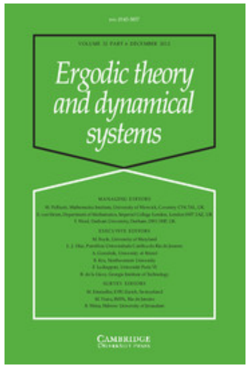

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Dynamical properties of profinite actions

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Abstract. We study profinite actions of residually finite groups in terms of weak containment. We show that two strongly ergodic profinite actions of a group are weakly equivalent if and only if they are isomorphic. This allows us to construct continuum many pairwise weakly inequivalent free actions of a large class of groups, including free groups and linear groups with property (T). We also prove that for chains of subgroups of finite index, Lubotzky's property (τ) is inherited when taking the intersection with a fixed subgroup of finite index. That this is not true for families of subgroups in general leads to the question of Lubotzky and Zuk: for families of subgroups, is property (τ) inherited by the lattice of subgroups generated by the family? On the other hand, we show that for families of normal subgroups of finite index, the above intersection property does hold. In fact, one can give explicit estimates on how the spectral gap changes when passing to the intersection. Our results also have an interesting graph theoretical consequence that does not use the language of groups. Namely, we show that an expanding covering tower of finite regular graphs is either bipartite or stays bounded away from being bipartite in the normalized edge distance.

1. *Introduction*

Let Γ be a countable group. A measure-preserving action f on a Borel probability space (X, μ) is *profinite* if there exists a sequence of finite Γ-invariant partitions P_n of X such that P_n consists of clopen sets, each P_n is a refinement of P_{n-1} and the union of P_n generates the topology on X. One can obtain all the ergodic profinite actions from the group itself as follows. A *chain* in Γ is a sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \cdots$ of subgroups of finite index in Γ. Let $T = T(\Gamma, (\Gamma_n))$ denote the coset tree of Γ with respect to (Γ_n) and let ∂T denote the boundary of T. Then Γ acts on ∂T by measure-preserving homeomorphisms; we call this action the *boundary action* of Γ with respect to (Γ_n). An especially nice case is when the chain consists of normal subgroups with trivial intersection. Here ∂T is a compact topological group, namely the profinite completion of Γ with respect to (Γ_n), endowed with the normalized Haar measure, and Γ maps in ∂T with a dense image.

Let f and g be measure-preserving actions of Γ on the Borel probability spaces (X, μ) and (Y, ν), respectively. Following [**18**], we say that f *weakly contains* g ($f \succeq g$) if for all measurable subsets $A_1, \ldots, A_n \subseteq Y$, finite sets $F \subseteq \Gamma$ and $\varepsilon > 0$ there exist measurable subsets $B_1, \ldots, B_n \subseteq X$ such that

$$|\mu(B_i^\gamma \cap B_j) - \nu(A_i^\gamma \cap A_j)| < \varepsilon \quad (1 \leq i, j \leq n, \gamma \in F).$$

This means that the action f can simulate g with arbitrarily small error. A natural example for weak containment is when g is a *factor* of f, that is, when there exists a Γ-equivariant surjective measure-preserving map from X to Y. We call f and g *weakly equivalent* if $f \succeq g$ and $g \succeq f$.

We say that f is *strongly ergodic* if it is ergodic and it does not weakly contain the trivial (non-ergodic) action of Γ on two points.

Our first theorem is a general weak containment rigidity result on strongly ergodic actions.

THEOREM 1. *Let Γ be a countable group, let f be a strongly ergodic measure-preserving action of Γ and g be a finite action of Γ. If f weakly contains g, then g is a factor of f.*

When applying this to profinite actions, we get the following rigidity result.

THEOREM 2. *Let f and g be profinite actions of Γ such that f is strongly ergodic. If f and g are weakly equivalent, then they are isomorphic.*

In terms of chains, the isomorphism of boundary actions means that all elements in one of the chains contains a conjugate of an element of the other chain. This result allows one to show that a natural class of groups has many weakly incomparable measure-preserving actions.

THEOREM 3. *Let Γ be a countable linear group with Kazhdan's property (T) or a finitely generated free group. Then Γ has continuum many, pairwise weakly incomparable free ergodic measure-preserving actions.*

The analogous question for orbit equivalence has been thoroughly investigated in the literature. Very recently, this culminated in proving that every countable, non-amenable group has continuously many, pairwise orbit inequivalent free ergodic measure-preserving actions (see [**10**]). Orbit equivalence rigidity has also been investigated specifically in the profinite case, mainly for Kazhdan groups; see the work of Ioana [**17**] and Ozawa–Popa [**24**].

Let Γ be a group generated by a finite symmetric set S. We say that a family of subgroups of finite index $\{H_n \mid n \geq 1\}$ has property (τ) if the family of Schreier graphs $\mathrm{Sch}(\Gamma/H_n, S)$ forms an expander family. It is easy to see that this property is independent of S. For chains, property (τ) is equivalent to saying that the boundary action has spectral gap. Although spectral gap implies strong ergodicity for arbitrary measure-preserving actions, an easy example of Schmidt [**25**] shows that this cannot be reversed in general. However, we can show that for boundary actions with respect to normal chains, the two properties are in fact equivalent.

THEOREM 4. *Let Γ be a finitely generated group. Let (Γ_n) be a normal chain in Γ and let f denote the boundary action of Γ with respect to (Γ_n). Then f is strongly ergodic if and only if it has spectral gap.*

What we actually show in this direction is that spectral gap and strong ergodicity are equivalent for compact topological groups acted on by their dense subgroups. Since (opposed to spectral gap, see [**16**]) strong ergodicity is an orbit equivalence invariant, we get that for these actions, having spectral gap is an orbit equivalence invariant as well.

Our next theorem shows that Theorem 4 does not hold for arbitrary chains. Let F_k denote the free group of rank k.

THEOREM 5. *For every $k \geq 2$, there exists a chain (Γ_n) in F_k such that the boundary action of Γ with respect to (Γ_n) is free and strongly ergodic but (Γ_n) does not have property (τ).*

The proof is probabilistic; it amalgamates the random lifting method of Friedman [**12**] with results on random actions on rooted trees investigated in [**4**].

Our next result shows that property (τ) of a chain is inherited when taking the intersection with a finite index subgroup.

THEOREM 6. *Let Γ be a finitely generated group and let (Γ_n) be a chain in Γ with property (τ). Let H be a subgroup of finite index in Γ. Then the chain $(H \cap \Gamma_n)$ has property (τ) in H.*

This is known for normal chains from the work of Shalom [**26**]. Note that he actually allows for the group H to be co-amenable rather than just finite index. In the case of normal subgroups, or, more generally, for compact metrizable topological groups acted on by their dense subgroups, we obtain a stronger result which gives an explicit lower estimate on how the spectral gap changes when passing to a subgroup of finite index.

THEOREM 7. *Let G be a compact metrizable topological group endowed with its normalized Haar measure μ and let Γ be a dense subgroup in G, generated by the finite symmetric set S. Let H be a subgroup of Γ of index k, let C be a coset representative system for H in Γ and let $T = N(S, C)$ be the Nielsen–Schreier generating set of H with respect to S and C. Let O be an ergodic component of G under the action of H. Then we have*

$$\mathrm{h}(O, T) > \frac{1}{8k^{Q}} \min\left\{\frac{2\mathrm{h}(G, S)}{k^2}, 1\right\},$$

where $\mathrm{h}(X, S)$ *denotes the Cheeger constant of the space X with respect to the set of maps S and* $Q = 1 + \log_2^{-1}(4/3)$.

This in turn implies the following for arbitrary *families* of normal subgroups.

THEOREM 8. *Let Γ be a finitely generated group and let $\{H_n \mid n \geq 1\}$ be a family of normal subgroups of finite index in Γ with property (τ). Let H be a subgroup of finite index in Γ. Then the family $\{H \cap H_n \mid n \geq 1\}$ has property (τ) in H.*

On the other hand, as we show in §7, Theorem 8 fails for a general family of subgroups of finite index. Together with Theorem 6, this can be used to answer a question of Lubotzky and Zuk [**23**, Question 1.14]. They asked the following: if $\{H_n \mid n \geq 1\}$ is a family of finite index subgroups in Γ with property (τ), does the set

$$\mathcal{L}(\{H_n\}) = \left\{ \bigcap_{j=1}^{k} H_{n_j}^{g_j} \,\middle|\, n_j \in \mathbb{N},\ g_j \in \Gamma \right\}$$

also have property (τ) (note that we denote a subgroup gHg^{-1} by H^g). The answer is negative.

COROLLARY 9. *There exists a family of finite index subgroups* $\{H_n \mid n \geq 1\}$ *in* F_4 *such that* $\{H_n\}$ *has property* (τ)*, but the chain* $\Gamma_n = \bigcap_{k=1}^n H_k$ *does not.*

The counterexample family $\{H_n \mid n \geq 1\}$ can be explicitly constructed. Note, however, that because of Theorem 8, we do not have a negative answer for the question of Lubotzky and Zuk if we restrict our attention to normal subgroups; so, for that case, the question is still open.

One can exploit Theorem 1 to obtain a purely graph theoretical result as well. By a *covering tower* of graphs, we mean a sequence G_n of graphs such that for all $n \geq 1$ there is a covering map from G_{n+1} to G_n.

THEOREM 10. *Let* G_n *be an expanding covering tower of* k*-regular graphs. Then exactly one of the following holds.*

(1) *All but finitely many of the* G_n *are bipartite.*
(2) *There exists* $r > 0$ *such that for all* n*, one needs to erase at least* $r|G_n|$ *edges of* G_n *to make it bipartite.*

Equivalently to (2), the so-called independence ratio of G_n is bounded away from $1/2$.

In spectral language, Theorem 10 takes the following equivalent form: let G_n be a covering tower of non-bipartite k-regular graphs. If $\lambda_1(G_n)$ is bounded away from k, then $\lambda_-(G_n)$ is bounded away from $-k$. Here λ_1 denotes the first non-trivial eigenvalue and λ_- the last eigenvalue in order. Trivially, these results are far from being true for an arbitrary expander family of k-regular graphs.

It would be interesting to see whether Theorem 10 holds for higher chromatic numbers as well.

Problem 1. Let G_n be an expanding covering tower of k-regular graphs such that G_n cannot be legally colored by c colors ($n \geq 0$). Is it true that there exists $r > 0$ such for all n and all c-colorings of G_n, the number of unicolored edges in G_n is at least $r|G_n|$?

As one would expect, almost covers of chains in amenable groups behave quite differently from groups having a chain with property (τ). Indeed, any two free ergodic actions of an amenable group are weakly equivalent (see [**11, 18**]), which implies that any free boundary action of a residually finite amenable group Γ weakly contains any finite action of Γ. This in turn enables us to show that every d-generated finite solvable group can be simulated by a d-generated finite p-group in terms of weak containment.

THEOREM 11. *Let* p *be a prime and let* F *be a finitely generated free group. Then the action of* F *on its pro-*p*-completion is weakly equivalent to the action of* F *on its pro-(finite solvable)-completion.*

Trivially, the pro-p-completion is a factor of the pro-solvable-completion, but the other direction is somewhat surprising. We suspect that the same result holds for the whole profinite completion.

This paper is organized as follows. In §2, we introduce our notions and state some of the results used later. In §3, we prove some general ergodic theoretical results needed later for profinite actions. Section 4 contains the proof of Theorem 7. In §5, we establish the weak equivalence rigidity results and prove Theorems 1, 2 and 3. In §6, we construct the example in Theorem 5. Section 7 contains the proof of Theorem 6, Theorem 8 and Corollary 9. Section 8 is about the graph theoretical consequences of our results; in particular, we prove Theorem 10 and its corollary on eigenvalues. Finally, in §9, we deal with amenable groups, prove Theorem 11 and show how to derive a recent result of Conley and Kechris [**7**] using our language.

2. *Preliminaries*

This section contains the general notations and some lemmas that will be used throughout the paper.

2.1. *Profinite and boundary actions.*

Let Γ be a group acting on the Borel probability space (X, μ) by measure-preserving transformations. We say that this action is *profinite* if there exists a sequence of finite Γ-invariant partitions P_n of X such that P_n consists of clopen sets, each P_n is a refinement of P_{n-1} and the union of P_n generates the topology on X.

