Mesoscopic Fluctuations of Conductance of a Helical Edge Contaminated by Magnetic Impurities

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Elastic backscattering of electrons moving along the helical edge is prohibited by time-reversal symmetry. We demonstrate, however, that an ensemble of magnetic impurities may cause time-reversal symmetry-preserving quasielastic backscattering, resulting in interference effects in the conductance. The characteristic energy transferred in a backscattering event is suppressed due to the Ruderman-Kittel-Kasuya-Yosida interaction of localized spins (the suppression is exponential in the total number of magnetic impurities). We predict the statistics of conductance fluctuations to differ from those in the conventional case of a one-dimensional system with quenched disorder.

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A two-dimensional topological insulator is defined by the presence in its electronic spectrum of helical edge states protected against elastic backscattering by time-reversal symmetry (TRS). One fundamental consequence, also perceived as a smoking-gun experimental signature of a topological insulator, is the universality of its zero-temperature conductance. Actual experiments clearly distinguish between topological insulators and highly resistive "conventional" ones [1]. However, the measured conductance approaches the universal value only in very short (less than 1 μm long) samples. In longer samples the conductance is suppressed, presumably by electron backscattering.

Mechanisms of such backscattering are a matter of ongoing debate. Current proposals involve the Coulomb interaction between the electrons [2–6] and electron scattering off a localized magnetic impurity [7–9]. In these theoretical models the electron backscattering is either deeply inelastic [2–6] or, as in the case of a magnetic impurity at temperatures exceeding the Kondo scale, quasielastic and incoherent [7,8]. As a result, none of the proposed mechanisms can lead to quantum interference effects (for example, mesoscopic fluctuations of the sample conductance G) ubiquitous in other conductors.

Here we show that the conductance of a helical edge can, in fact, display pronounced quantum interference effects. This requires the sample to be contaminated with magnetic impurities of large spin ($S > 1$) experiencing uniaxial single-ion anisotropy. An ensemble of such impurities, coupled by the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction, forms a rigid block seen as one composite scatterer by an electron at the edge. The form factor of such an extended object contains interference terms leading to mesoscopic fluctuations in the backscattering rate as a function of the electron energy. Respecting the TRS, the electron scattering remains inelastic; however, the associated energy transfer is strongly suppressed setting a very mild lower bound for the temperature at which the considered mechanism is effective.

In this work we focus on the interference-induced mesoscopic fluctuations of the conductance of a helical edge as a function of the chemical potential of electrons. In conventional one-dimensional wires such fluctuations arise from rapidly varying interference conditions in the electron scattering amplitudes and are pronouncedly non-Gaussian. In particular, when backscattering is weak they obey the Rayleigh statistics. In contrast, we find that the fluctuations of G caused by an ensemble of magnetic impurities are nearly Gaussian except for temperatures close to the spin-glass crossover.

We consider a topological insulator with a simple helical edge [1,2] such that the spin $\mathbf{s}$ of an electron occupying an edge state has a conserved component s_z in some fixed direction $\hat{z}$. A magnetic impurity in the vicinity of the edge experiences a local single-ion anisotropy induced by the bulk spin-orbit coupling and a local anisotropic exchange coupling to the electrons of the edge. The effective low-energy Hamiltonian describing the helical edge with N magnetic impurities is

$$\mathcal{H} = \mathcal{H}_0 - \sum_{i=1}^{N} K(\mathbf{n}\cdot\mathbf{S}^i)^2 + \hbar v \sum_{i=1}^{N} \kappa_{ab}^i S_a^i \sigma_b(x_i), \quad (1)$$

$$\mathcal{H}_0 = \hbar v \int dx \psi^\dagger(x)(-i\hat{\sigma}_3 \nabla)\psi(x). \quad (2)$$

