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WEAK CONVERGENCE OF THE LOCALIZED DISTURBANCE FLOW TO THE COALESCING BROWNIAN FLOW

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We define a new state-space for the coalescing Brownian flow, also known as the Brownian web, on the circle. The elements of this space are families of order-preserving maps of the circle, depending continuously on two time parameters and having a certain weak flow property. The space is equipped with a complete separable metric. A larger state-space, allowing jumps in time, is also introduced, and equipped with a Skorokhod-type metric, also complete and separable. We prove that the coalescing Brownian flow is the weak limit in this larger space of a family of flows which evolve by jumps, each jump arising from a small localized disturbance of the circle. A local version of this result is also obtained, in which the weak limit law is that of the coalescing Brownian flow on the line. Our set-up is well adapted to time-reversal and our weak limit result provides a new proof of time-reversibility of the coalescing Brownian flow. We also identify a martingale associated with the coalescing Brownian flow on the circle and use this to make a direct calculation of the Laplace transform of the time to complete coalescence.

1. Introduction. This paper is a contribution to the theory of stochastic flows in one dimension. The main result is Theorem 6.2. It establishes weak convergence of a certain class of discrete-time stochastic flows on the circle, which we call disturbance flows, to the coalescing Brownian flow. This is motivated by a surprising connection with a model of Hastings and Levitov [9] for planar aggregation, which is worked out in our companion paper [15]. In this model, the flow of harmonic measure on the cluster boundary is a disturbance flow, and our convergence theorem then shows that the random structure of fingers in the Hasting–Levitov cluster is well described in the small-particle limit by the coalescing Brownian flow.

A disturbance flow is a composition of independent and identically distributed random maps of the circle to itself. We do not assume that the maps are homeomorphisms, but do require that they preserve order. We consider the limit where the maps are close to the identity and are well localized. In this limit, we show that the trajectories of points in the flow converge weakly to coalescing Brownian motions. Further, we obtain a corresponding result at the level of flows. In formulating this,

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we define some new metric spaces, which we call the continuous weak flow space and the cadlag weak flow space. These spaces have a number of convenient properties, which we prove. In particular, the continuous weak flow space provides a state-space for the coalescing Brownian flow where its independent-increment and reversibility properties are transparently expressed. The cadlag weak flow space provides a good framework for weak convergence of one-dimensional stochastic flows with jumps.

The coalescing Brownian flow is, loosely speaking, a family of one dimensional Brownian motions, one for each space–time starting point, which evolve independently up to collision and coalesce thereafter. The possibility to identify a precise mathematical object corresponding to this idea was shown by Arratia in 1979 in his Ph.D. thesis [1]. Beginning with Arratia, and more recently pursued by Le Jan and Raimond [11] and Tsirelson [18], one line of work has focused on the possibility to define a family of random measurable functions $(\phi_{ts} : s, t \in \mathbb{R}, s \leq t)$, having the flow property

$$\phi_{ts} \circ \phi_{sr} = \phi_{tr}, \qquad r \leq s \leq t$$

and such that any finite collection of trajectories $(\phi_{ts}(x) : t \geq s)$ performs coalescing Brownian motions. It is known that the functions ϕ_{ts} cannot be chosen to be right-continuous (or left-continuous) and this presents an obstacle in identifying a suitable metrizable state-space. A second line of work, initiated by Fontes et al. [7], overcomes this difficulty by completing the set of trajectories to form a compact set of continuous paths (for a well-chosen topology on paths). The space of these compact sets of paths is then complete and separable for the Hausdorff metric. Depending on exactly which completion is chosen, this leads to a number of canonical versions of Arratia's flow, known as Brownian webs.

In this paper, we follow the flow-type picture, but in order to overcome the problem of having multiple choices for the value of $\phi_{ts}(x)$ at points of discontinuity, we work instead with the pairs $\{\phi^-, \phi^+\}$ of left-continuous and right-continuous modifications of the Arratia flow. This is not far from the viewpoint of Tóth and Werner [17], who however did not address questions of weak convergence. In forgetting the values of ϕ at jumps, our state-space becomes less informative about path properties, but more regular. We are obliged to relax the flow condition to a "weak flow" property that we define in Section 3, where we also show how to define a suitable metric on this space. This gives us an alternative state-space to [7], where independent increment and time reversibility properties are, we think, more naturally expressed; indeed, time-reversal appears as an isometry. Moreover, we have been able to develop a Skorokhod-type state-space for flows which evolve by jumps. This then dispenses with the need to embed jump flows in continuous flows by interpolation.

We envisage that there are many natural stochastic flow processes, which have jumps in time for which continuous interpolation may be problematic. Our topology provides a convenient framework in which to characterize these processes and study convergence. A limitation of our framework is that it requires the flows to have noncrossing trajectories. In addition, in our formulation, one does not see so clearly the possible varieties of path. In models where these properties are important, the topology in [7] may be more appropriate, however, this needs to be weighed against the complications that may arise from the interpolation process. An early version of some parts of the present paper, along with its companion paper [15], appeared in [14]. A discussion on the relation between our work and the well-established framework from [7] can also be found in this paper.

The paper is organized as follows. In Section 2, we introduce disturbance flows, and we prove weak convergence for the trajectories from countably many points, in the limit as the disturbances become small and well-localized. In Section 3, we define the continuous weak flow space and show that it provides a canonical space for the coalescing Brownian flow. Section 4 is a short digression on the distribution of the time taken for the coalescing Brownian flow on the circle to coalesce completely. In Section 5, the larger, cadlag weak flow space, of Skorokhod type, is introduced. The convergence of the disturbance flow to the coalescing Brownian flow is shown in Section 6. In Section 7, we take advantage of the approximation by disturbance flows to give a new proof of the time-reversibility of the coalescing Brownian flow. We prove in Section 8 a local limit for scales intermediate between the disturbance and the whole circle, the limit object being the coalescing Brownian flow on the line. The more technical proofs can be found in the Appendix, and a list of notation is provided at the end of the paper.