Let (Γ_n) be a chain in Γ. Then the *coset tree* $T = T(\Gamma, (\Gamma_n))$ of Γ with respect to (Γ_n) is defined as follows. The vertex set of T equals

$$T = \{\Gamma_n g \mid n \geq 0,\ g \in \Gamma\}$$

and the edge set is defined by inclusion, that is,

$$(\Gamma_n g, \Gamma_m h) \text{ is an edge in } T \text{ if } m = n + 1 \text{ and } \Gamma_n g \supseteq \Gamma_m h.$$

Then T is a tree rooted at Γ and every vertex of level n has the same number of children, equal to the index $|\Gamma_n : \Gamma_{n+1}|$. The right actions of Γ on the coset spaces Γ/Γ_n respect the tree structure and so Γ acts on T by automorphisms.

The boundary ∂T of T is defined as the set of infinite rays starting from the root. The boundary is naturally endowed with the product topology and product measure coming from the tree. More precisely, for $t = \Gamma_n g \in T$, let us define $\mathrm{Sh}(t) \subseteq \partial T$, the *shadow* of t, as

$$\mathrm{Sh}(t) = \{x \in \partial T \mid t \in x\},$$

the set of rays going through t. Set the base of topology on ∂T to be the set of shadows and set the measure of a shadow to be

$$\mu(\mathrm{Sh}(t)) = 1/|\Gamma : \Gamma_n|.$$

This turns ∂T into a totally disconnected compact space with a Borel probability measure μ. The group Γ acts ergodically on ∂T by measure-preserving homeomorphisms; we call this action the *boundary action of* Γ with respect to (Γ_n). See [**14**] where these actions were first investigated in a measure-theoretic sense.

Another way to obtain boundary actions of a finitely generated group Γ is to consider its profinite completion G. For every closed subgroup H of G, the right coset space G/H is a compact topological space with a normalised Haar measure on which Γ acts from the right. One can get a chain leading to this action by using that H is an intersection of open subgroups in G. It will be convenient to use this notation in §5.

It is easy to see that a profinite action can be obtained as a boundary action if and only if it is ergodic.

2.2. *Cheeger constant, spectral gap and strong ergodicity.* Let (X, μ) be a probability space and let S be a set of measure-preserving bijections. Let us define the *Cheeger constant* of X with respect to S as

$$\mathrm{h}(X, S) = \inf\left\{ \frac{\mu(AS\backslash A)}{\mu(A)} \;\middle|\; A \subseteq X, 0 < \mu(A) \leq 1/2 \right\},$$

where

$$AS = \{as \mid a \in A, s \in S\}.$$

Note that for a finite graph G, the Cheeger constant of G is defined as

$$\mathrm{Ch}(G) := \inf_{A \subset G, |A| \leq \frac{1}{2}|G|} \frac{|L(A)|}{|A|},$$

where $L(A)$ denotes the number of edges between A and its complement.

Let Γ be a group acting on the probability space (X, μ) by measure-preserving transformations. A sequence A_n of measurable subsets of positive measure is called an I-sequence, if for all $\gamma \in \Gamma$, we have

$$\lim_{n\to\infty} \frac{\mu(A_n\gamma\backslash A_n)}{\mu(A_n)} = 0.$$

Following [**25**], we say that the action of Γ has *spectral gap*, if it admits no I-sequences. When Γ is generated by a finite symmetric set S, the action of Γ has spectral gap if and only if $h(X, S) > 0$.

A sequence of subsets $A_n \subseteq X$ is *almost invariant*, if

$$\lim_{n\to\infty} \mu(A_n \setminus A_n\gamma) = 0 \quad \text{for all } \gamma \in \Gamma.$$

The sequence is trivial if $\lim_{n\to\infty} \mu(A_n)(1 - \mu(A_n)) = 0$. We say that the action is *strongly ergodic* if every almost invariant sequence is trivial.

In this paper, we will subsequently make use of the following lemma of Schmidt [**25**]. Let Id_Γ denote the trivial action of Γ on one point and let $\frac{1}{2}\mathrm{Id}_\Gamma + \frac{1}{2}\mathrm{Id}_\Gamma$ denote its trivial action on two points, both of measure $\frac{1}{2}$.

LEMMA 2.1. (Schmidt) *Let Γ act on a probability space (X, μ) by measure-preserving maps. If the action is ergodic, but not strongly ergodic, then for all $\lambda \in (0, 1)$, there exists an almost invariant sequence $A_n \subseteq X$ such that $\mu(A_n) = \lambda$ $(n \geq 0)$. In particular, an ergodic action is strongly ergodic if and only if it does not contain $\frac{1}{2}\mathrm{Id}_\Gamma + \frac{1}{2}\mathrm{Id}_\Gamma$ weakly.*

2.3. *Schreier graphs, Cayley graphs and property (τ).* Let Γ be a group acting on the set X by permutations and let S be a subset of Γ. Then we define the Schreier graph $\mathrm{Sch}(X, S)$ as follows: its vertex set is X and for every $s \in S$, $x \in X$, there is an s-labeled edge going from x to xs. When S is symmetric, that is, $S = S^{-1}$, we can think of $\mathrm{Sch}(X, S)$ as an undirected graph. A special case is when S generates Γ and $X = \Gamma/H$, the set of right cosets for a subgroup H of Γ; in this case, $\mathrm{Sch}(\Gamma/H, S)$ is connected. When, moreover, H is normal, we define the Cayley graph $\mathrm{Cay}(\Gamma/H, S) = \mathrm{Sch}(\Gamma/H, S)$. Cayley graphs are vertex-transitive, that is, their automorphism groups act transitively on the set of vertices.

Let Γ be a finitely generated group. A set $\{\Gamma_n\}$ of subgroups of finite index in Γ has *Lubotzky's property* (τ) if for some finite, symmetric generating set S of Γ, the sequence of Schreier graphs $\mathrm{Sch}(\Gamma/\Gamma_n, S)$ forms an *expander family*, that is, there exists $c > 0$ such that

$$h(\mathrm{Sch}(\Gamma/\Gamma_n, S)) > c \quad (n \geq 0),$$

where the measure on Γ/Γ_n is defined to be uniformly random. For chains, property (τ) can be expressed as follows.

LEMMA 2.2. *Let (Γ_n) be a chain in Γ. Then (Γ_n) has property (τ) if and only if the boundary action of Γ with respect to (Γ_n) has spectral gap.*

Proof. Let S be a finite symmetric generating set for Γ and let $T = T(\Gamma, (\Gamma_n))$ be the coset tree. Since the set of shadows generates the topology on ∂T, one gets that

$$h(\partial T, S) = \inf_{n \geq 0} h(\mathrm{Sch}(\Gamma/\Gamma_n, S)).$$

This proves the lemma. □

2.4. *A covering lemma.* We will use the following lemma from [**2**]. Since we cite it in modified form, we include a short proof.

LEMMA 2.3. *Let G be a compact topological group with normalized Haar measure μ and let $A, B \subseteq G$ be measurable subsets of positive measure. Let g be a μ-random element of G. Then the expected value satisfies*

$$E(\mu(Ag \cap B)) = \mu(A)\mu(B).$$

In particular, for any natural number k, there exists a subset X of size k such that

$$\mu(AX) \geq 1 - (1 - \mu(A))^k.$$

For $k = \lceil 1/\mu(A) \rceil$, this gives

$$\mu(AX) > 1 - \frac{1}{e}.$$

Proof. Let $U = \{(a, g) \in G \times G \mid a \in A, ag \in B\}$. Then U is measurable in $G \times G$ and, using Fubini's theorem both ways, we get

$$\mu(A)\mu(B) = \int_{a \in A} \mu(a^{-1}B) = \mu^2(U) = \int_{g \in G} \mu(Ag \cap B) = E(\mu(Ag \cap B)).$$

The equality $E(\mu(AX)) = 1 - (1 - \mu(A))^k$ follows by induction on k. This implies both inequalities. □

3. *Strong ergodicity and spectral gap for finite index subgroups*

This section analyzes what happens to the strong ergodicity and spectral gap properties for a general measure-preserving action when restricting it to a subgroup of finite index.

LEMMA 3.1. *Let Γ act ergodically on a probability space (X, μ) by measure-preserving maps. Let $H \leq \Gamma$ be a subgroup of finite index and let O be an ergodic component of X for the action of H. Then $\mu(O)$ is a multiple of $1/|\Gamma : H|$ and the action of Γ on X is strongly ergodic if and only if the action of H on O is strongly ergodic.*

Proof. Let $H' = \{\gamma \in \Gamma \mid O\gamma = O\}$ be the setwise stabilizer of O. Let C be a coset representative system for H' in Γ. Then OC is invariant under Γ and hence is equal to X. For $x \in X$, let $f(x)$ be the number of sets in the form of Oc, $c \in C$, that contain x. Clearly, f is a measurable function. On the other hand, if $\gamma \in \Gamma$, then $f(x) = f(x\gamma)$. Indeed, if x is covered by $Oc_1, Oc_2, \ldots, Oc_i$, then $x\gamma$ is covered by $Od_1, Od_2, \ldots, Od_i$, where d_j is the coset representative of $H'c_j\gamma$. Since the Γ-action is ergodic, the function f is almost everywhere equal to a constant l. Therefore $l = \mu(O)|\Gamma : H'|$. Since $H' \geq H$, $\mu(O)$ is a constant multiple of $1/|\Gamma : H|$.

Assume that the action of H on O is not strongly ergodic but the action of Γ on X is strongly ergodic. Let T be a coset representative system for H in Γ. Then, by Schmidt's lemma, there exists an almost H-invariant sequence of measurable subsets $A_n \subseteq O$ such that

$$\mu(A_n) = \frac{1}{2|\Gamma : H|} \quad (n \geq 0).$$

For $n \geq 0$, let $B_n = A_n T$ be the union of T-translates of A_n. Let $\gamma \in \Gamma$, and for $t \in T$, let $\bar{t} \in T$ such that $t\gamma\bar{t}^{-1} \in H$. Then we have $1/2|\Gamma : H| \leq \mu(B_n) \leq 1/2$ and

$$B_n\gamma \backslash B_n \subseteq \bigcup_{t\in T} (A_n t\gamma \backslash A_n \bar{t}) \subseteq \bigcup_{t\in T} (A_n t\gamma\bar{t}^{-1} \backslash A_n)\bar{t},$$

hence

$$\mu(B_n\gamma \backslash B_n) \leq \sum_{t\in T} \mu(A_n t\gamma\bar{t}^{-1} \backslash A_n).$$

The latter converges to zero as n tends to infinity, so B_n is a non-trivial almost Γ-invariant sequence, a contradiction. We get that strong ergodicity of Γ on X implies strong ergodicity of H on O.

Assume that the action of Γ on X is not strongly ergodic. Following the proof of Schmidt's lemma, let $\phi \in L^\infty(X, \mu)$ be the weak $*$-limit of a subsequence of $\{\chi_{A_n}\}_{n=1}^\infty$. Here we use the Banach–Alaoglu theorem and the fact that $L^\infty(X, \mu)$ is the dual of the separable Banach space $L^1(X, \mu)$.

We claim that the function ϕ is invariant under the action of Γ. It is enough to see that for any Borel set $B \subseteq X$ and $\gamma \in \Gamma$,

$$\int_X (\phi \circ \gamma)\chi_B \, d\mu = \int_X \phi\chi_B \, d\mu.$$

The right-hand side equals $\lim_{k\to\infty} \mu(A_{n_k} \cap B)$. The left-hand side equals

$$\int_X \phi(\chi_B \circ \gamma^{-1}) \, d\mu = \lim_{k\to\infty} \mu(A_{n_k} \cap B\gamma) = \lim_{k\to\infty} \mu(A_{n_k}\gamma^{-1} \cap B),$$

and our claim follows from the almost invariance of $\{A_n\}_{n=1}^\infty$.

By ergodicity, ϕ is a constant $1/2$-function. This means that

$$\lim_{k\to\infty} \mu(O \cap A_{n_k}) = \mu(O)/2.$$

But then $O \cap A_{n_k}$ is a non-trivial almost invariant sequence with respect to H. We get that the action of H on O is not strongly ergodic, a contradiction. □

Now we will show the corresponding theorem for spectral gap. First we need a lemma showing that if we pass to a subgroup of finite index, small sets keep expanding.