Here the two-component spinor $\psi(x)$ represents the smooth, at Fermi wavelength scale $2\pi/k_F$, envelope of the electron field operators, $\boldsymbol{\sigma}(x) = \psi^\dagger(x)e^{-ik_F\sigma_z x}\hat{\boldsymbol{\sigma}}e^{ik_F\sigma_z x}\psi(x)$ is the electron spin density operator and $\hat{\boldsymbol{\sigma}}$ is the spin vector composed of the three Pauli matrices, v is the electron velocity, and $\mathbf{S}^i$ is the ith impurity spin. We will see that the appearance of interference effects caused by magnetic doping requires the magnetic anisotropy to be uniaxial ($K > 0$) with an axis $\mathbf{n}$ different from $\hat{z}$, and the impurities to have spin $S > 1$. We assume the magnetic impurities to be randomly distributed along the sample length L, resulting in random positions x_i and coupling constants κ^i_{ab}. In general, the exchange tensors κ^i_{ab} and the tensor of single-ion anisotropy should be considered as running coupling constants, depending on the choice of the bandwidth cutoff. The renormalization of the anisotropy is not infrared divergent resulting in a ultraviolet correction to Eq. (1)

$$\mathcal{H}_A = \sum_{i=1}^{N} \delta K^i_{ab} S^i_a S^i_b, \qquad |\delta K^i| \sim |\kappa^i|^2 E_G, \tag{3}$$

where the insulator band gap E_G sets the ultraviolet cutoff scale. Renormalization of $\hat{\kappa}^i$ is infrared divergent; however, it can be neglected if the associated Kondo scale $T_K \sim E_G e^{-1/|\kappa|}$ is smaller than the characteristic energy of the RKKY exchange mediated by the itinerant electrons $N\hbar v|\kappa|^2/L$.

We begin our analysis by considering two impurities at a distance x from each other. Assuming that $|\hat{\kappa}^i| \ll 1$, the RKKY exchange between two localized spins is (see the Supplemental Material [10])

$$\mathcal{H}_{\mathrm{RKKY}} = -\frac{\hbar v}{4\pi|x|} S^1_a \kappa^1_{ab} \omega_{bc}(x) P_{cd} \kappa^2_{ed} S^2_e. \tag{4}$$

Here $\omega_{bc}(x)$ is the orthogonal matrix of counterclockwise rotation through angle $2k_F x$ about the z axis and $P_{cd} = \delta_{cd} - \delta_{cz}\delta_{dz}$ is the matrix of orthogonal projection onto the xy plane. Next, we determine the low-energy spectrum of the two-spin system, assuming that $\mathcal{H}_{\mathrm{RKKY}}$ [Eq. (4)] and $\mathcal{H}_A$ [Eq. (3)] are small as compared to the easy-axis anisotropy K. In the zeroth-order perturbation theory the ground state of the two-spin system is fourfold degenerate with the corresponding eigenspace spanned by four vectors, $|\pm S\rangle_1 \otimes |\pm S\rangle_2$, where $|s\rangle_i$ denotes an eigenstate of $\mathbf{S}^i \cdot \mathbf{n}$ with an eigenvalue $s \in \{-S, -S+1, \ldots, S-1, S\}$ and the subscript $i = 1, 2$ labels the Hilbert space attached to the ith spin. The secular matrix of the perturbation [Eq. (4)] written in terms of the operators

$$\hat{s}_i = |S\rangle_i\langle S|_i - |-S\rangle_i\langle -S|_i, \qquad i = 1, 2, \tag{5}$$

takes the form of the Ising Hamiltonian

$$\mathcal{H}_I = -\delta E \cos(2k_F x)\hat{s}_1\hat{s}_2, \tag{6}$$

where $\delta E = (\hbar v/4\pi|x|) n_a \kappa^1_{ab} P_{bc} \kappa^2_{cd} n_d$. The spectrum of the Hamiltonian [Eq. (6)] consists of two doublets $|S\rangle_1 \otimes |S\rangle_2$, $|-S\rangle_1 \otimes |-S\rangle_2$ and $|S\rangle_1 \otimes |-S\rangle_2$, $|-S\rangle_1 \otimes |S\rangle_2$ separated by the energy $2\delta E$, with the ground state doublet chosen by the sign of $\cos(2k_F x)$. Note that for any $S > 1$ the perturbation [Eq. (3)] has no effect on the splitting of the ground level.