2. The disturbance flow on the circle.

We introduce a class of random flows on the circle, whose distributions are invariant under rotations of the circle and under which each point on the circle performs a random walk. The flow maps are in general not continuous on the circle but have an order-preserving property. In a certain asymptotic regime, the motion of the flow from a countable family of starting points is shown to converge weakly to a family of coalescing Brownian motions.

We specify a particular flow by the choice of a nondecreasing, right-continuous function $f^+ : \mathbb{R} \to \mathbb{R}$ with the following *degree* 1 property[2]

$$f^+(x+1) = f^+(x) + 1, \qquad x \in \mathbb{R}. \tag{1}$$

Denote the set of such functions by $\mathcal{R}$ and write $\mathcal{L}$ for the analogous set of left-continuous functions. Each $f^+ \in \mathcal{R}$ has a left-continuous modification $f^- \in \mathcal{L}$, given by $f^-(x) = \lim_{y \uparrow x} f^+(y)$. Write $\mathcal{D}$ for the set of all pairs $f = \{f^-, f^+\}$. When f^+ is continuous, we also write $f = f^+$ and, generally, we write f in place

[2]These functions can be considered as liftings of maps from the circle $\mathbb{R}/\mathbb{Z}$ to itself having an order-preserving property. In the limiting regime which we consider, the circle map is a perturbation of the identity map and our basic map f^+ is the unique lifting which is close to the identity map on $\mathbb{R}$.

of $f^{\pm}$ in expressions where the choice of left or right-continuous modification makes no difference to the value. The sets $\mathcal{R}$ and $\mathcal{L}$ are closed under composition, but $\mathcal{D}$ is not. In fact, if $f_1, f_2 \in \mathcal{D}$, then $f_2^- \circ f_1^-$ is the left-continuous modification of $f_2^+ \circ f_1^+$ if and only if f_1 sends no interval of positive length to a point of discontinuity of f_2. We say in this case that $f_2 \circ f_1 \in \mathcal{D}$, denoting by $f_2 \circ f_1$ the pair $\{f_2^- \circ f_1^-, f_2^+ \circ f_1^+\}$. Write $\tilde{f}^{\pm}$ for the periodic functions $\tilde{f}^{\pm}(x) = f^{\pm}(x) - x$. Define $\mathrm{id}(x) = x$ and set

$$\mathcal{D}^* = \left\{ f \in \mathcal{D} \setminus \{\mathrm{id}\} : \int_0^1 \tilde{f}(x)\,dx = 0 \right\}.$$

We assume throughout that our basic map $f \in \mathcal{D}^*$.

Let us suppose we are given a sequence $(\Theta_n : n \in \mathbb{Z})$ of independent random variables, all distributed uniformly on $(0, 1]$. For $f \in \mathcal{D}^*$ and $\theta \in (0, 1]$, define $f_\theta(x) = f(x - \theta) + \theta$. Then define, for $m, n \in \mathbb{Z}$ with $m < n$,

$$\Phi^{\pm}_{n,m} = f^{\pm}_{\Theta_n} \circ \cdots \circ f^{\pm}_{\Theta_{m+1}}. \tag{2}$$

Set $\Phi_{n,n} = \mathrm{id}$ for all $n \in \mathbb{Z}$. Thus, for $l \le m \le n$, we have $\Phi^{\pm}_{n,l} = \Phi^{\pm}_{n,m} \circ \Phi^{\pm}_{m,l}$. Since f can have at most countably many points of discontinuity and intervals of constancy, we have $\Phi_{n,m} = \{\Phi^-_{n,m}, \Phi^+_{n,m}\} \in \mathcal{D}$ almost surely. We call the function f the *disturbance* and we call $(\Phi_{n,m} : m, n \in \mathbb{Z}, m \le n)$ the *discrete disturbance flow*.[3] Define $\rho = \rho(f) \in (0, \infty)$ by

$$\rho \int_0^1 \tilde{f}(x)^2\,dx = 1. \tag{3}$$

We embed the discrete-time flow in continuous-time using a Poisson random measure N on $\mathbb{R}$ of intensity ρ. Write $(T_n : n \in \mathbb{Z})$ for the ordered sequence of atoms of N, labeled so that $T_0 \le 0 < T_1$. Then, for each bounded interval $I \subseteq \mathbb{R}$, set $\Phi_I = \mathrm{id}$ if $N(I) = 0$, and otherwise set

$$\Phi_I = \Phi_{n,m},$$

where T_{m+1} and T_n are the smallest and largest atoms of N in I. Write $\Phi = (\Phi_I : I \subseteq \mathbb{R})$ for the family of maps Φ_I where I ranges over all bounded intervals in $\mathbb{R}$. We call Φ the *Poisson disturbance flow with disturbance* f. A second embedding in continuous time, without additional randomness, will also be considered. By the *lattice disturbance flow with disturbance* f, we mean the family

[3] In the case where f is a homeomorphism, the restriction of the flow to $m, n \ge 0$ can be recovered from the process $(\Phi_{n,0} : n \ge 0)$. This is a random walk on the group of homeomorphisms of the circle. The structure of this group is a rich area of mathematics. See, for example, [3, 8, 12, 13]. The present paper can be seen as an investigation of scaling limits for such random walks with small localized steps. Our conclusion is then that one has to complete the homeomorphism group to the space of weak flows in order to support the limit measure, and then that, within the class we consider, the limit is universal.