Let Γ be a group generated by a symmetric set S. Let H be a subgroup of Γ and let C be a coset representative system for H in Γ. Then for each $s \in S$ and $c \in C$, there exists a unique $p_{c,s} \in C$ satisfying $csp_{c,s}^{-1} \in H$. Let

$$N(S, C) = \{csp_{c,s}^{-1} \mid s \in S, c \in C\}.$$

It is well known that $N(S, C)$ generates H.

LEMMA 3.2. *Let Γ act ergodically on a probability space (X, μ) by measure-preserving maps. Let S be a finite symmetric generating set for Γ and let H be a subgroup of Γ of index k. Let C be a coset representative system for H in Γ and let $T = N(S, C)$. Then, for all measurable subsets $A \subseteq X$ with $0 < \mu(A) \leq 1/2k$, we have*

$$\frac{\mu(AT\backslash A)}{\mu(A)} \geq \frac{\mathrm{h}(X, S)}{k}.$$

Proof. Let $B = AC$. Then by straightforward set manipulations we get

$$\begin{aligned} BS\backslash B &= \bigcup_{c\in C, s\in S} Acs \Big\backslash \bigcup_{c\in C} Ac = \Bigg(\bigcup_{d\in C} \bigcup_{\substack{c\in C, s\in S\\ p_{c,s}=d}} Acs\Bigg) \Big\backslash \bigcup_{d\in C} Ad \\ &\subseteq \bigcup_{d\in C}\Bigg(\bigcup_{\substack{c\in C, s\in S\\ p_{c,s}=d}} Acs\backslash Ad\Bigg) = \bigcup_{d\in C}\Bigg(\bigcup_{\substack{c\in C, s\in S\\ p_{c,s}=d}} Acsd^{-1}\backslash A\Bigg)d \\ &\subseteq \bigcup_{d\in C}(AT\backslash A)\, d = (AT\backslash A)C, \end{aligned}$$

which, using the definition of the Cheeger constant and $\mu(A) \leq \mu(B) \leq 1/2$, yields

$$k\mu(AT\backslash A) \geq \mu(BS\backslash B) \geq \mathrm{h}(X, S)\mu(B) \geq \mathrm{h}(X, S)\mu(A),$$

from which the lemma follows. □

LEMMA 3.3. *Let Γ be a finitely generated group acting ergodically on a probability space (X, μ) by measure-preserving maps. Let $H \leq \Gamma$ be a subgroup of finite index and let O be an ergodic component of X for the action of H. Then the action of Γ on X has spectral gap if and only if the action of H on O has spectral gap.*

Proof. Let S be a finite symmetric generating set for Γ, let C be a coset representative system for H in Γ and let $T = N(S, C)$.

Assume that the action of Γ on X has spectral gap. Then it is also strongly ergodic. So, by Lemma 3.1, the action of H on O is strongly ergodic. Assume it does not have spectral gap. Then there exists an I-sequence A_n in O. So, for all $h \in H$, we have

$$\lim_{n\to\infty} \frac{\mu(A_n h\backslash A)}{\mu(A_n)} = 0.$$

But that implies $\lim_{n\to\infty} \mu(A_n) = 0$, otherwise a suitable subsequence would be a non-trivial almost invariant sequence for H in O. Now, by Lemma 3.2, we have

$$\sum_{t\in T} \frac{\mu(A_n t\backslash A)}{\mu(A_n)} \geq \frac{\mu(A_n T\backslash A)}{\mu(A_n)} \geq \frac{\mathrm{h}(X, S)}{|\Gamma : H|} > 0$$

for all large enough n, a contradiction.

Now assume that the action of H on O has spectral gap. Then it is also strongly ergodic, so, by Lemma 3.1, the action of Γ on X is strongly ergodic. Assume it does not have spectral gap. Then there exists an I-sequence $A_n \subseteq X$. By strong ergodicity, we have $\lim_{n\to\infty} \mu(A_n) = 0$. Since $OC = X$, there exists $c \in C$ and a subsequence B_n of A_n such that

$$\mu(B_n \cap Oc) \geq \frac{1}{k}\mu(B_n).$$

But then $(O \cap B_n c^{-1})$ is an I-sequence for H, a contradiction. So, the action of Γ on X has spectral gap. □

Now we can prove a general result that will lead to Theorem 4.

PROPOSITION 3.1. *Let G be a compact topological group endowed with its normalized Haar measure μ and let Γ be a dense subgroup in G. Then the right action of Γ on G is strongly ergodic if and only if it has spectral gap.*

Proof. Spectral gap implies strong ergodicity in general, as, clearly, any non-trivial almost invariant sequence is an I-sequence. Assume that the action of Γ on G is strongly ergodic but has no spectral gap. Then there exists an I-sequence $A_n \subseteq G$. Now, by Lemma 2.3, for any $n \geq 1$, we have a subset X_n of size $k_n = \lceil 1/2\mu(A_n)\rceil$ such that

$$\left(1 - \frac{1}{e}\right)^2 < \mu(X_n A_n) \leq \frac{1}{2}.$$

On the other hand, for all $\gamma \in \Gamma$, we have

$$X_n A_n \gamma \backslash X_n A_n \subseteq \bigcup_{x\in X_n} x(A_n \gamma \backslash A_n),$$

which yields

$$\mu(X_n A_n \gamma \backslash X_n A_n) \leq k_n \mu(A_n \gamma \backslash A_n) \leq \frac{\mu(A_n \gamma \backslash A_n)}{2\mu(A_n)}$$

for any $\gamma \in \Gamma$. Since A_n is an I-sequence, the last expression converges to zero in n. This means that $X_n A_n$ is a non-trivial almost invariant sequence, a contradiction. We have proved that the action of Γ has spectral gap. □

4. *Distortion of the Cheeger constant for compact groups*

In this section, we prove Theorem 7 by giving an explicit estimate of how the Cheeger constant is distorted when passing to a finite index subgroup. We start with a general lemma on finite graphs.

LEMMA 4.1. *Let $G = (V, E)$ be a finite, undirected connected graph. For a function $f : V \to [0, 1]$, let $f' : V \to [0, 1]$ be defined as follows. For $v \in V$, let*

$$f'(v) = \max\{0, \max\{f(w) - f(v) \mid (v, w) \in E\}\},$$

and let

$$F(f) = \sum_{v \in V} f'(v).$$

Then

$$F(f) \geq \max\{f(v) \mid v \in V\} - \min\{f(v) \mid v \in V\}.$$

Proof. It is easy to see that F is not increasing if we restrict f on a subgraph. Hence, by taking a path between a minimal and a maximal element, it is enough to prove the lemma for segments. We will proceed by erasing one of the endpoints and using induction. If any end of the segment is not minimal or maximal, then erasing it does not change the minimum and the maximum. The same happens if both ends are maximal. Let v be an endpoint where f is minimal. By erasing v, we may increase the minimum, but by at most $f'(v)$. □

Proof of Theorem 7. We can assume that Γ acts with spectral gap on G, otherwise the theorem is trivial. Let A be a measurable subset of O with $0 < \mu(A) \leq \mu(O)/2$. We can also assume $\mu(A) > 1/2k$, otherwise we are done, by Lemma 3.2. Let a and b be parameters to be set later, satisfying $0 < a < 1/2k$ and $1 - 1/2k < b < 1$. Using $k \geq 2$, this implies $2b - 1 \geq a$.

Let $A_0 = A$, and for $l \geq 0$, let us define A_{l+1} as follows. If there exists $g \in G$ such that

$$a\mu(A_l) \leq \mu(gA_l \cap A_l) \leq b\mu(A_l),$$

then let $A_{l+1} = gA_l \cap A_l$. We do this until there is no such $g \in G$ or $\mu(A_{l+1}) \leq 1/2k$. Let t be the last index and let $B = A_t$. Then, trivially, $\mu(B) > a/2k$.

Case 1. If $\mu(B) \leq 1/2k$, then, using

$$(X \cap Y)T \backslash (X \cap Y) \subseteq (XT \backslash X) \cup (YT \backslash Y),$$

we get

$$\mu(BT \backslash B) \leq 2^t \mu(AT \backslash A),$$

which gives

$$\frac{\mu(AT \backslash A)}{\mu(A)} \geq \frac{1}{2^t} \frac{\mu(BT \backslash B)}{\mu(B)} \frac{\mu(B)}{\mu(A)} \geq \frac{a}{2^t k} \frac{\mu(BT \backslash B)}{\mu(B)}.$$

Using Lemma 3.2, this yields

$$\frac{\mu(AT \backslash A)}{\mu(A)} \geq \frac{a}{2^t k^2} \mathrm{h}(G, S). \tag{1}$$

Case 2. If $\mu(B) > 1/2k$, then, for all $g \in G$, we have $\mu(gB \cap B) < a\mu(B)$ or $\mu(gB \cap B) > b\mu(B)$. Let

$$K = \{g \in G \mid \mu(gB \cap B) > b\mu(B)\}.$$

Let f, $g \in K$; then, using

$$fgB \backslash B \subseteq f(gB \backslash B) \cup (fB \backslash B)$$

and $2b - 1 \geq a$, we get

$$\mu(fgB \cap B) \geq (2b-1)\mu(B) \geq a\mu(B).$$

This means that $fg \in K$, so K is a subgroup of G.

We claim that K is closed. Indeed, if we approximate the indicator function of B on G with a continuous function $F : G \to \mathbb{R}$ in L^2 norm well enough, then for all $g \in G$, we have

$$\mu(gB \cap B) > b\mu(B) \text{ if and only if } \int_{x \in G} F^g(x)F(x)\, d\mu \geq \frac{a+b}{2}\mu(B).$$

But the integral above is a continuous function of g, so our claim holds. The same argument gives that K is open. By compactness, K has finite index in G.

Let l be the index of K in G. Let $g \in G$ be a random element according to μ. Then, by Lemma 2.3, the expected measure

$$\begin{aligned}\mu(B)^2 &= E(\mu(gB \cap B)) = \int_{x \in K} \mu(xB \cap B)\, d\mu + \int_{x \in G \backslash K} \mu(xB \cap B)\, d\mu \\ &\leq \frac{1}{l}\mu(B) + \frac{l-1}{l} a\mu(B),\end{aligned}$$

which gives

$$l \leq \frac{1-a}{\mu(B)-a} < \frac{2k(1-a)}{1-2ka}.$$

Let $K_1, K_2, \ldots, K_n$ be the right cosets of K in G that intersect O non-trivially (in measure). Then $O \subseteq \bigcup_1^n K_i$, so $\mu(O) \leq n/l$. Let $B_i = K_i \cap B$ and let $p_i = l\mu(B_i)$ $(1 \leq i \leq n)$. Let

$$m = \min\{p_i \mid 1 \leq i \leq n\} \quad \text{and} \quad M = \max\{p_i \mid 1 \leq i \leq n\}.$$

Then

$$m \leq \frac{1}{n}\sum_{i=1}^{n} p_i = \frac{l}{n}\mu(B) \leq \frac{1}{2}\frac{l\mu(O)}{n} \leq \frac{1}{2}.$$

Let g be a random element of K according to its normalized Haar measure $l\mu$. Again using Lemma 2.3 for each $B_i \subseteq K_i$ separately, the expected measure satisfies

$$\sum_{i=1}^{l} p_i^2 = E(l\mu(gB \cap B)) = \int_{x \in K} l\mu(xB \cap B)\, dl\mu \geq bl\mu(B) = b\sum_{i=1}^{l} p_i.$$

This gives

$$M\sum_{i=1}^{l} p_i \geq \sum_{i=1}^{l} p_i^2 \geq b\sum_{i=1}^{l} p_i,$$

which implies $M \geq b$.