In higher-order perturbation theory further splitting of the two doubly degenerate energy levels occurs. The dominant effect here is due to the perturbation [Eq. (3)] Indeed, it has nonvanishing matrix elements for transitions in which the projection of one impurity spin increases (or decreases) by 1 or 2. For $S > 1$ any such transition takes a vacuum state to a virtual state having the energy of the order K. At least $[S - 1/2]$ such consecutive transitions (here the symbol $[\ldots]$ stands for the integer part of S) are needed in order to flip one impurity spin from $-S$ to S, that is, to bring the system from a ground state to a state in the excited doublet with the energy $2\delta E$. Taking such processes into account amounts to introducing an off-diagonal correction to the Hamiltonian [Eq. (6)]

$$\Delta\mathcal{H} = \delta K \left(\frac{\delta K}{K}\right)^{[S-1/2]} \sum_{i=1,2} r_i |S\rangle_i \langle -S|_i, \tag{7}$$

where r_i are some constants of the order of unity. This will result in a finite energy spacing between levels in each doublet

$$\varepsilon \sim \delta K \left(\frac{\delta K}{K}\right)^{2[S-1/2]} \frac{\delta K}{|\delta E|}, \tag{8}$$

which will remain small compared to δE as long as $|x| \ll (\hbar v/\Delta) \times (K/\delta K)^{[S-1/2]}$. We emphasize that the appearance of two nearly degenerate doublets in the low-energy spectrum is the direct result of large spin ($S > 1$) and uniaxial anisotropy of sufficient strengths $\delta K \lesssim K$.

Consider now the combined dynamics of electrons and spins at energies $E \ll \min(\hbar v/x, K)$. It is described by the Hamiltonian $\mathcal{H}_{\mathrm{eff}} = \mathcal{H}_0 + \mathcal{H}_I + \Delta\mathcal{H} + \mathcal{U}$ with

$$\mathcal{U} = \hbar v \sum_{i=1,2} \hat{s}_i \psi^\dagger(x_i)(\boldsymbol{\xi}_i \cdot \boldsymbol{\tau})\psi(x_i), \tag{9}$$

where $\boldsymbol{\xi}_i = [P\omega(x_i)\kappa^i]\mathbf{n}$. The z component of $\boldsymbol{\xi}_i$ has no effect on electron backscattering; therefore, we discard it, and

$$\boldsymbol{\xi}_i = \xi_i(\cos 2k_F x_i, \sin 2k_F x_i, 0). \tag{10}$$

At temperatures $T \gg \varepsilon$ we may neglect the term $\Delta\mathcal{H}$ in $\mathcal{H}_{\mathrm{eff}}$. In this limit, variables $\hat{s}_i$ are constants of motion and can be treated as numbers. The backscattering current is then found as the Gibbs average over four distinct configurations of $\hat{s}_i$. In the Born approximation this gives the conductance correction

$$\delta G = -\frac{e^2}{h}[\xi_1^2 + \xi_2^2 + 2(\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2)\eta]. \tag{11}$$

The term $2(\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2)\eta$ here comes from the interference between the electron waves reflected by the two local magnetic moments. The factor $(\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2)$ experiences the conventional Fabry-Pérot oscillations as a function of the Fermi momentum k_F with the period $2\pi/|x|$. The factor $\eta = \tanh[\delta E \cos(2k_F x)/T]$ oscillates with the same period and results in a deviation of the overall oscillatory behavior from the simple harmonic law.

At temperatures, $T \lesssim \varepsilon$, effects of $\Delta\mathcal{H}$ [Eq. (7)] become important. The degeneracy of the ground state is lifted and electron backscattering becomes deeply inelastic with the energy transfer ε in a chirality flip process. In this regime the conductance correction evaluated by means of nonequilibrium perturbation theory is (see the Supplemental Material [10])

$$\delta G = -\frac{e^2}{h}[\xi_1^2 + \xi_2^2 + 2(\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2)\eta]F\left(\frac{\varepsilon}{T}\right) \tag{12}$$

with

$$F\left(\frac{\varepsilon}{T}\right) = \int_{-\infty}^{\infty} d\lambda \frac{2\cosh^2(\lambda)}{[\cosh(\varepsilon/T) + \cosh(2\lambda)]^2}. \tag{13}$$

At $T \ll \varepsilon$ the backscattering correction is exponentially suppressed, $F(\varepsilon/T) \sim \exp(-\varepsilon/T)$. In the opposite limit $\varepsilon \to 0$ the function $F \to 1$ in agreement with Eq. (11) obtained in the $\Delta\mathcal{H} = 0$ approximation. Hereinafter we assume [11] $T \gg \varepsilon$.