$(\Phi_I : I \subseteq \mathbb{R})$, where $\Phi_I = \mathrm{id}$ if $\rho I \cap \mathbb{Z} = \varnothing$ and otherwise $\Phi_I = \Phi_{n,m}$ with $m+1$ the smallest integer and n the largest integer in the interval ρI. In each embedding, the time-scale has been chosen to normalize the mean square displacement per unit time. Unless otherwise mentioned, our discussion refers to the Poisson case, which is slightly cleaner, but the variations needed for the lattice case are slight and we shall end up with the same asymptotic results in both cases.

Write $I = I_1 \oplus I_2$ if I_1, I_2 and I are intervals with $\sup I_1 = \inf I_2$, $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 = I$. Note that Φ has the following properties:

(4) $\Phi_I^+(x)$ and $\Phi_I^-(x)$ are random variables for all bounded intervals I and all $x \in \mathbb{R}$,

$$\Phi_I^+ = \Phi_{I_2}^+ \circ \Phi_{I_1}^+ \quad \text{and} \quad \Phi_I^- = \Phi_{I_2}^- \circ \Phi_{I_1}^- \qquad \text{whenever } I = I_1 \oplus I_2, \tag{5}$$

(6) for all $t \in \mathbb{R}$ there exists $\delta > 0$ such that for all $s \in (t-\delta, t)$ and all $u \in (t, t+\delta)$, $\Phi_{(s,t)} = \Phi_{(t,u)} = \mathrm{id}$.

For $e = (s, x) \in \mathbb{R}^2$ and $t \in [s, \infty)$, set

$$X_t^{e,\pm} = \Phi_{(s,t]}^{\pm}(x).$$

For each e, almost surely,

$$X_t^{e,-} = X_t^{e,+} \qquad \text{for all } t \geq s. \tag{7}$$

We will therefore drop the $\pm$ and write simply $X^e = (X_t^e : t \geq s)$. We call X^e the *trajectory of the flow starting from* e. The $\pm$ will reappear in any statement requiring specification of a version of X^e for uncountably many e. Write μ_e^f for the distribution of X^e on the Skorokhod space $D_e = D_x([s, \infty), \mathbb{R})$ of cadlag paths starting from x at time s. Write d_e for the Skorokhod metric on D_e and write μ_e for the distribution on D_e of a standard Brownian motion starting from e.

PROPOSITION 2.1. *The trajectory X^e of the Poisson disturbance flow with disturbance f converges weakly to Brownian motion on D_e, uniformly in $f \in \mathcal{D}^*$ as $\rho(f) \to \infty$.*

PROOF. Write X for X^e within the proof to lighten the notation. Note that X is a compound Poisson process, making jumps distributed as $\tilde{f}(\Theta_1)$ at rate ρ. So, for $t \geq s$,

$$\mathbb{E}(X_t - X_s) = \rho(t-s) \int_0^1 \tilde{f}(\theta)\, d\theta = 0,$$

$$\mathbb{E}\big((X_t - X_s)^2\big) = \rho(t-s) \int_0^1 \tilde{f}(\theta)^2\, d\theta = t - s.$$

Hence, the processes $(X_t)_{t\geq s}$ and $(X_t^2-t)_{t\geq s}$ are martingales. A standard criterion (see, e.g., [2], page 143 or [10], page 355) allows us to deduce that the family of laws $\{\mu_e^f : f\in\mathcal{D}^*\}$ is tight in D_e. Now f is nondecreasing so

$$\tilde f(\theta)\geq \tilde f(\theta_0)-(\theta-\theta_0),\qquad \theta\geq\theta_0$$

and so, if $\tilde f(\theta_0)\geq 0$ for some θ_0, then

$$\rho^{-1}=\int_0^1 \tilde f(\theta)^2\,d\theta \geq \int_{\theta_0}^{\theta_0+\tilde f(\theta_0)} \bigl(\tilde f(\theta_0)-(\theta-\theta_0)\bigr)^2\,d\theta = \bigl|\tilde f(\theta_0)\bigr|^3/3$$

and a similar argument leads to the same estimate also when $\tilde f(\theta_0)\leq 0$. Hence,

$$\bigl|\tilde f(\theta)\bigr|\leq (3/\rho)^{1/3},\qquad \theta\in(0,1]. \tag{8}$$

So the jumps of $(X_t)_{t\geq s}$ are bounded in absolute value by $(3/\rho)^{1/3}$. Let μ be any weak limit law for the limit $\rho(f)\to\infty$. Write $(Z_t)_{t\geq s}$ for the coordinate process on D_e. Then, by standard arguments, μ is supported on continuous paths and under μ both $(Z_t)_{t\geq s}$ and $(Z_t^2-t)_{t\geq s}$ are local martingales in the natural filtration of $(Z_t)_{t\geq s}$. Hence $\mu=\mu_e$ by Lévy's characterization of Brownian motion. □

Given a sequence $E=(e_k : k\in\mathbb{N})$ in $\mathbb{R}^2$, set

$$D_E=\prod_{k=1}^{\infty} D_{e_k}$$

and define a metric d_E on D_E by

$$d_E(z,z')=\sum_{k=1}^{\infty}2^{-k}\bigl(d_{e_k}(z_k,z'_k)\wedge 1\bigr),\qquad z=(z_k:k\in\mathbb{N}), z'=(z'_k:k\in\mathbb{N}). \tag{9}$$

Then (D_E,d_E) is a complete separable metric space and $(X^{e_k}:k\in\mathbb{N})$ is a random variable in D_E. Write μ_E^f for the distribution of $(X^{e_k}:k\in\mathbb{N})$ on D_E.