Now, H acts on the partition $P = (K_1, K_2, \ldots, K_n)$ transitively from the right since it acts ergodically on O. Let $W = \mathrm{Sch}(P, T)$ be the Schreier graph for this action. We will use Lemma 4.1 on W. Let $f : P \to [0, 1]$ be defined by $f(K_i) = p_i$. Then, for all i, we have

$$K_i \cap (BT \backslash B) = \bigcup_{\substack{x \in T \\ K_j x = K_i}} B_j x \backslash B_i \supseteq B_j x \backslash B_i$$

for all $j \leq n$ and $x \in T$ such that $K_j x = K_i$. This implies

$$l\mu(K_i \cap (BT \backslash B)) \geq f'(K_i)$$

and, hence, using Lemma 4.1, we get

$$\mu(BT \backslash B) \geq \frac{1}{l} F(f) \geq \frac{M - m}{l} \geq \frac{2b - 1}{2l}.$$

Again, using $\mu(A) \leq 1/2$ and the upper estimate on l, we get

$$\frac{\mu(AT \backslash A)}{\mu(A)} \geq \frac{1}{2^t} \frac{\mu(BT \backslash B)}{\mu(A)} > \frac{2b - 1}{2^t l} > \frac{(2b - 1)(1 - 2ka)}{2^{t+1} k(1 - a)}. \tag{2}$$

Now we summarize the two cases. First we estimate the t-part. If $\mu(B) > 1/2k$, then

$$\frac{1}{2k} < \mu(A_t) < b^t \mu(A) \leq \frac{1}{2} b^t,$$

which implies $b^t > 1/k$ and so $2^t < k^{\log_2 1/b}$. Otherwise, $\mu(A_{t-1}) > 1/2k$, which gives $2^{t-1} < k^{\log_2^{-1} 1/b}$. Let us set the parameters as $a = 1/4k$ and $b = 3/4$. We get that if $\mu(B) > 1/2k$, then, by (2), we have

$$\frac{\mu(AT \backslash A)}{\mu(A)} > \frac{1}{8k} \frac{1}{2^t} > \frac{1}{8k^{1 + \log_2^{-1}(4/3)}}$$

and if $\mu(B) < 1/2k$, then, by (1), we have

$$\frac{\mu(AT \backslash A)}{\mu(A)} \geq \frac{a}{2^t k^2} \mathrm{h}(G, S) = \frac{1}{8k^3} \frac{1}{2^{t-1}} \mathrm{h}(G, S) > \frac{1}{8k^{3 + \log_2^{-1}(4/3)}} 2\mathrm{h}(G, S),$$

which in general yields

$$\frac{\mu(AT \backslash A)}{\mu(A)} > \frac{1}{8k^Q} \min\left\{\frac{2\mathrm{h}(G, S)}{k^2}, 1\right\},$$

where $Q = 1 + \log_2^{-1}(4/3)$. The theorem is proved. □

Remark. One can probably improve the exponent of k in Theorem 7 with a more careful analysis.

5. *Weak containment rigidity of profinite actions*

Throughout this section we fix the following notation. Let Γ be a countable group. Let (X, μ) be a standard Borel probability space and (Y, ν) be a probability space. We allow Y to be finite. Let f be a strongly ergodic measure-preserving action of Γ on (X, μ) and let g be a measure-preserving action of Γ on (Y, ν).

Let us recall the definition of weak containment for measure-preserving actions. We say that f *weakly contains* g ($f \succeq g$) if, for all measurable subsets $A_1, \ldots, A_n \subseteq Y$, finite sets $F \subseteq \Gamma$ and $\varepsilon > 0$, there exist measurable subsets $B_1, \ldots, B_n \subseteq X$ such that

$$|\mu(B_i^\gamma \cap B_j) - \nu(A_i^\gamma \cap A_j)| < \varepsilon \quad (1 \leq i, j \leq n, \gamma \in F).$$

We say that f *contains* g ($f \geq g$, or g is a *factor* of f) if there exists a map $\Phi : X \to Y$ which is Γ-equivariant and $\Phi^{-1}(\nu) = \mu$.

Note that the terms 'weakly contains' and 'contains' can be somewhat misleading for measure-preserving actions. The reason they were named in this way comes from the realm of unitary representations. In fact, $f \succeq g$ implies that the Koopman representation of f weakly contains the Koopman representation of g in the unitary sense, but the reverse implication does not hold. For details, see the recent book by Kechris [**18**].

The first lemma says that weak containment preserves strong ergodicity and the Cheeger constant is monotonic with respect to it.

LEMMA 5.1. *Let Γ be a countable group and let f and g be measure-preserving actions on the spaces (X, μ) and (Y, ν), respectively. If $f \succeq g$ and f is strongly ergodic, then g is strongly ergodic as well. Also, for any finite subset S of Γ, we have*

$$\mathrm{h}(X, S) \leq \mathrm{h}(Y, S).$$

Proof. Assume g is not strongly ergodic. Let A_n be an almost Γ-invariant sequence of measurable subsets of Y such that $\lim_n \nu(A_n) = \alpha$ with $0 < \alpha < 1$ (A_n can be a fixed set if g is not ergodic). Enumerate the elements of Γ such that 1 is the first element and let F_n be the set of the first n elements of Γ.

Using the weak containment condition with A_n, F_n and $\varepsilon = 1/n$, we get that there exist measurable subsets $B_n \subseteq X$ such that

$$|\mu(B_n) - \nu(A_n)| < 1/n$$

and, for all $\gamma \in \Gamma$, for all large enough n, we have

$$|\mu(B_n^\gamma \cap B_n) - \nu(A_n^\gamma \cap A_n)| < 1/n.$$

This gives us that $\lim_n \nu(B_n) = \alpha$ and hence (B_n) is a non-trivial almost invariant sequence in X, which contradicts the strong ergodicity of f. The statement on Cheeger constants follows similarly. □

Now we can prove Theorem 1.

Proof of Theorem 1. Let f and g be measure-preserving actions on the spaces (X, μ) and (Y, ν), respectively. Assume that Y is finite, $\nu(y) \neq 0$ ($y \in Y$), f is strongly ergodic and weakly contains g. Then, by Lemma 5.1, g is strongly ergodic, in particular, ergodic and hence transitive. Let $k = |Y|$, let $y \in Y$ and let H be the stabilizer of y in Γ. Let F_n be the first n elements of H. Using the weak containment condition for $\{y\}$, F_n and $\varepsilon = 1/n$, we get that there exists a measurable $B_n \subseteq X$ such that

$$|\mu(B_n) - \nu(\{y\})| = |\mu(B_n) - 1/k| < 1/n,$$

and for all $\gamma \in H$, for all large enough n, we have

$$|\mu(B_n^\gamma \cap B_n) - 1/k| < 1/n,$$

that is, B_n is a non-trivial almost invariant sequence for H such that $\lim_n \mu(B_n) = 1/k$. Now, let $O_1, \ldots, O_m$ be the ergodic components of X under the action of H. Then, for all $l \leq m$, $B_n \cap O_l$ is an almost H-invariant sequence in O_l, hence, by Lemma 3.1, it has to be trivial. Since $\mu(O_l)$ is a multiple of $1/k$, we get that there exists a unique component O of measure exactly $1/k$ such that $\lim_n(O \backslash B_n) = 0$.

Now we define the map $\Phi : X \to Y$ as follows. For $x \in X$, there exists $\gamma \in \Gamma$ such that $x\gamma^{-1} \in O$. Let

$$\Phi(x) = y\gamma.$$

It is easy to check that Φ is well defined, measure preserving and Γ-equivariant. Hence g is a factor of f. □

When f is the boundary action of Γ with respect to a chain (Γ_n), then we can say more.

LEMMA 5.2. *Let f be the boundary action of Γ with respect to a chain (Γ_n) and let g be a finite action of Γ. If f is strongly ergodic and weakly contains g, then there exists n such that g is a factor of the action of Γ on Γ/Γ_n. In particular, there exist $y \in Y$ and $n \in \mathbb{N}$ such that the stabilizer of y in Γ contains Γ_n.*

Proof. By Theorem 1, g is a factor of f. Let $o \in O$ and let U_n be the H-orbit on the nth level of $T(\Gamma, (\Gamma_n))$ that o passes through. The sets (U_n) define a level-transitive boundary action of H, hence this action is ergodic and equal to the H-action on O. Note that the measure of U_n is a multiple of $1/k$ and $\mu(U_n)$ converges to $1/k$, so there exists n such that U_n has measure exactly $1/k$. But then the Γ-translates of U_n form a Γ-invariant partition, so g (which is isomorphic to the action of Γ on Γ/H) is a factor of the action of Γ on Γ/Γ_n and, for any $u \in U_n$, the stabilizer of u in Γ is contained in H. □

Let (A_n) and (B_n) be chains in Γ. We say that (A_n) *dominates* (B_n) if, for all n, there exist k and $x \in \Gamma$ such that $A_n^x \supseteq B_k$.

LEMMA 5.3. *If (A_n) dominates (B_n), then the boundary action of Γ with respect to (A_n) is a factor of the boundary action of Γ with respect to (B_n). If (B_n) also dominates (A_n), then the boundary actions are isomorphic, that is, there exists a measure- preserving Γ-equivariant homeomorphism between $\partial T(\Gamma, (A_n))$ and $\partial T(\Gamma, (B_n))$.*

Proof. Let G be the profinite completion of Γ [**27**]. Then G acts on $T(\Gamma, (A_n))$ by automorphisms and on $\partial T(\Gamma, (A_n))$ transitively by measure-preserving homeomorphisms. Let $\overline{A_n}$ be the closure of A_n in G, let $a = (A_n) \in \partial T(\Gamma, (A_n))$ and let $A = \bigcap_n \overline{A_n}$. Then A equals the stabilizer of a in G and the action of G on $\partial T(\Gamma, (A_n))$ is isomorphic to the coset space action on G/A. Let us define $\overline{B_n}$, b and B similarly, using the chain (B_n).

Let

$$O_n = \{g \in G \mid \overline{A_n^g} \supseteq B\}.$$

Then O_n is a descending chain of non-empty closed subsets in G, so it has non-trivial intersection by compactness. Thus there exists $g \in \cap O_n$ such that $A^g \supseteq B$. But then the map $F : G/B \to G/A$ defined by

$$F(Bx) = Ag^{-1}x \quad (x \in G)$$

is measure preserving, Γ-equivariant and surjective.

Now, if (A_n) and (B_n) both dominate each other, we get that A can be conjugated into B and *vice versa*. Since both A and B are closed, they must be conjugate in G (since the same is true in any finite quotient group). Hence F, defined above, is a bijection and the lemma holds. □

We are ready to prove the weak containment rigidity theorem.

Proof of Theorem 2. Let f and g be profinite actions for Γ. Since f is strongly ergodic and is weakly equivalent to g, g is strongly ergodic as well and they are both boundary actions for some chain. Let (F_n) be such a chain for f and (G_n) be such a chain for g. Let g_n be the the action of g on the nth level of $T(\Gamma, (G_n))$. Then g_n is a factor of g, so it is also weakly contained in f, so, by Lemma 5.2, it is a factor of f. We get that every G_n contains a suitable conjugate of some F_k. Similarly, every F_n contains a suitable conjugate of some G_k. Using Lemma 5.3, we obtain that the two profinite actions are isomorphic. □

Now we will construct many non-weakly comparable free boundary actions of a wide class of groups. The following lemma will be useful.

LEMMA 5.4. *Let Γ be a residually finite group and let G be its profinite completion. Let A, B be closed normal subgroups in G. Then the action of Γ on G/A is a factor of the action of Γ on G/B if and only if $A \supseteq B$.*

Proof. If $A \supseteq B$, then, from the above proof, G/A is a factor of G/B.

Assume G/A is a factor of G/B. Then there exists a chain (A_n) in Γ such that $A = \bigcap_n \overline{A_n}$. Now, $G/\overline{A_n} = \Gamma/A_n$ is a factor of G/B, and since Γ/A_n is finite and Γ is dense in G, Γ-equivariance translates to being a homomorphism. We get that $\overline{A_n} \supseteq B$, which yields $A \supseteq B$. □

Now we will start proving Theorem 3. We will need a general lemma on product actions that weakly contain a finite action and then a general theorem that produces many weakly incomparable free actions of a wide class of groups.

Let Γ be a countable group and let f and g be measure-preserving actions of Γ. Then by the product action $f \times g$ we mean the following: the underlying probability space is the product of the underlying spaces of f and g and the action of Γ on this space is the diagonal action. The following lemma is in the genre of the classical result that the product of a weakly mixing and an ergodic Z-action is ergodic.