Next, we consider a system of $N > 2$ magnetic impurities statistically uniformly distributed with average density $n = N/L$ along the edge of length L. Focusing on the energy scales $E \lesssim \hbar v n$ we write the effective Hamiltonian as $\mathcal{H}_{\text{eff}} = \mathcal{H}_0 + \mathcal{H}_{\text{Ising}} + \Delta\mathcal{H} + \mathcal{U}$, where $\Delta\mathcal{H}$ and $\mathcal{U}$ are defined by Eqs. (7) and (9) (with extension of the summation to N), and

$$\mathcal{H}_{\text{Ising}} = -\frac{\hbar v}{4\pi}\sum_{i<j}\frac{\xi_i\xi_j \cos 2k_F(x_i - x_j)}{|x_i - x_j|}\hat{s}_i\hat{s}_j, \tag{14}$$

with ξ_i defined in Eq. (10). The conductance correction evaluated in the Born approximation is

$$\delta G = -\frac{e^2}{h}\sum_{i,j}\xi_i\xi_j \cos 2k_F(x_i - x_j)\langle\hat{s}_i\hat{s}_j\rangle, \tag{15}$$

where $\langle\hat{s}_i\hat{s}_j\rangle$ is the equilibrium spin-spin correlation function for the Hamiltonian [Eq. (14)]. The Born approximation is valid for $N\xi^2 \ll 1$, where ξ is the typical value of ξ_i.

It follows from Eq. (15) that in a given sample the conductance correction δG should exhibit pseudorandom fluctuations with changing k_F due to the oscillatory factors in both Eq. (15) and $\mathcal{H}_{\text{Ising}}$ [see Eq. (14)]. The statistical properties of such fluctuations are fully encoded in the cumulants

$$G_m = \lim_{\lambda\to 0}\frac{d^m}{d\lambda^m}\ln\overline{\exp(\lambda\cdot\delta G)}, \tag{16}$$

where the overline represents the statistical average. We now demonstrate that the statistical properties of conductance fluctuations of the magnetically contaminated helical edge are drastically different from those of a conventional one-dimensional conductor.

First, we recall the structure of conductance fluctuations caused by an ensemble of weak quenched scatterers in a usual one-dimensional conductor. At temperatures such that the coherence length and the thermal length $\hbar v/T$ are both greater than the sample length the conductance is temperature independent. The electron reflection amplitude is, in the Born approximation, a linear superposition of N random complex numbers $e^{ik_F x_i}$, where x_i is the position of the ith impurity. Consequently, in the large-N limit the conductance correction obeys the Rayleigh distribution, which is essentially non-Gaussian. In particular, for the second and third cumulants one has $G_3 = -2G_2^{3/2}$.

In contrast, fluctuations of the conductance [Eq. (15)] exhibit strong temperature dependence in the whole validity range of the model [Eq. (14)]. We note that the model possesses an intrinsic temperature scale

$$T_{\text{SG}} = \frac{\hbar v \xi^2 n}{4\pi} \tag{17}$$

defining a crossover from the high-temperature regime with thermally disordered spins to the "spin glass" regime with spin correlations spanning the sample.

At $T \gg T_{\text{SG}}$ the model can be investigated by means of the virial expansion. We assume that the coupling constants ξ_i have a Gaussian distribution with the average ξ and introduce the normalized cumulants g_m such that

$$G_m = \left(\frac{\xi^2 e^2}{h}\right)^m [g_m(\tau, N) - \delta_{m,1}N]. \tag{18}$$

The normalized cumulants are functions of the dimensionless temperature $\tau = T/T_{\text{SG}}$ and the number of impurities $N = nL$. At $T \gg T_{\text{SG}}$ the spin-spin correlation function in Eq. (15) is given by

$$\langle\hat{s}_i\hat{s}_j\rangle = \tanh\left(\frac{\hbar v \xi_i\xi_j \cos 2k_F(x_i - x_j)}{4\pi|x_i - x_j|T}\right). \tag{19}$$

Substituting Eq. (19) in Eq. (15) and averaging over the uniform distribution of magnetic impurities we find (see the Supplemental Material [10])

$$g_1(\tau, N) = -\frac{N}{\tau}\ln(\tau N). \tag{20}$$

The logarithm in Eq. (20) is due to the $1/|x_i - x_j|$ dependence in the large-distance expansion of Eq. (19). The upper cutoff for this dependence is provided by the system size N/n, while the lower cutoff is defined by the distance x at which the exchange energy $\hbar v\xi^2/x$ equals the temperature. For the higher cumulants the virial expansion yields (see the Supplemental Material [10])

$$g_m(\tau, N) = C_m\frac{N}{\tau}, \qquad m > 1. \tag{21}$$

Here C_m are constants, in particular $C_2 = 0.812$ and $C_3 = 0.0994$. At $NT_{\text{SG}} \gg T \gg T_{\text{SG}}$ the higher cumulants satisfy

$g_m^2 \ll g_2^m$; therefore, the distribution of conductance fluctuations is close to Gaussian.