Write $e_k=(s_k,x_k)$ and denote by $(Z_t^k)_{t\geq s_k}$ the kth coordinate process on D_E, given by $Z_t^k(z)=z_t^k$. Consider the filtration $(\mathcal{Z}_t)_{t\in\mathbb{R}}$ on D_E, where $\mathcal{Z}_t$ is the σ-algebra generated by $(Z_s^k : s_k<s\leq t\vee s_k, k\in\mathbb{N})$. Write C_E for the (measurable) subset of D_E where each coordinate path is continuous. Define on C_E

$$T^{jk}=\inf\{t\geq s_j\vee s_k : Z_t^j-Z_t^k\in\mathbb{Z}\}.$$

We sometimes think of the paths $(Z_t^k)_{t\geq s_k}$ as liftings of paths in the circle $\mathbb{R}/\mathbb{Z}$. Then the times T^{jk} are collision times of the circle-valued paths. The following is a variant of a result of Arratia [1]. It provides a useful martingale characterization corresponding to the intuitive idea of coalescing Brownian motions on the circle.

PROPOSITION 2.2. *There exists a unique Borel probability measure μ_E on D_E under which, for all j,k, the processes $(Z_t^k)_{t\geq s_k}$ and $(Z_t^jZ_t^k-(t-T^{jk})^+)_{t\geq s_j\vee s_k}$ are both continuous local martingales in the filtration $(\mathcal{Z}_t)_{t\in\mathbb{R}}$.*

We sketch a proof. For existence, one can take independent Brownian motions from each of the given time–space starting points and then impose a rule of coalescence on collision, deleting the path of lower index. The law of the resulting process has the desired properties. On the other hand, given a probability measure such as described in the proposition, on some larger probability space, one can use a supply of independent Brownian motions to resurrect the paths deleted at each collision. Then Lévy's characterization can be used to see that one has recovered the set-up used for existence. This gives uniqueness.

Consider now a limit in which the basic map f is an increasingly well localized perturbation of the identity, where we quantify this property in terms of the smallest constant $\lambda = \lambda(f) \in (0, 1]$ such that

$$\rho \int_0^1 \big|\tilde{f}(x+a)\tilde{f}(x)\big|\,dx \le \lambda, \qquad a \in [\lambda, 1-\lambda]. \tag{10}$$

PROPOSITION 2.3. *The joint distribution μ_E^f of the family of trajectories $(X^e : e \in E)$ in the Poisson disturbance flow with disturbance f converges weakly to the coalescing Brownian law μ_E on D_E, uniformly in $f \in \mathcal{D}^*$, as $\rho(f) \to \infty$ and $\lambda(f) \to 0$.*

PROOF. We write X^k for X^{e_k} within the proof. For each k, the family of marginal laws $\{\mu_{e_k}^f : f \in \mathcal{D}^*\}$ is tight, as in Proposition 2.1. Hence, the family of laws $\{\mu_E^f : f \in \mathcal{D}^*\}$ is also tight. Let μ be any weak limit law for $\{\mu_E^f : f \in \mathcal{D}^*\}$ under the limits $\rho = \rho(f) \to \infty$ and $\lambda = \lambda(f) \to 0$. Then μ is supported on C_E. For all j, k the process

$$X_t^j X_t^k - \int_{s_j \vee s_k}^t b\big(X_s^j, X_s^k\big)\,ds, \qquad t \ge s_j \vee s_k,$$

is a martingale,[4] where

$$b\big(x, x'\big) = \rho \int_0^1 \tilde{f}(x-\theta)\tilde{f}\big(x'-\theta\big)\,d\theta.$$

We have $|b(x, x')| \le \lambda$ whenever $\lambda \le |x - x'| \le 1 - \lambda$. Hence, by standard arguments, under μ, the process $(Z_t^j Z_t^k : s_j \vee s_k \le t < T^{jk})$ is a local martingale. We know from the proof of Proposition 2.1 that, under μ, the processes $(Z_t^j : t \ge s_j)$, $((Z_t^j)^2 - t : t \ge s_j)$ and $(Z_t^k : t \ge s_k)$ are continuous local martingales. But μ inherits from the laws μ_E^f the property that, almost surely, for all

[4] In the lattice case, a similar argument can be based on the martingale

$$X_t^j X_t^k - \frac{1}{\rho} \sum_{n=\lfloor \rho(s_j \vee s_k)\rfloor}^{\lfloor \rho t\rfloor - 1} b\big(X_{n/\rho}^j, X_{n/\rho}^k\big), \qquad t \ge s_j \vee s_k,$$

$n \in \mathbb{Z}$, the process $(Z_t^j - Z_t^k + n : t \geq s_j \vee s_k)$ does not change sign. Hence, by an optional stopping argument, $Z_t^j - Z_t^k$ is constant for $t \geq T^{jk}$. It follows that $(Z_t^j Z_t^k - (t - T^{jk})^+)_{t \geq s_j \vee s_k}$ is a continuous local martingale. Hence, $\mu = \mu_E$, by Proposition 2.2. □

3. A new state-space for the coalescing Brownian flow. The weak convergence result for trajectories, obtained in Proposition 2.3, suggests the possibility of a deeper result at the level of flows, independent of the choice of starting points for trajectories. This would be of interest to understand what statistics of the disturbance flows, beyond trajectories, have weak limits, for example, trajectories of the inverse, reverse-time flow. For such a flow-level result, we first specify a state-space and metric for the notion of weak convergence, and then identify a limit object, which we call the coalescing Brownian flow.