PROPOSITION 5.1. *Let f, g be measure-preserving actions of the countable group Γ. Assume that f is strongly ergodic and profinite and g is mixing. Then $f \times g$ is ergodic. Suppose that f, g are as above and $f \times g$ is strongly ergodic and contains the finite action h. Then f contains h as well.*

Proof. Let f be a boundary action on (X, μ) associated to the chain (H_n) and let g be a mixing Γ-action on (Y, ν). Suppose that there exists an invariant subset A in $X \times Y$ such that

$$0 < \lambda = (\mu \times \nu)(A) < 1.$$

Recall that the Borel sets of $X \times Y$ can be approximated in measure by finite union of product sets and that the Borel structure of X is generated by the shadows of the H_n cosets. Hence there exists a sequence (B_n) of H_n-cylindrical sets such that

$$\lim_{n\to\infty} (\mu \times \nu)(B_n \triangle A) = 0. \tag{3}$$

Note that an H_n-cylindrical set is in the form

$$B_n = \bigcup_{x\in\Gamma/H_n} \mathrm{Sh}(x) \times T_n^x,$$

where $\mathrm{Sh}(x)$ is the shadow of the coset x and $T_n^x \subset Y$ is a Borel set. Let $J_n \subset H_n$ be the normal core of H_n, that is, the intersection of all the conjugates of H_n. Clearly, J_n stabilizes all the cosets in Γ/H_n.

LEMMA 5.5.

$$\lim_{n\to\infty} \frac{|\{x \in \Gamma/H_n \mid \lambda/10 < \nu(T_n^x) < 1 - \lambda/10\}|}{|\Gamma : H_n|}.$$

Proof. Let $\{k_i\}_{i=1}^{\infty}$ be a subset of J_n. By the mixing property,

$$\lim_{i\to\infty} (B_n k_i \triangle B_n) > \left(1 - \frac{\lambda}{10}\right) \frac{\lambda}{10} \frac{|\{x \in \Gamma/H_n \mid \lambda/10 < \nu(T_n^x) < 1 - \lambda/10\}|}{|\Gamma : H_n|}.$$

On the other hand, by (3) and the invariance of A,

$$\lim_{n\to\infty} \sup_{\gamma\in\Gamma} (\mu \times \nu)(B_n\gamma \triangle B_n) = 0.$$

Thus the lemma follows. □

Let $R_n \subset X$ be the union of the shadows of all the $x \in \Gamma/H_n$ for which $\nu(T_n^x) \geq 1 - \lambda/10$. Let $Q_n \subset X$ be the union of the shadows for which $\nu(T_n^x) \leq \lambda/10$. Clearly, $\mu(R_n)$ does not tend to zero since the measure of A is λ. Observe that the sets $\{R_n\}_{n=1}^{\infty}$ form a non-trivial almost invariant system. Indeed, it is easy to see that, for any $\gamma \in \Gamma$,

$$\tfrac{1}{2}\mu(R_n\gamma \cap Q_n) < (\mu \times \nu)(B_n\gamma \triangle B_n).$$

Since $\mu(X \backslash (R_n \cup Q_n))$ tends to 0 as n tends to ∞, the almost invariance of $\{R_n\}_{n=1}^{\infty}$ follows. This is in contradiction to the strong ergodicity of f. Therefore $f \times g$ is ergodic.

Now let us turn to the second part of our proposition. Let h be a finite action on the set Z and $z \in Z$. Let $\mathrm{Stab}_\Gamma(z) = H$. Since h is a factor of $f \times g$, an H-ergodic component of $f \times g$ has measure $1/k$. Assume that h is not a factor of f. By the proof of Theorem 1, it follows that all the H-ergodic components in X have measure larger than $1/k$. Let $O \subset X$ be such an H-ergodic component. The action on O is profinite and strongly ergodic, the H-action on Y is mixing, thus $O \times Y$ is H-ergodic. Therefore the H-ergodic components of $X \times Y$ have measure greater than $1/k$, leading to a contradiction. □

PROPOSITION 5.2. *Let $\{G_n\}$ be an infinite family of non-isomorphic, non-Abelian finite simple groups and let Γ be a dense subgroup of $G = \prod_{n=1}^{\infty} G_n$ such that the right action of Γ on G has spectral gap. Then Γ has continuously many boundary actions that are pairwise weakly incomparable. If such a Γ has property (T) then Γ has continuously many free ergodic measure-preserving actions that are pairwise weakly incomparable.*

Proof. For a subset α of the natural numbers, let

$$G_\alpha = \prod_{n \in \alpha} G_n.$$

Then G_α is a continuous image of the profinite completion of Γ with kernel K_α; let f_α denote the profinite action of Γ on G_α. Then f_α is a factor of the action of Γ on G. In particular, f_α is ergodic and has spectral gap.

Let I be a collection of continuously many infinite subsets of the natural numbers such that no two contain one another. Then, for $\alpha, \beta \in I$ with $\alpha \neq \beta$, we get that K_α is not a subset of K_β. So, by Lemma 5.4, f_α is not a factor of f_β and hence, by Theorem 2, f_α does not weakly contain f_β.

The actions f_α are not free in general. Let as assume that Γ has property (T). Let b denote a Bernoulli action of Γ on the product space $\{0, 1\}^\Gamma$. We claim that the set

$$\{f_\alpha \times b \mid \alpha \in I\}$$

consists of pairwise weakly incomparable free actions. Freeness is trivial, since the action b is free. Let $\alpha, \beta \in I$ be distinct and assume that $f_\alpha \times b \succeq f_\beta \times b$. Let g_n denote the action of f_β on the nth level of the corresponding coset tree. Then g_n is a factor of $f_\beta \times b$, so $f_\alpha \times b \succeq g_n$. Then, by Kazhdan's property (T), $f_\alpha \times b \succeq g_n$ is strongly ergodic. Hence we can apply Proposition 5.1 and get that $f_\alpha \succeq g_n$ for all n. But then $f_\alpha \succeq f_\beta$, a contradiction. So, the claim holds. □

Proof of Theorem 3. First, let Γ be a linear group with property (T). Then, by strong approximation (see [**22**, p. 401]), Γ has infinitely many pairwise non-isomorphic non-Abelian finite simple quotient groups. This gives a homomorphism of Γ to the product of these groups, and since any subdirect product of non-isomorphic non-Abelian finite simple groups equals their direct product, the image of Γ in the product is dense. By property (T), any ergodic measure-preserving action of Γ has spectral gap, so the assumptions of Proposition 5.2 hold.

Now let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. It is well known that $\mathrm{SL}(2, \mathbb{Z})$ has property (τ) with respect to its congruence subgroup chain. The boundary action associated to this chain is just the natural action of $\mathrm{SL}(2, \mathbb{Z})$ on the product of finite simple groups,

$$G = \prod \mathrm{SL}(2, q),$$

where q runs through all the prime powers except $q = 2, 3$. Therefore $\mathrm{SL}(2, \mathbb{Z})$ is a dense subgroup of G and the action has spectral gap. Also, the action is free since, if $g \in \mathrm{SL}(2, \mathbb{Z})$ is in the kernel of all the maps $\pi_p : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, p)$, then q must be the unit element. By the previous proposition, $\mathrm{SL}(2, \mathbb{Z})$ has continuously many pairwise non-weakly equivalent free ergodic actions.

Now let $H \subseteq \mathrm{SL}(2, \mathbb{Z})$ be a finite index subgroup. Observe that there exist only finitely many qs for which H does not surject onto $\mathrm{SL}(2, q)$. Indeed, the normal core N_A of A has finite index and N_A either surjects onto $\mathrm{SL}(2, q)$ or in the kernel of the quotient map $\pi_q : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, q)$. In the latter case, $|\mathrm{SL}(2, \mathbb{Z}) : N_A| \geq \mathrm{SL}(2, q)$. Therefore H acts densely on an infinite product of simple groups. Since the action of $\mathrm{SL}(2, \mathbb{Z})$ on this product has a spectral gap, by Lemma 3.3, the action of H has spectral gap as well. Therefore all the finite index subgroups of $\mathrm{SL}(2, \mathbb{Z})$ have continuously many pairwise non-weakly equivalent free ergodic actions. Now we finish the proof of the theorem by noting the well-known fact that $\mathrm{SL}(2, \mathbb{Z})$ contains all the finitely generated free groups as subgroups of finite index. □

6. *A free strongly ergodic boundary action that is not (τ)*

In this section, we first introduce covers and random covering towers, then prove Theorem 5. Let us outline the strategy. We will construct two infinite covering towers of graphs G_n and K_n. The graphs G_n and K_n will have the same vertex set ($n \geq 1$) and they will stay close in the edge metric. The tower K_n will consist of disconnected graphs, but with a large connected component that is an expander, while the tower G_n will have girth tending to infinity, but it will not be an expander family. However, using its small distance from K_n in the edge metric, we will conclude that big sets still expand in G_n. Hence the corresponding boundary action for (G_n) will be strongly ergodic but not with spectral gap.

We will find our towers by iterating two steps. In the first step, we perform a suitable random cover of G_n and K_n, which does not change the spectral gap of the large component of K_n but increases the girth of G_n. It is important to note that we use the *same* cover of G_n and K_n—this makes sense because covers can be defined using only the vertex set. Since simple random covers do not increase the girth, we will use a sequence of iterated covers that does. Friedman's theorem will control expansion. The girth of iterated random covers has already been analyzed in [**4**]; here we use a variant that is described in [**3**]. In the second step, we kill the Cheeger constant by using a specific gluing technique and thus obtain G_{n+1} and K_{n+1}.

Let S be a set of size k. By an *S-labeled graph* we mean a finite Schreier graph for the free group F_S on the alphabet S, that is, a finite directed graph where the edges are labeled by elements of S in a way such that for each vertex v and $s \in S$, there is a unique s-labeled directed edge leaving v and another one entering v. We emphasize that the label set S is not symmetric; on the contrary, the formal inverses of elements of S in F_S are not in S. When needed (especially when considering the graph as a group action), we can extend the labeling by putting a reverse edge for each s-labeled edge and labeling it by s^{-1}. Finally, when we talk about spectra or girth (the smallest length of a cycle), we forget the direction and the labels and consider the undirected graph obtained this way.

Now we will define covers for S-labeled graphs. The only non-standard thing here is that we define covers just for the underlying vertex set in such a way that it simultaneously extends to any S-labeled graph on the set.

6.1. *Covers, random covers and covering towers.* Let X be a finite set and let $d > 1$ be an integer. Let $\mathrm{Sym}(d) = \mathrm{Sym}(\{1, \ldots, d\})$ be the symmetric group on d points. Let

$$Y = X \times \{1, \ldots, d\}.$$

For a map

$$f : S \times X \to \mathrm{Sym}(d)$$

and an S-labeled graph R on X, let us define the S-labeled graph $C_f(R)$ on Y as follows. For $x \in X$, $k \in \{1, \ldots, d\}$ and $s \in S$, let

$$(x, k) \cdot s = (x \cdot s, k \cdot f(s, x)).$$

Then it is easy to check that $C_f(R)$ is an S-labeled graph and the map

$$\phi : (x, k) \mapsto x$$

extends to a d-sheeted covering from $C_f(R)$ to R.

A *random d-cover* of R is defined as $C_f(R)$, where $f : S \times X \to \mathrm{Sym}(d)$ is chosen uniformly randomly.

Let $d_1, d_2, \ldots$ be a sequence of natural numbers. A random $(d_1, d_2, \ldots, d_n)$-cover is defined recursively as follows. For $n = 1$, let it be a random d_1-cover of R and for $n > 1$ let it be a random d_n-cover of a random $(d_1, d_2, \ldots, d_{n-1})$-cover.