With decreasing temperature virial corrections to the spin-spin correlation function $\langle \hat{s}_i \hat{s}_j \rangle$ become increasingly important. To explore this effect we calculate (see the Supplemental Material [10]) the first two terms in the virial expansion of the correlator

$$\overline{\langle \hat{s}_i \hat{s}_j \rangle^2} = \frac{1}{2\tau^2 n^2 |x_i - x_j|^2}\left(1 + \frac{2}{\tau}\ln \tau n|x_i - x_j| + \cdots\right), \quad (22)$$

where ... stand for the higher-order terms in $1/\tau$. One can see that no matter how large the temperature τ is, the virial expansion breaks down at sufficiently large distances. For a given system size L one can define the crossover temperature τ_* such that $1 = \tau_*^{-1}\ln(nL\tau_*)$. Below this temperature Eqs. (20) and (21) are not valid. The temperature τ_* is a monotonically increasing function of the system size. In large systems there exists a parametric window $1 < \tau < \tau_*$ where the long-range spin-glass correlations are absent, yet the virial expansion is invalid.

In order to investigate $g_m(\tau, N)$ at $\tau < \tau_*$ we performed Monte Carlo (MC) simulations for $N = 25$, 50, 100, and 200. For each N we consider 10 random realizations of quenched impurity positions, assuming ξ to be the same for all impurities. We observe a considerable slowdown of the convergence of the Metropolis algorithm for $T < 5T_{\rm SG}$ caused by the onset of spin glass correlations. To overcome this difficulty we employ parallel tempering, which works efficiently down to $T = 0.5T_{\rm SG}$. Numerical results for $g_{1,2}(\tau, N)$ are presented in Fig. 1. In the temperature window $5 \lesssim \tau \lesssim 10$ the cumulants experience a sharp increase from $g_m \sim N$ [see Eq. (21)] to $g_m \sim N^{m+1}$ [see Fig. 1(c)] with decreasing temperature. At $\tau \lesssim 5$ correlations between moments located near the opposite ends of the edge set in (see Fig. 2). Note that at all temperatures the magnitude of the skewness of the conductance distribution is less than the universal value $2\sqrt{2}$ predicted by the Rayleigh distribution. Moreover, at both high and low temperatures away from the "spin glass" transition the skewness is suppressed indicating a symmetric distribution of conductance, quite unlike the quenched case.

Making material-specific estimates, we consider a CdTe/HgTe quantum well [12] doped with Mn. We assume values $d \sim 7$ nm, $E_G \sim 10$ meV, and $v \sim 5 \times 10^7$ cm/sec for the thickness, energy gap, and edge state velocity, respectively. These values are typical for a quantum well in a topologically nontrivial state [13]. We estimate the characteristic depth of the edge state as $\ell \sim \hbar v/E_G \approx 10$ nm. The typical value of the exchange constant per volume of the elementary cell a_0^3 in bulk $Cd_{1-x}Mn_xTe$ or $Hg_{1-x}Mn_xTe$ is [14] $J/a_0^3 \sim 0.5$ eV, and the lattice constant in these materials is $a_0 \approx 0.65$ nm. With the above parameters, we find for the dimensionless exchange constant in Eq. (1) $|\kappa_{ab}^i| \sim J/(\hbar v \ell d) \sim 3 \times 10^{-2}$. For such $|\kappa_{ab}^i|$ the