We begin by defining a metric on $\mathcal{D}$. Let $\mathcal{S}$ denote the set of all periodic contractions on $\mathbb{R}$ having period 1. Each $f \in \mathcal{D}$ can be identified with some $f^\times \in \mathcal{S}$ by drawing new axes at an angle $\pi/4$ with the old, and scaling appropriately. See Figure 1. More formally, since $x + f^+(x)$ is strictly increasing in x, there is for each $t \in \mathbb{R}$ a unique $x \in \mathbb{R}$ such that

$$\frac{x + f^-(x)}{2} \leq t \leq \frac{x + f^+(x)}{2}. \tag{11}$$

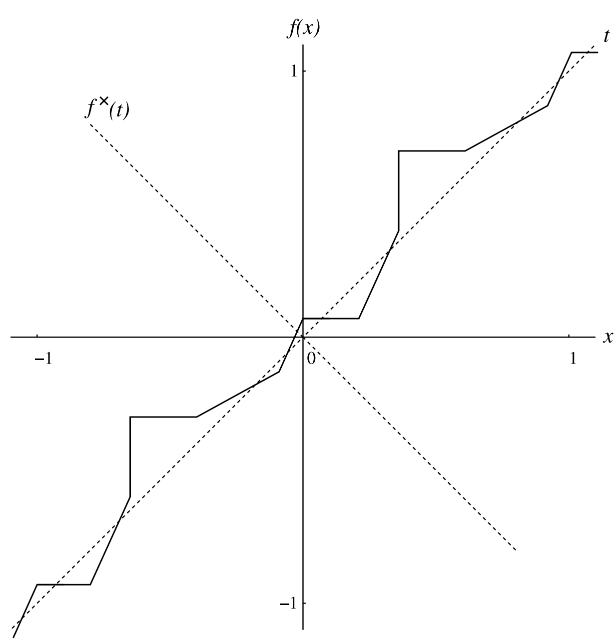

FIG. 1. *The map $f^\times$ obtained from f by rotating the axes by $\frac{\pi}{4}$.*

Define $f^\times(t) = t - x$. Note that $\mathrm{id}^\times = 0$. Then *the map* $f \mapsto f^\times : \mathcal{D} \to \mathcal{S}$ *is a bijection*, so we can define a metric $d_\mathcal{D}$ on $\mathcal{D}$ by

$$d_\mathcal{D}(f, g) = \|f^\times - g^\times\| = \sup_{t\in[0,1)} \left|f^\times(t) - g^\times(t)\right|. \tag{12}$$

A proof of the italicized assertion is given in the Appendix. The same is true for some further technical assertions which will be made below, written also in italics. The metric space $(\mathcal{S}, \|\cdots\|)$ is complete and locally compact, so the same is true for $(\mathcal{D}, d_\mathcal{D})$. An alternative characterization[5] of the metric $d_\mathcal{D}$ is as follows: *for* $f, g \in \mathcal{D}$ *and* $\varepsilon > 0$, *we have*

$$d_\mathcal{D}(f, g) \le \varepsilon \quad \iff \quad f^-(x-\varepsilon) - \varepsilon \le g^-(x) \le g^+(x) \le f^+(x+\varepsilon) + \varepsilon \qquad \text{for all } x \in \mathbb{R}.$$

We deduce that, for $f, g \in \mathcal{D}$,

$$d_\mathcal{D}(f, g) \le \|f - g\|, \qquad 2d_\mathcal{D}(f, \mathrm{id}) = \|f - \mathrm{id}\|$$

and

$$d_\mathcal{D}(f, g\circ f) \le \|g - \mathrm{id}\| \qquad \text{when } g\circ f \in \mathcal{D},$$
$$d_\mathcal{D}(f, f\circ g) \le \|g - \mathrm{id}\| \qquad \text{when } f\circ g \in \mathcal{D}.$$

Moreover, *for any sequence* $(f_n : n \in \mathbb{N})$ *in* $\mathcal{D}$,

$$f_n \to f \quad \iff \quad f_n(x) \to f(x) \qquad \text{at every point } x \text{ where } f \text{ is continuous.}$$

Here and below, we write $f_n \to f$ to mean convergence in the metric $d_\mathcal{D}$.

We now define our space of flows. We call them weak flows to emphasize that the usual flow property may fail at points of spatial discontinuity. Consider $\phi = (\phi_{ts} : s, t \in \mathbb{R}, s < t)$, with $\phi_{ts} \in \mathcal{D}$ for all s, t. Say that ϕ is a *weak flow* if

$$\phi_{ut}^- \circ \phi_{ts}^- \le \phi_{us}^- \le \phi_{us}^+ \le \phi_{ut}^+ \circ \phi_{ts}^+, \qquad s < t < u. \tag{13}$$

Say that ϕ is *continuous* if, for all $t \in \mathbb{R}$,

$$\phi_{ts} \to \mathrm{id} \qquad \text{as } s \uparrow t, \qquad \phi_{ut} \to \mathrm{id} \qquad \text{as } u \downarrow t.$$

Write $C^\circ(\mathbb{R}, \mathcal{D})$ for the set of all continuous weak flows. It will be convenient sometimes to extend a continuous weak flow ϕ to the diagonal, which we do by setting $\phi_{ss} = \mathrm{id}$ for all $s \in \mathbb{R}$. Then, *for any* $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$, *the map*

$$(s, t) \mapsto \phi_{ts} : \{(s, t) : s \le t\} \to \mathcal{D} \tag{14}$$

[5] Thus, $d_\mathcal{D}$ is a close relative of the Lévy metric sometimes used on the set of distribution functions for real random variables. This choice of topology is insensitive to the value of a function at its jump discontinuities, only keeping track of its left and right continuous versions. The relationships of such a metric to the operations of composition and inversion in $\mathcal{D}$, which are significant for us, do not appear to have been studied.

is continuous.