THEOREM 12. *Let X be a finite set and S be an alphabet of size d. Then there exists a constant $b < d$ such that for all $\varepsilon > 0$, there exist $k > 0$ and a sequence $d_1, d_2, \ldots, d_k$ of natural numbers such that the following holds. Let R be an S-labeled graph on X and let R' be a random $(d_1, d_2, \ldots, d_k)$-cover of R with the covering map $\phi : R' \to R$. Then with probability at least $1 - \varepsilon$, it holds that*

$$\mathrm{girth}(R') > \mathrm{girth}(R)$$

and

$$\lambda_1(\phi^{-1}(G)) \leq \max\{\lambda_1(G), b\}$$

for all non-trivial connected components G of R, where λ_1 denotes the second-largest eigenvalue of the adjacency matrix.

We will use two non-trivial results for the proof. The first one is essentially proved in [**3**] using the language of random automorphisms acting on an infinite rooted tree.

PROPOSITION 6.1. *Let $d_1, d_2, \ldots$ be a sequence of natural numbers such that $d_n \geq 2$ ($n \geq 1$) and let G be a finite S-labeled graph. Then, for all $\varepsilon > 0$, there exists k such that for a random $(d_1, d_2, \ldots, d_k)$-cover G' of G, the probability*

$$\mathcal{P}(\mathrm{girth}(G') > \mathrm{girth}(G)) > 1 - \varepsilon.$$

Proof. Let T be the rooted tree such that the root has $|G|$ children and every vertex of level $n > 0$ has d_n children. For each $s \in S$, assign an independent random element of the automorphism group $\mathrm{Aut}(T)$ (in Haar measure). Let G_n be the Schreier graph of the action of S on the nth level of T. Then G_n is a covering tower and the following two random variables have the same distribution for all n:

(1) a random $(d_1, d_2, \ldots, d_n)$-cover of G; and
(2) the graph G_n, conditioned on $G_1 = G$.

Now, it has been proved in [**3**] that, almost surely, the automorphisms assigned to S generate the free group F_S and, moreover, the action of F_S on the boundary of T is free. This is equivalent to saying that, almost surely, we have

$$\mathrm{girth}(G_n) \longrightarrow \infty.$$

Since $G_1 = G$ with positive probability and the girth is non-decreasing for any covering tower, we get that for all $K > 0$, the probability

$$\mathcal{P}(\mathrm{girth}(G_n) > K) \longrightarrow 1$$

as $n \to \infty$, and so the proposition is proved. □

Remark. It is worth noting that a single random cover does not increase the girth almost surely (as the degree of the cover tends to infinity). Indeed, a random cover of the trivial graph (a vertex with a loop) is just a random permutation, which has a fixed point with probability bounded away from 0.

The second result we will use for Theorem 12 is due to Friedman [**12**, Theorem 1.2]. Let the finite graph H cover the graph G. Then, trivially, all the eigenvalues of the adjacency matrix of G are also eigenvalues for H. These are called the old eigenvalues of the covering map, and the rest of the eigenvalues are called the new ones.

PROPOSITION 6.2. (Friedman) *Let G be a fixed graph, let λ_0 denote the largest eigenvalue of G and let ρ denote the spectral radius of the universal cover of G. Let $R_n(G)$ denote a uniform random n-fold covering of G. Then there exists a positive function $\alpha(n)$, where $\alpha(n) \to 0$ with $n \to \infty$, such that the probability that $H \in R_n(G)$ has all its new eigenvalues inside the interval*

$$[-\sqrt{\lambda_0 \rho} - \alpha(n), \sqrt{\lambda_0 \rho} + \alpha(n)]$$

goes to 1 *as $n \to \infty$.*

Proof of Theorem 12. For d-regular graphs, the parameters in Friedman's theorem give $\lambda_0 = d$ and $\rho = 2\sqrt{d-1}$. This gives

$$\sqrt{\lambda_0 \rho} = \sqrt{2d\sqrt{d-1}},$$

which is bounded away from d. Let $a = d - \sqrt{\lambda_0 \rho}$ and let $b = d - a/2$. Now, using Friedman's theorem, we get that there exists d_1, such that with probability at least $1 - \varepsilon/4$, a uniform random d_1-cover of any S-labeled graph G on $X_0 = X$ will have all its new eigenvalues inside $[-b, b]$. Let $X_1 = X_0 \times \{1, \ldots, d_1\}$ be the new underlying set. Iterating this, we get that there exists d_k, such that with probability at least $1 - \varepsilon/2^{k+1}$, a uniform random d_1-cover of any S-labeled graph G on X_{k-1} will have all its new eigenvalues inside $[-b, b]$. Let $X_k = X_{k-1} \times \{1, \ldots, d_k\}$ be the new underlying set.

This will give us an infinite sequence $d_1, d_2, \ldots$ that satisfies the following. For an S-labeled graph R on X, let R_k be the random $(d_1, d_2, \ldots, d_k)$-cover of R. Then with probability at least $1 - \varepsilon/2$, for any such R, all the new eigenvalues of any of the R_k will be inside $[-b, b]$. In particular, for all connected components G of R, we have

$$\lambda_1(\phi^{-1}(G)) \leq \max\{\lambda_1(G), b\}.$$

Now let us use Proposition 6.1 with the sequence $d_1, d_2, \ldots$ and $\varepsilon/2$. We get that there exists k such that with probability at least $1 - \varepsilon/2$, for any S-labeled graph R on X, the $(d_1, d_2, \ldots, d_k)$-cover R' of R will have larger girth than R.

Putting the two probabilities together, the theorem is proved. □

6.2. *Gluing step.* Let G, P_1, P_2 be S-labeled graphs with covering maps

$$\pi_i : P_i \to G \quad (i = 1, 2).$$

Let $s \in S$ and $p_1 \in P_1$, $p_2 \in P_2$ such that $\pi_1(p_1) = \pi_2(p_2)$. Let the S-labeled graph P be defined as follows. First, take the disjoint union of P_1 and P_2. Let $\pi : P \to G$ be the union of π_1 and π_2. Now erase the s-labeled edges $(p_1, p_1 s)$ and $(p_2, p_2 s)$ and glue in the s-labeled edges $(p_1, p_2 s)$ and $(p_2, p_1 s)$.

LEMMA 6.1. *Assume that* $\mathrm{girth}(P_i) > 2$ *($i = 1, 2$). Then P is an S-labeled graph, π is a covering map, and we have*

$$\mathrm{girth}(P_i) \geq \min(\mathrm{girth}(P_1), \mathrm{girth}(P_2))$$

and

$$\mathrm{Ch}(P) \leq \frac{2}{\min(|P_1|, |P_2|)}.$$

Proof. It is easy to check that π is a covering map. Since $\mathrm{girth}(P_i) > 2$, both of the removed edges are in an s-labeled cycle of length at least 3, hence, by removing the edges, P_i stays connected and the new edges make the whole P connected. Thus P is an S-labeled graph. A cycle in P either stays in one component, or, by putting back the old edges, it becomes the disjoint union of two cycles, hence its size is at least $\mathrm{girth}(P_1) + \mathrm{girth}(P_2)$. The estimate on the Cheeger constant follows trivally by considering the partition $P_1 \cup P_2$. □

Let G and H be graphs on the same vertex set X. Then their edit distance is defined as

$$d_e(G, H) := \frac{|E(G) \triangle E(H)|}{|X|}.$$

Let $f : S \times X \to \mathrm{Sym}(d)$ as above and G, H be S-labeled graphs on X. Then, by definition, $d_e(G, H) = d_e(C_f(G), C_f(H))$. Recall that for a finite graph G,

$$\mathrm{Ch}(G) = \sup_{0<|A|\leq\frac{1}{2}|V(G)|, A\subset V(G)} \frac{|L(A)|}{|A|},$$

where $L(A)$ is the set of edges between A and its complement. Clearly, (G_n) is an expander sequence if and only if $\liminf_{n\to\infty} \mathrm{Ch}(G_n)$. It is well known that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\lambda_0(G) - \lambda_1(G) > \epsilon$ for a d-regular graph G, then $\mathrm{Ch}(G) > \delta$.

Let G_1 be an arbitrary d-regular connected F_S-labeled graph ($d = 2|S|$). Let q be its second-largest eigenvalue. Now let b be the constant in Theorem 12, and let $\delta > 0$ be such a number that $\mathrm{Ch}(G) > \delta$ if G is a connected, d-regular graph with second-largest eigenvalue not greater than $\max(q, b)$.

LEMMA 6.2. *There exist two covering towers of F_S-Schreier graphs*

$$G_1 \leftarrow G_2 \leftarrow \cdots \quad \text{and} \quad K_1 \leftarrow K_2 \leftarrow \cdots$$

such that the following properties are satisfied.

- $G_1 = K_1$ *is a connected graph.*
- G_n *and* K_n *are defined on the same vertex set and* $d_e(G_n, K_n) < \delta/50$.
- $\mathrm{girth}(G_n) \to \infty$, $\mathrm{girth}(K_n) \to \infty$.
- $\mathrm{Ch}(G_n) \to 0$.
- *If* $T_1 \subset K_1, T_2 \subset K_2, \ldots$ *are the largest components, then* $\mathrm{Ch}(T_n) > \delta$ *and* $|T_n|/|X_n| > 1 - \delta/100d$ *for any* $n \geq 1$.

Before proving the lemma, let us see how it implies Theorem 5.

Proof of Theorem 5. Since $\mathrm{Ch}(G_n) \to 0$, the boundary action associated to the covering tower does not have a spectral gap. In order to prove that the action is strongly ergodic, it is enough to see that if $A_n \subset X_n$ and

$$\tfrac{1}{4}|X_n| \leq |A_n| \leq \tfrac{1}{3}|X_n|,$$

then there exists at least $(\delta/20)|X_n|$ edges between A_n and its complement in G_n. Observe that

$$\frac{|X_n|}{10} \leq |A_n \cap T_n| \leq \frac{1}{2}|T_n|.$$

Hence there are at least $\delta|X_n|/10$ edges between $A_n \cap T_n$ and its complement in T_n. Since $|K_n \backslash T_n| < (\delta/100d)|X_n|$ and $|E(G_n) \triangle E(K_n)| < (\delta/50)|X_n|$, we obtain that $|E(K_n) \triangle E(T_n)| < (\delta/20)|X_n|$. Therefore there are at least $(\delta/20)|X_n|$ edges between A_n and its complement in K_n. □

Proof of Lemma 6.2. We construct the towers inductively. Suppose we have already constructed G_n and K_n. Then we pick some iterated coverings $\kappa_{n+1} : M_{n+1} \to K_n$, respectively $\kappa_{n+1} : L_{n+1} \to G_n$ of K_n, respectively G_n (on the same vertex set using the same $\mathrm{Sym}(d_i)$-valued functions) in such a way that:

- $\mathrm{girth}(M_{n+1}) > \mathrm{girth}(K_n)$;
- $\mathrm{girth}(L_{n+1}) > \mathrm{girth}(G_n)$;
- L_{n+1} is connected; and
- $\lambda_1(\kappa_{n+1}^{-1}(G)) \leq \max\{\lambda_1(G), b\}$ for any connected component of K_n. Particularly, each $\kappa_{n+1}^{-1}(G)$ is connected.

The existence of such a construction easily follows from Proposition 6.1 and Proposition 6.2. Now we pick other coverings $\kappa'_{n+1} : M'_{n+1} \to K_n$, respectively $\kappa'_{n+1} : L'_{n+1} \to G_n$, satisfying the same properties. We choose M'_{n+1} so large that the size of the greatest component of M'_{n+1} is still larger than $(|M_{n+1}| + |M'_{n+1}|)(1 - \delta/100d)$. Now let K_{n+1} be the disjoint union of M_{n+1} and M'_{n+1} and let G_{n+1} be the union of L_{n+1} and L'_{n+1} glued together. It is easy to see that if M'_{n+1} is large enough, then $d_e(G_{n+1}, K_{n+1})$ is still smaller than $\delta/50$. □

7. *Subgroups and property (τ)*

Let us first outline the contents of this section using graph theoretical language. Let S be a finite alphabet and let G be an S-labeled graph (see the previous section for the definition). Label the inverses of edges by formal inverses of elements of S. Now fix a symmetric set of words $w_1, w_2, \ldots, w_n \in F_S$. Then we can define a new graph G' on V by drawing a w_i-labeled edge from $v \in V$ to $v \cdot w_i$ ($v \in V$, $1 \leq i \leq n$). This section investigates how the expansion of G' is related to the expansion of G.