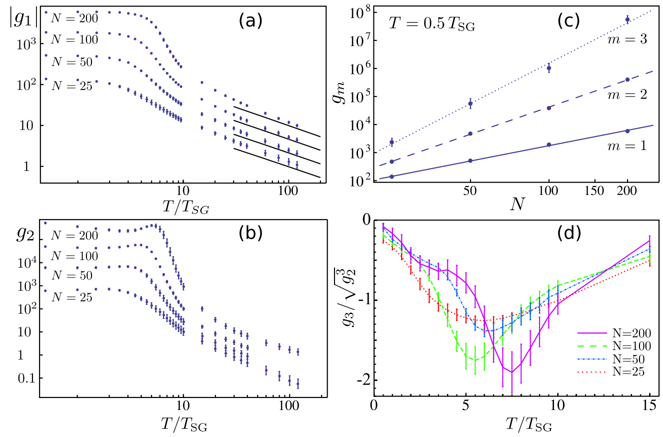

FIG. 1 (color online). MC data for the statistics of the mesoscopic conductance fluctuations at different temperatures and system sizes. Panel (a) shows the average conductance g_1 as a function of temperature. The asymptotes given by Eq. (20) are shown for comparison as solid black lines. Panel (b) shows the second cumulant g_2 of the conductance distribution. Panel (c) shows the dependence of the first three cumulants at $T = 0.5T_{\rm SG}$ on the system size. The log-log plot shows a good fit with the $g_m \sim N^{m+1}$ scaling. Panel (d) shows the skewness of the distribution of conductance values as a function of temperature.

Born approximation is valid if $N \lesssim 10^3$. The magnetic anisotropy for Mn ions $S = 5/2$ in materials with a zinc blende structure is sensitive to deformation and possibly to the nonmagnetic doping level, making an estimate of K difficult. The existing experiments and band structure calculations point to an easy-axis anisotropy [15] with characteristic [16] values $K \sim 0.1$ K. (We expect magnetic ions with nonzero orbital angular momenta, such as Co, to have larger values of K.) Replacing for estimates $|\xi|$ with $|\kappa|$ in Eq. (17) and expressing n there in terms of the bulk doping level $n = (\ell d/a_0^3)x$, we may rewrite Eq. (17) as

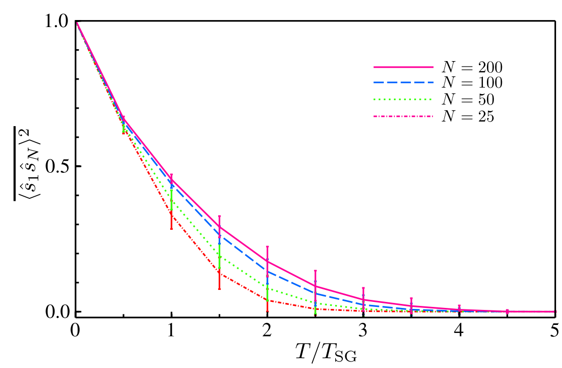

FIG. 2 (color online). Spin-spin correlation function at the opposite ends of the sample as a function of temperature for various systems sizes. Note that larger system sizes result in stronger correlations at a given temperature. This effect is caused by the long rangedness of the RKKY exchange resulting in the logarithmic renormalization of the spin-spin interaction constant with increasing system size.

$T_{SG} \sim \kappa^2 \hbar v(\ell d/4\pi a_0^3)x \approx 20x$ [K]. That yields a reachable value of the characteristic temperature $T_{SG} \approx 100$ mK at a fairly low [17] magnetic doping level $x = 0.005$, while the assumption $\delta K \lesssim K$ is marginally satisfied.

To conclude, the purpose of this work is to reconcile the possibility of mesoscopic fluctuations of the conductance of a helical edge with the exclusion of coherent backscattering by time-reversal symmetry. We find that scattering off an ensemble of large-spin ($S > 1$) magnetic impurities may open a temperature window in which the conductance fluctuations are appreciable. The existence of such a window is provided by a relatively strong effect of single-ion anisotropy which prevents easy flips of the impurity spins. It is further enhanced by the RKKY interaction between the spins. The latter interaction depends on the Fermi momentum of the helical edge, bringing ergodicity in the conductance fluctuations as a function of the helical edge chemical potential. We elucidated the signatures of the described mechanism in the distribution function of conductance fluctuations.

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[10] See the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.206803 for the details of the derivation.

[11] Similar to δG, one may consider the differential conductance at $T = 0$. The differential conductance correction $-\partial I_{BS}/\partial V$ is exponentially small at $eV \ll \varepsilon$ and oscillates with k_F at a fixed $eV \gtrsim \varepsilon$.

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