Define, for $\phi, \psi \in C^\circ(\mathbb{R}, \mathcal{D})$,

$$d_C(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \{d_C^{(n)}(\phi, \psi) \wedge 1\}, \tag{15}$$

where

$$d_C^{(n)}(\phi, \psi) = \sup_{s,t \in (-n,n), s<t} d_{\mathcal{D}}(\phi_{ts}, \psi_{ts}). \tag{16}$$

Then d_C is a metric on $C^\circ(\mathbb{R}, \mathcal{D})$, *under which* $C^\circ(\mathbb{R}, \mathcal{D})$ *is complete and separable*. Define, for $e = (s, x) \in \mathbb{R}^2$ and $t \geq s$, *evaluation maps* $Z_t^{e,+}$ and $Z_t^{e,-}$ on $C^\circ(\mathbb{R}, \mathcal{D})$ by

$$Z_t^{e,\pm}(\phi) = \phi_{ts}^{\pm}(x).$$

Then, *for all* $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$, *the maps* $t \mapsto Z_t^{e,\pm}(\phi) : [s, \infty) \to \mathbb{R}$ *are continuous.* So we can consider the *left and right coordinate processes* $Z^{e,\pm} = (Z_t^{e,\pm} : t \geq s)$ as C_e-valued random variables on $C^\circ(\mathbb{R}, \mathcal{D})$. Write $Z^e = Z^{e,+}$ to lighten the notation. Define a σ-algebra $\mathcal{F}$ and a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ on $C^\circ(\mathbb{R}, \mathcal{D})$ by

$$\mathcal{F} = \sigma(Z_t^e : e \in \mathbb{R}^2, t \geq s(e)), \qquad \mathcal{F}_t = \sigma(Z_r^e : e \in \mathbb{R}^2, r \in (-\infty, t] \cap [s(e), \infty)),$$

where $s(e)$ is the first component of e. Then $\mathcal{F}_t$ *is generated by the random variables* Z_r^e *with* $e \in \mathbb{Q}^2$ *and* $r \in (-\infty, t] \cap [s(e), \infty)$, *and* $\mathcal{F}$ *is the Borel* σ*-algebra of the metric* d_C. Define for $e = (s, x)$ and $e' = (s', x')$ the *collision time* $T^{ee'} : C^\circ(\mathbb{R}, \mathcal{D}) \to [0, \infty]$ by

$$T^{ee'}(\phi) = \inf\{t \geq s \vee s' : Z_t^e(\phi) - Z_t^{e'}(\phi) \in \mathbb{Z}\}.$$

The following result is a variant, stated in the language of continuous weak flows, of a result of Tóth and Werner [17], Theorem 2.1, which itself was a variant of a result of Arratia [1]. The characterizing martingale properties may be expressed less formally as saying that there exists a unique probability measure on $C^\circ(\mathbb{R}, \mathcal{D})$ under which the left and right coordinate processes $Z^{e,\pm}$ agree almost surely for all $e \in \mathbb{R}^2$ and behave as Brownian motions coalescing on the circle. We shall give a complete proof, in part because we need most components of the proof also for our main convergence result, and in part because our framework leads to some simplifications, for example in the probabilistic underpinnings contained in Proposition A.10. The formulation in terms of continuous weak flows has advantages in leading to a unique object, with a natural time-reversal invariance (for which see Section 7), and for the derivation of weak limits (see Section 6).

THEOREM 3.1. *There exists a unique Borel probability measure* μ_A *on* $C^\circ(\mathbb{R}, \mathcal{D})$ *under which, for all* $e, e' \in \mathbb{R}^2$, *the processes* $(Z_t^e)_{t \geq s(e)}$ *and* $(Z_t^e Z_t^{e'} - (t - T^{ee'})^+)_{t \geq s(e) \vee s(e')}$ *are continuous local martingales for* $(\mathcal{F}_t)_{t \in \mathbb{R}}$. *Moreover, for all* $e \in \mathbb{R}^2$, *we have* $Z^{e,+} = Z^{e,-}$ μ_A*-almost surely.*

PROOF. We first show that there exists a unique probability measure μ_A on $C^\circ(\mathbb{R}, \mathcal{D})$ under which the above property holds for all $e, e' \in \mathbb{Q}^2$. This essentially amounts to showing that if we have a family of coalescing Brownian motions starting from every point in $\mathbb{Q}^2$, then there exists a unique continuous weak flow under which the motions of each point in $\mathbb{Q}^2$ are the given coalescing Brownian motions.

Fix an enumeration $E = (e_k : k \in \mathbb{N})$ of $\mathbb{Q}^2$. Define the evaluation map $Z^{E,\pm} : C^\circ(\mathbb{R}, \mathcal{D}) \to C_E$ by $Z^{E,\pm}(\phi) = (Z^{e_k,\pm}(\phi) : k \in \mathbb{N})$. Then, *we have* $\mathcal{F}_t = \{(Z^{E,+})^{-1}(B) : B \in \mathcal{Z}_t\}$, where $(\mathcal{Z}_t)_{t\in\mathbb{R}}$ is the filtration on C_E generated by projection mappings as in Proposition 2.2. Therefore, if μ is any probability measure on $C^\circ(\mathbb{R}, \mathcal{D})$ with the property that for all $j, k \in \mathbb{N}$, the processes $(Z_t^{e_k})_{t\ge s_k}$ and $(Z_t^{e_j} Z_t^{e_k} - (t - T^{e_j e_k})^+)_{t \ge s_j \vee s_k}$ are continuous local martingales for $(\mathcal{F}_t)_{t\in\mathbb{R}}$; then by Proposition 2.2, $\mu \circ (Z^{E,+})^{-1} = \mu_E$.