If $w_1, w_2, \ldots, w_n$ generate F, then it is easy to see that the graph metric on G' is bi-Lipschitz to the one on G with a bounded Lipschitz constant, and hence G' is also connected and expansion is distorted in a bounded way. When $H = \langle w_1, w_2, \ldots, w_n \rangle$ is a proper subgroup of finite index in F, G' may or may not stay connected. We shall present an example for a sequence of graphs G_n, where H has index 2, G'_n stays connected, but expansion vanishes. Surprisingly, however, when G_n comes from a chain of subgroups, or a family of normal subgroups, expansion stays bounded away from zero—of course, in the light of the previous claim, for chains, the bound is not absolute. This directly leads us to answering the question of Lubotzky and Zuk.

We start with the construction of 'bad' S-labeled graphs. As an input, we use a (τ) chain in F_2. These exist, by various arguments (see [**21, 23**]); one of the constructions is outlined at the end of §5.

7.1. *Construction of bad Schreier graphs.*

Let F_2 be generated by x_1 and x_2. Let (H_n) be a chain in F_2 with property (τ). Let C denote the cyclic group of 2 elements generated by t, let $\Delta = F_2 \times C$ and let $H'_n = H_n \times 1 \leq \Delta$. Let

$$E_n = \mathrm{Sch}(\Delta / H'_n, \{x_1, x_2, t\}).$$

Then E_n is a union of two subgraphs $E_{n,1}$ and $E_{n,t}$, both isomorphic to

$$\mathrm{Sch}(F_2/H_n, \{x_1, x_2\})$$

plus the action of t, which is a perfect matching between the two subgraphs. Now we introduce a new generator c that acts on the vertex set of E_n as follows. Let $e_{n,1}, e_{n,2} \in E_{n,1}$ and let $e_{n,3} = e_{n,2} \cdot t \in E_{n,t}$. Let

$$c = (e_{n,1}, e_{n,2}, e_{n,3})$$

be the 3-cycle moving only these points. Let G_n be E_n plus the additional c-edges.

Let Γ be the free group on the generating set $\{x_1, x_2, t, c\}$. Then Γ acts transitively on G_n. Let

$$\Gamma_n = \mathrm{Stab}_\Gamma(e_{n,2}).$$

By transitivity, we have

$$G_n = \mathrm{Sch}(\Gamma / \Gamma_n, \{x_1, x_2, t, c\}).$$

Note that Γ_n is not a chain anymore, as c ruins G_n being a covering tower.

Let H be the kernel of the projection $\phi : \Gamma \to C$ defined by $\phi(x_1) = \phi(x_2) = \phi(c) = 1$ and $\phi(t) = t$. Then H is a normal subgroup of Γ of index 2 and, by the Nielsen–Schreier theorem, it is generated by

$$T = \{x_1, x_2, c, tx_1t^{-1}, tx_2t^{-1}, tct^{-1}, t^2\}.$$

PROPOSITION 7.1. *The family Γ_n has property (τ) in Γ but the family $H \cap \Gamma_n$ does not have property (τ) in H.*

Proof. The sequence E_n is an expander family, hence G_n is an expander family as well. Thus the family Γ_n has property (τ) in Γ.

For all $n \geq 1$, the element ct^{-1} fixes $e_{n,2}$, so $ct^{-1} \in \Gamma_n$ but $ct^{-1} \notin H$. This shows that $\Gamma_n \nleq H$. Since H has index 2 in Γ, we get $\Gamma_n H = \Gamma$, and so $\mathrm{Sch}(H/H \cap \Gamma_n, T)$ is isomorphic to $\mathrm{Sch}(\Gamma/\Gamma_n, T)$. Moreover, in this action we have

$$x_1 = tx_1t^{-1}, \quad x_2 = tx_2t^{-1} \quad \text{and} \quad t^2 = 1.$$

Let us look at the set $E_{n,1} \subseteq \Gamma/\Gamma_n$. Both x_1 and x_2 fix $E_{n,1}$ as a set, so there are exactly 4 edges in $\mathrm{Sch}(\Gamma/\Gamma_n, T)$ that leave $E_{n,1}$, that is, the edges coming from c and tct^{-1}. This implies that the Cheeger constant satisfies

$$\mathrm{Ch}(\mathrm{Sch}(H/H \cap \Gamma_n, T)) \leq \frac{4}{|E_{n,1}|} = \frac{2}{|\Gamma : \Gamma_n|}.$$

Hence $\mathrm{Sch}(H/H \cap \Gamma_n, T)$ is not an expander family in H and so the family $H \cap \Gamma_n$ does not have property (τ) in H. □

We are ready to prove Theorem 6. Note that we are not aware of any proof that does not use compactness in some form; the fact that there are no bounds on how badly expansion can be distorted makes it dubious that such proof exists. Even for normal chains, where by Theorem 7 there is an explicit lower bound on distortion, the only other proof we know [**26**] uses invariant means.

Proof of Theorem 6. Let S be a finite symmetric generating set for Γ and let $k = |\Gamma : H|$. Let $T = T(\Gamma, (\Gamma_n))$ be the coset tree and let $t_n \in T$ be the vertex representing the subgroup Γ_n. Then $\{t_n\}$ forms a ray in T since (Γ_n) is a chain. Let $t \in \partial T$ denote this ray as a boundary point. Let T_n denote the nth level of T. Let O_n be the orbit of t_n in T_n under the action of H. Then the permutation action of H on O_n is isomorphic to the coset action of H on $H/H \cap \Gamma_n$. Also, the union of O_n forms a subtree that is isomorphic to the coset tree $T(H, (H \cap \Gamma_n))$ and the limit of the O_n equals the ergodic component of ∂T under the action of H that contains t. Let us call this component O. Now, (Γ_n) has property (τ) in Γ, so, by Lemma 2.2, the action of Γ on ∂T has spectral gap. Now, using Lemma 3.3, we get that the action of H on O also has spectral gap. But the action of H on O is isomorphic to the boundary action of H with respect to $(H \cap \Gamma_n)$, so, again by Lemma 2.2, $(H \cap \Gamma_n)$ has property (τ) in H. □

Theorem 6 has been proved for normal chains by Shalom [**26**] using invariant means. Theorem 7 allows us to extend his result to arbitrary families of normal subgroups, as stated in Theorem 8.

Proof of Theorem 8. Let S be a finite symmetric generating set for Γ, let $k = |\Gamma : H|$, let C be a coset representative system for H in Γ and let $T = N(S, C)$. Let $G_n = \Gamma/\Gamma_n$. Then G_n is a compact (in fact, finite) topological group and the image of Γ in G_n is dense (being equal to G_n). Let O_n be the orbit of H in G_n containing the identity of G_n. Then we can invoke Theorem 7 and get

$$\mathrm{h}(O_n, T) > \frac{1}{8k^{3-\log_2 3}} \min\left\{ \frac{\mathrm{h}(G_n, S)}{k^2}, 1 \right\}.$$

In particular, since $\mathrm{h}(G_n, S)$ is bounded below and k is fixed, the family of Cayley graphs $\mathrm{Cay}(H/H \cap \Gamma_n, T)$ $(n \geq 1)$ is an expander family and so the theorem holds. □

Finally, Proposition 7.1 and Theorem 6 together allow us to answer the question of Lubotzky and Zuk.

Proof of Corollary 9. Let $\Gamma = F_4$ and let $H \leq \Gamma$ and $\Gamma_n \leq \Gamma$ $(n \geq 1)$ be defined as in the construction above. Then, by Proposition 7.1, the family Γ_n has property (τ). For $n \geq 1$, let $H_n = \bigcap_{i=1}^n \Gamma_i$. Assume that the chain (H_n) has property (τ). Then, using Theorem 6, the chain $(H \cap H_n)$ has property (τ) in H. But $H \cap H_n \leq H \cap \Gamma_n$ $(n \geq 1)$, which implies that the family $H \cap \Gamma_n$ $(n \geq 1)$ also has property (τ) in H. This contradicts Proposition 7.1.

Hence the chain (H_n) does not have property (τ) and the corollary is proved. □

8. *Almost covers of graphs and the distance from being bipartite*

For general unlabeled graphs, weak containment translates as follows.

Definition 8.1. Let G and H be finite k-regular graphs. A map $f : E(G) \to E(H)$ is an *ε-covering* if f is surjective and there exists $X \subseteq V(G)$ with $|X| > (1 - \varepsilon)|V(G)|$ such that, for all $x \in X$, there exists $y \in V(H)$ such that f is a bijection between the set of edges leaving x and the set of edges leaving y.

So, f is a local isomorphism at most vertices of G. Note that for $\varepsilon = 0$, we get back the original notion of a finite sheeted covering map and, by our definition, y is a unique function of x, that is, f induces a map $V(G) \to V(H)$. It is easy to see that if H is connected, then every vertex in H has the same number of preimages.

A sequence of finite graphs (G_n) *almost covers* a finite graph H if, for all $\varepsilon > 0$, there exists n_0 such that, for all $n > n_0$, G_n has an ε-covering to H.

By a *covering tower of graphs*, we mean a sequence (G_n, f_n) of graphs and maps such that, for all $n \geq 1$, f_n is a covering map from G_{n+1} to G_n. Let (G_n, f_n) be a covering tower of connected k-regular graphs. Then we define the *covering tree* $T = T(G_n)$ as follows. Let the vertex set of T be the disjoint union of the $V(G_n)$ and, for all $n > 1$ and $x \in V(G_n)$, connect x to its image under the covering map. Then T is a spherically homogeneous rooted tree. Let $\partial(G_n) = \partial T$ be the boundary of the tree, that is, the set of infinite rays in T, endowed with the product topology and measure. The boundary ∂T is naturally endowed with a graph structure: we connect $(x_n), (y_n) \in \partial T$ if x_n and y_n are connected in G_n for every n. This gives us a k-regular graphing, which we call the *boundary graphing* of (G_n, f_n) and denote by $\partial(G_n, f_n)$. By composing covering maps and taking a limit, we get a continuous covering map from the boundary graphing to G_n.

We are ready to prove Theorem 10 after a lemma that is folklore in graph theory.

LEMMA 8.1. *Let G be a finite undirected k-regular graph and let S be an alphabet on k letters. Then G can be turned into an S-labeled graph such that every edge of G is used exactly once in each direction.*

Proof. Let A be the adjacency matrix of G. Then let us look at A as the adjacency matrix of a bipartite graph obtained by doubling the vertices of G. It is k-regular, so it is a disjoint union of k perfect matchings, that is, A is the sum of k permutation matrices. Let us label the directed edges of G according to these permutations. This gives the required decomposition. □

Using Lemma 8.1 and putting in formal inverses of elements of S, one can turn a k-regular graph G to a Schreier graph for F_S such that each edge is used exactly twice by the generators and its inverses (in each direction). Note that if the directed edge (x, y) is labeled by s and (y, x) is labeled by t, then for the associated F_S-action $xs = y$ and $xt^{-1} = y$.

Proof of Theorem 10. Let (G_n) be an expanding covering tower of graphs. Consider the associated F_S-action on G_1. Using the covering maps, this defines an F_S-action on all the G_n by pulling back the Schreier graphs according to the covering maps. Then we obtain a F_S-chain with boundary action on ∂T. Let us consider the homomorphism $\phi : F_S \to C = \{1, t\}$, where $\phi(s) = t$ for all the generators. Let H be the kernel of ϕ, a subgroup of index 2. Observe that the F_S-action f on ∂T has a spectral gap since (G_n) is an expander system, that is, f is strongly ergodic. Now consider ∂T as an H-space.

Case 1. Suppose that the H-action on ∂T is ergodic. Then, by Lemma 3.3, it has a spectral gap. Let g be the F_S-action on the set $\{1, t\}$ induced by ϕ. By the ergodicity assumption, g is not a factor of f. Hence, by Theorem 1, f does not contain g weakly. Let $r_n|V(G_n)|$ be the minimal number of edges one needs to erase to make G_n bipartite (with partition sets A_n,B_n). Clearly, $r_1 \geq r_2 \geq \cdots$. Suppose that $\lim_{n\to\infty} r_n = 0$. Let C_n be the shadow of A_n and D_n be the shadow of B_n. It is easy to see that $\mu(C_n) \to 1/2$, $\mu(D_n) \to 1/2$ and, for any $\gamma \in H$, $\mu(C_n\gamma \cap C_n) \to 0$ and $\mu(D_n\gamma \cap D_n) \to 0$. Hence f weakly contains g, leading to a contradiction. Therefore $\lim_{n\to\infty} r_n > 0$.