To show existence and uniqueness, it is therefore sufficient to show that $Z^{E,+}$ is bijective, or rather that there exists some μ_E-almost sure subset on which $Z^{E,+}$ is bijective. Let the images of the evaluation maps be

$$C_E^{\circ,\pm} = \{Z^{E,\pm}(\phi) : \phi \in C^\circ(\mathbb{R}, \mathcal{D})\}.$$

Then *the sets* $C_E^{\circ,\pm}$ *are measurable subsets of* C_E *with* $\mu_E(C_E^{\circ,\pm}) = 1$. Moreover, $Z^{E,\pm}$ *maps* $C^\circ(\mathbb{R}, \mathcal{D})$ *bijectively to* $C_E^{\circ,\pm}$ *and the inverse bijections* $C_E^{\circ,\pm} \to C^\circ(\mathbb{R}, \mathcal{D})$, which we denote by $\Phi^{E,\pm}$, *are measurable*. Write Z^E for $Z^{E,+}$ and Φ^E for $\Phi^{E,+}$. Then, on $C_E^{\circ,+}$, for all $j, k \in \mathbb{N}$, we have

$$Z^{e_k} \circ \Phi^E = Z^k, \qquad T^{e_j e_k} \circ \Phi^E = T^{jk},$$

where Z^k and T^{jk} are the projections and stopping times from Proposition 2.2, and for all $t \in \mathbb{R}$ and $B \in \mathcal{F}_t$ we have $1_B \circ \Phi^E = 1_{B'}$ for some $B' \in \mathcal{Z}_t$. Thus, we can uniquely define $\mu_A = \mu_E \circ (\Phi^E)^{-1}$ as required.

To complete the proof, we need to show that μ_A has the required properties for any given $e, e' \in \mathbb{R}^2$. Observe that all the assertions above hold also when E is replaced by the sequence $E' = (e, e', e_1, e_2, \ldots)$. We repeat the steps taken to obtain a probability measure $\mu'_A = \mu_{E'} \circ (\Phi^{E'})^{-1}$ on $C^\circ(\mathbb{R}, \mathcal{D})$. Then, under μ'_A, the processes $(Z_t^e)_{t\ge s(e)}$ and $(Z_t^e Z_t^{e'} - (t - T^{ee'})^+)_{t\ge s(e)\vee s(e')}$ are continuous local martingales for $(\mathcal{F}_t)_{t\in\mathbb{R}}$. But also, under μ'_A, for all $j, k \in \mathbb{N}$, the processes $(Z_t^{e_k})_{t\ge s_k}$ and $(Z_t^{e_j} Z_t^{e_k} - (t - T^{e_j e_k})^+)_{t\ge s_j \vee s_k}$ are continuous local martingales for $(\mathcal{F}_t)_{t\in\mathbb{R}}$, so $\mu_A = \mu'_A$.

Finally, *we have* $\Phi^{E',+} = \Phi^{E',-}$ *on* $C_{E'}^{\circ,-} \cap C_{E'}^{\circ,+}$, so

$$Z^{e,-}(\Phi^{E'}) = Z^{e,-}(\Phi^{E',-}) = Z^{e,+}(\Phi^{E'}),$$

$\mu_{E'}$-almost surely, and so $Z^{e,-} = Z^{e,+}$, μ_A-almost surely, as claimed. □

We call any $C^\circ(\mathbb{R}, \mathcal{D})$-valued random variable with law μ_A a *coalescing Brownian flow on the circle*.

4. Complete coalescence time. In this section, we digress to discuss the complete coalescence time T of a coalescing Brownian flow Φ on the circle, given by

$$T = \inf\{t \geq 0 : \Phi^+_{t0}(x) = y + n \text{ for some } n \in \mathbb{Z}, \text{ for all } x \in \mathbb{R}, \text{ for some } y \in \mathbb{R}\}.$$

It is known that

$$\mathbb{E}(e^{\lambda T}) = \sqrt{\lambda}/\sin\sqrt{\lambda}, \qquad \lambda < \pi^2. \tag{17}$$

Cox [4] showed this by an indirect argument. More recently, Zhou [19] gave a direct proof. We give an alternative and simpler proof.

Fix $N \in \mathbb{N}$ and define for $t \geq 0$

$$B^k_t = \Phi_{t0}(k/N) - \Phi_{t0}((k-1)/N), \qquad k = 1, \dots, N.$$

Then each process B^k is a Brownian motion of diffusivity 2, starting from $1/N$ and stopped on hitting 0 or 1. Consider the stopping time $S = \inf\{t \geq 0 : B^k_t = 1 \text{ for some } k\}$ and note that $B^k_S = 0$ for all but one random value, $k = K$ say, for which $B^K_S = 1$. Define

$$M_t = M^{(N)}_t = e^{\lambda t} \sum_{k=1}^N \sin\{\sqrt{\lambda} B^k_t\}$$

then the stopped process $(M^S_t)_{t \geq 0} = (M_{S \wedge t})_{t \geq 0}$ is a martingale so, for all $t \geq 0$,

$$\begin{aligned} N \sin\{\sqrt{\lambda}/N\} &= M_0 \\ &= \mathbb{E}(M_{S \wedge t}) \\ &= \mathbb{E}\left(e^{\lambda(S \wedge t)} \sum_{k=1}^N \sin\{\sqrt{\lambda} B^k_{S \wedge t}\}\right) \\ &\geq \mathbb{E}(e^{\lambda(S \wedge t)}) \sin\sqrt{\lambda}. \end{aligned}$$

For $\lambda < \pi^2$ the final inequality allows us to see that $\mathbb{E}(e^{\lambda S}) < \infty$, so we can let $t \to \infty$ to obtain

$$N \sin\{\sqrt{\lambda}/N\} = \mathbb{E}(e^{\lambda S}) \sin\sqrt{\lambda}.$$

On letting $N \to \infty$, we obtain (17).

In fact, it is not hard to see that $M^{(N)}_t$ increases with N for all $t \geq 0$ and is eventually constant for all $t > 0$. The limit process $M^{(\infty)}$ is also a martingale with $M^{(\infty)}_0 = \sqrt{\lambda}$ and $M^{(\infty)}_T = e^{\lambda T} \sin\sqrt{\lambda}$, and the optional stopping argument can alternatively be applied directly to $M^{(\infty)}$.