Case 2. There exists an H-ergodic component O of size 1/2. Similarly, as in Lemma 5.2, this implies that if n is larger than some constant n_k, there are exactly two H-orbits on the nth level, that is, G_n is bipartite if $n > n_k$. □

Theorem 1 suggests the following problem.

Problem 2. Let (G_n) be an expanding covering tower of k-regular graphs and H a finite graph such that (G_n) almost covers H. Does it follow that there exists n such that G_n covers H?

By Theorem 10, the answer is affirmative when H is a graph with two points and k edges going between them.

In spectral language, Theorem 10 takes the following equivalent form. For a k-regular undirected graph G on v vertices, let $\lambda_0(G) \geq \lambda_1(G) \geq \cdots \geq \lambda_{v-1}(G) = \lambda_-(G)$ denote the eigenvalues of the adjacency matrix of G. Then $\lambda_0(G) = k$ and $\lambda_-(G) \geq -k$. Assuming that G is connected, $\lambda_-(G) = -k$ if and only if G is bipartite.

COROLLARY 13. *Let (G_n) be a covering tower of non-bipartite k-regular graphs. If $\lambda_1(G_n)$ is bounded away from k, then $\lambda_-(G_n)$ is bounded away from $-k$.*

Proof. Let $G(V, E)$ be a finite d-regular connected graph. If S is a subset of V, then let $e(S)$ be the minimal number of edges to be removed from the graph spanned by S to make it bipartite. Let $k(S)$ be the number of edges to be removed to disconnect S from $V - S$. Let $r(G) := e(G)/|V|$ and $c(G) := \min_{S\subset V, |S|\leq \frac{1}{2}|V|} k(S)/|S|$. Desai and Rao [**9**] introduced the constant

$$\psi(G) = \min_{S\subset V} \frac{e(S) + k(S)}{|S|}$$

and proved (Theorem 3.2) that for the smallest eigenvalue of the adjacency matrix of G, $q_n(G)$ satisfies

$$q_n(G) \geq -d + \frac{\psi^2(G)}{4d}. \qquad \square$$

LEMMA 8.2. *We have*

$$\psi(G) \geq \min\left\{c(G), \frac{r(G)c(G)}{2d}, \frac{r(G)}{4}\right\}.$$

Proof. Let $w(S) = (e(S) + k(S))/|S|$. First, let $|S| \leq |V|/2$, and then $c(G) \leq w(S)$. Now let $|V|/2 \leq |S| \leq (1 - r(G)/2d)|V|$. Then $k(S) \geq r(G)c(G)|V|/2d$, that is, $w(S) \geq r(G)c(G)/2d$. Finally, let $(1 - r(G)/2d)|V| \leq |S| \leq |V|$. Then the number of edges in the span of $V - S$ is at most $r(G)|V|/4$ and the number of edges between S and $V(S)$ is at most $r(G)|V|/2$. Hence, in order to make S bipartite, one needs to remove at least $r(G)|V|/4$ edges. Otherwise, one can make G bipartite by removing less than $e(G)$ edges. Consequently, $w(S) \geq r(G)/4$. This ends the proof of our lemma. $\square$

Trivially, all these results are far from being true for an arbitrary expander sequence of k-regular graphs.

Remark. A standard example for a sequence of finite k-regular graphs where the girth (the minimal size of a cycle) tends to infinity and the independence ratio is bounded away from 1/2 is due to Bollobás [**5**], who showed that large random k-regular graphs satisfy these properties. Now, Theorem 10 allows us to find these sequences in abundance. Indeed, take the free product $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ (with k factors), or, alternatively, for an even $k \geq 4$, the free group $\Gamma = F_{k/2}$. Let S be a standard generating set of Γ and let N be the kernel of the homomorphism $\Gamma \to \mathbb{Z}/2\mathbb{Z}$ that sends all elements of S to the non-trivial element. Then, by Theorem 10, for any chain (Γ_n) in Γ which has property (τ) and satisfies $\Gamma_n \not\leq N$ for all n, the sequence of Schreier graphs $\mathrm{Sch}(\Gamma/\Gamma_n, S)$ will have an independence ratio bounded away from 1/2.

9. *Amenable groups and free groups*

In this section, we discuss weak containment in the realm of amenable groups and then apply the result for free groups. We also show how to derive a recent theorem of Conley and Kechris on the maximal measure of independent subsets for measure-preserving actions.

LEMMA 9.1. *Let Γ be an amenable group and let g be a measure-preserving action of Γ on a finite set. Then every free measure-preserving ergodic action of Γ weakly contains g.*

Proof. It is known (see [**18**, 13.2] and [**11**]) that any two free measure-preserving actions of an amenable group are weakly equivalent. Let $b = \{0, 1\}^{\Gamma}$ denote the standard Bernoulli action of Γ and let $g_0 = g \times b$. Then g_0 is weakly contained in any measure-preserving free action, so the same holds for its factor g. □

LEMMA 9.2. *Let Γ be a countable group and let f be a measure-preserving action of Γ. Let (Γ_n) be a chain in Γ, let g_n be the coset action of Γ on Γ/Γ_n and let g be the boundary action of Γ with respect to (Γ_n). Then f weakly contains g if and only if f weakly contains g_n for all n.*

Proof. Since g_n is a factor of g, if f weakly contains g, then it also weakly contains g_n for all n. In the other direction, every finite measurable partition of the underlying measure space of g can be approximated by partitions of the underlying sets of g_n, projected to the boundary of the coset tree of g with arbitrarily small error. Hence, if f weakly contains all the g_n, then it can simulate any partition of the underlying set of g as well, and so it weakly contains g. □

Proof of Theorem 11. Let F be a free group of rank d and let p be a prime. Let $\mathcal{N}$ be the set of normal subgroups of F with finite p-power index and let $\mathcal{K}$ be the set of normal subgroups of F, where the quotient group is finite and solvable. For $l > 0$ let $\mathcal{K}_l \subset \mathcal{K}$ consist of normal subgroups, where the quotient group has derived length at most l, and let $\mathcal{N}_l = \mathcal{N} \cap \mathcal{K}_l$. Let G denote the inverse limit of F with respect to $\mathcal{N}$; G is called the pro-p-completion of F. Let g denote the left action of F on G. Since F is residually a p-group, g is ergodic and free. Similarly, let S denote the pro-(finite solvable)-completion of F, that is, the inverse limit of F with respect to $\mathcal{K}$, and let s denote the left action of F on S.

Since every finite p-group is solvable, $\mathcal{N}$ is a subset of $\mathcal{K}$ and so g is a factor of s. In particular, s weakly contains g.

In the other direction, let h be a finite action of F with solvable image. Let l be the derived length of the image of h and let $\Gamma = F/F^{(l)}$ be the free solvable group of derived length l, where $F^{(l)}$ denotes the lth element of the derived series of F. Let G_l denote the inverse limit of F with respect to $\mathcal{N}_l$ and let g_l denote the left action of F on G_l. Let S_l denote the inverse limit of F with respect to $\mathcal{K}_l$ and let s_l denote the left action of F on S_l. It is easy to see that $F^{(l)} \leq \mathrm{Ker}(s_l) \leq \mathrm{Ker}(g_l)$, so, in fact, s_l and g_l can also be regarded as Γ-actions. Again, g_l is a factor of s_l. Now, by a result of Gruenberg [**15**], Γ is residually p, which implies that g_l (and hence s_l) are free as Γ-actions. Since Γ is amenable, g_l weakly contains s_l as a Γ-action. But then g_l weakly contains s_l as an F-action as well. Now h is a factor for s_l, so g_l weakly contains h. But then g weakly contains h as well, since g_l is a factor of g. Since F is finitely generated, it has finitely many subgroups of a given index. Hence $\mathcal{K}$ is countable, and so it is generated by a chain in F. In particular, s is a boundary action with respect to a chain in $\mathcal{K}$. Using Lemma 9.2, g weakly contains s.

The theorem is proved. □

Remark. One can ask whether the whole profinite completion of a finitely generated free group is weakly equivalent to its pro-p-completion. To prove this, it would suffice to show the following: if F is a finitely generated free group and N is a normal subgroup of finite index in F, then there exists a normal subgroup $K \leq N$ in F such that F/K is amenable and residually p.

Now we present how to derive the following recent results of Conley and Kechris [**7**, Theorems 0.5 and 0.6], using the language established in this paper. For a measure-preserving action a of Γ on (X, μ) and a finite generating set S of Γ, we call a subset $Y \subseteq X$ S-independent, if, for all $y \in Y$ and $s \in S$, $ys \notin Y$. Let $i(S, a)$ denote the supremum of μ-measures of S-independent Borel subsets. In the same way, we call a c-coloring $f : X \to \{1, \ldots, c\}$ S-legal, if, for all $x \in X$ and $s \in S$, $f(x) \neq f(xs)$. Let $\varkappa(S, a)$ denote the minimal c such that X has an S-legal c-coloring.

THEOREM 14. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite symmetric set of generators with* $\mathrm{Cay}(\Gamma, S)$ *bipartite. Then the following are equivalent.*
(i) Γ *is amenable.*
(ii) $i(S, a)$ *is constant for any free, measure-preserving action a of* Γ.
(iii) $i(S, a) = 1/2$ *for any free, measure-preserving action a of* Γ.
(iv) $\varkappa(S, a)$ *is constant for any free, measure-preserving action a of* Γ.
(v) $\varkappa(S, a) = 2$ *for every free, measure-preserving action a of* Γ.

THEOREM 15. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite symmetric set of generators with* $\mathrm{Cay}(\Gamma, S)$ *bipartite. Then the following are equivalent.*
(i)′ Γ *has property (T).*
(ii)′ $i(S, a) < 1/2$ *for any free, measure-preserving, weakly mixing action a of* Γ.
(iii)′ $\varkappa(S, a) \geq 3$ *for every free, measure-preserving, weakly mixing action a of* Γ.

Proof. Since $\mathrm{Cay}(\Gamma, S)$ is bipartite, Γ acts on the two-point set such that no element of S fixes a point. Let us call this action g and its kernel N. Let f be the Bernoulli action of Γ on $\{0, 1\}^\Gamma$ endowed with the product measure. Then g is not a factor of f since the action of N on $\{0, 1\}^\Gamma$ is isomorphic to $\{0, 1, 2, 3\}^N$ and hence is ergodic. Let h be the induced action of the Bernoulli action of N on $\{0, 1\}^N$ to Γ. Then h factors on g, so $i(S, h) = 1/2$ and $\varkappa(S, h) = 2$.

If Γ is amenable, then any two free, measure-preserving, ergodic actions of Γ are weakly equivalent. Hence (ii) holds and the constant has to be $1/2$ by considering h. So, (ii), (iii), (iv) and (v) all hold. If Γ is non-amenable, then, by [**20**], f is strongly ergodic, so, by Theorem 1, f does not even weakly contain g. In particular, $i(S, f) < 1/2$ and $\varkappa(S, f) > 2$. Again considering h, we see that (ii), (iii), (iv) and (v) all fail.

If Γ has property (T), then, by [**25**], any free, measure-preserving, ergodic action a of Γ is strongly ergodic, and, by weak mixing, the restriction of a to N stays ergodic, so a does not factor on g. Hence, by Theorem 1, a does not even weakly contain g. In particular, $i(S, a) < 1/2$ and $\varkappa(S, a) \geq 3$. So, both (ii)′ and (iii)′ hold. If Γ does not have property (T), then, by [**13**], there exists a free, measure-preserving, weakly mixing action a of Γ that is not strongly ergodic. By weak mixing, a does not factor on g, hence the restriction of a to N stays ergodic, but not strongly ergodic, and so, by Schmidt's lemma, it weakly contains $\frac{1}{2}\mathrm{Id}_N + \frac{1}{2}\mathrm{Id}_N$, which is equivalent to saying that a weakly contains g. So, both (ii)′ and (iii)′ fail. □

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