From (17), we can identify T as having the same law as one-half of the time $\tilde{T}$ taken for a BES(3) to get from 0 to 1. This can also be seen directly using the relation

$$S = \sum_{k=1}^N S_k 1_{\{B^k(S_k)=1\}},$$

where $S_k = \inf\{t \geq 0 : B_t^k \in \{0, 1\}\}$. Then, for any bounded measurable function f,

$$\mathbb{E}\big(f(S)\big) = \sum_{k=1}^{N} \mathbb{E}\big(f(S_k) 1_{\{B^k(S_k)=1\}}\big) = \mathbb{E}\big(f(S_1) | B^1(S_1) = 1\big)$$

and, on letting $N \to \infty$, we obtain $\mathbb{E}(f(T)) = \mathbb{E}(f(\tilde{T}/2))$. We thank Neil O'Connell and Marc Yor for this observation.

5. A Skorokhod-type space of nondecreasing flows on the circle. Since the disturbance flow is not continuous in time, it will be necessary to introduce a larger flow space to accommodate it. Consider now $\phi = (\phi_I : I \subseteq \mathbb{R})$, where $\phi_I \in \mathcal{D}$ and I ranges over all nonempty bounded intervals. Recall that we write $I = I_1 \oplus I_2$ if I, I_1, I_2 are intervals with $\sup I_1 = \inf I_2$, $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 = I$. Say that ϕ is a *weak flow* if

$$\phi_{I_2}^- \circ \phi_{I_1}^- \leq \phi_I^- \leq \phi_I^+ \leq \phi_{I_2}^+ \circ \phi_{I_1}^+, \qquad I = I_1 \oplus I_2. \tag{18}$$

Say that ϕ is *cadlag*[6] if, for all $t \in \mathbb{R}$,

$$\phi_{(s,t)} \to \mathrm{id} \qquad \text{as } s \uparrow t, \qquad \phi_{(t,u)} \to \mathrm{id} \qquad \text{as } u \downarrow t.$$

Write $D^\circ(\mathbb{R}, \mathcal{D})$ for the set of cadlag weak flows. It will be convenient to extend a cadlag weak flow ϕ to the empty interval by setting $\phi_\varnothing = \mathrm{id}$. Given a bounded interval I and a sequence of bounded intervals $(I_n : n \in \mathbb{N})$, write $I_n \to I$ if the indicator functions $1_{I_n} \to 1_I$ pointwise as $n \to \infty$. *For any $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$, we have*

$$\phi_{I_n} \to \phi_I \qquad \text{as } I_n \to I. \tag{19}$$

Let ϕ be a cadlag weak flow and suppose that $\phi_{\{t\}} = \mathrm{id}$ for all $t \in \mathbb{R}$. Then, using (18), we have $\phi_{(s,t)} = \phi_{(s,t]} = \phi_{[s,t)} = \phi_{[s,t]}$ for all $s < t$ and, denoting all these functions by ϕ_{ts},[7] the family $(\phi_{ts} : s, t \in \mathbb{R}, s < t)$ is a continuous weak flow in the sense of the preceding section.

For $\phi, \psi \in D^\circ(\mathbb{R}, \mathcal{D})$ and $n \geq 1$, define

$$d_D^{(n)}(\phi, \psi) = \inf_\lambda \Big\{ \gamma(\lambda) \vee \sup_{I \subseteq \mathbb{R}} \big\| \chi_n(I) \phi_I^\times - \chi_n\big(\lambda(I)\big) \psi_{\lambda(I)}^\times \big\| \Big\}, \tag{20}$$

where the infimum is taken over the set of increasing homeomorphisms λ of $\mathbb{R}$, where

$$\gamma(\lambda) = \sup_{t \in \mathbb{R}} \big|\lambda(t) - t\big| \vee \sup_{s,t \in \mathbb{R}, s<t} \left| \log\left(\frac{\lambda(t) - \lambda(s)}{t - s} \right) \right|, \tag{21}$$

[6] This definition is more symmetric in time than is usual for "cadlag": a more accurate acronym would be *laglad*.

[7] Note the reversal of the order of s and t. This was chosen to make the weak flow property (13) appear neater.

and where χ_n is the cutoff function[8] given by

$$\chi_n(I) = 0 \vee (n+1-R) \wedge 1, \qquad R = \sup I \vee (-\inf I).$$

Then define

$$d_D(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n}\{d_D^{(n)}(\phi, \psi) \wedge 1\}. \tag{22}$$

Then d_D is a metric on $D^\circ(\mathbb{R}, \mathcal{D})$ *under which* $D^\circ(\mathbb{R}, \mathcal{D})$ *is complete and separable.* Moreover, *the metrics* d_C *and* d_D *generate the same topology on* $C^\circ(\mathbb{R}, \mathcal{D})$. *For the metric* d_D*, for all bounded intervals* I *and all* $x \in \mathbb{R}$*, the* evaluation map

$$\phi \mapsto \phi_I^+(x) : D^\circ(\mathbb{R}, \mathcal{D}) \to \mathbb{R}$$

is Borel measurable. Moreover, the Borel σ*-algebra on* $D^\circ(\mathbb{R}, \mathcal{D})$ *is generated by the set of all such evaluation maps with* $I = (s, t]$ *and* s, t *and* x *rational.*

6. Convergence to the coalescing Brownian flow. We now give a criterion for weak convergence on $D^\circ(\mathbb{R}, \mathcal{D})$ and use it to show that the disturbance flow converges to the coalescing Brownian flow.

For $e = (s, x) \in \mathbb{R}^2$ *and* $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$*, the maps*

$$t \mapsto \phi_{(s,t]}^{\pm}(x) : [s, \infty) \to \mathbb{R}$$

are cadlag. Hence, we can extend the maps $Z^e = Z^{e,+}$ and $Z^{e,-}$, which we defined on $C^\circ(\mathbb{R}, \mathcal{D})$ in Section 3, to measurable maps $Z^{e,\pm} : D^\circ(\mathbb{R}, \mathcal{D}) \to D_e$ by setting

$$Z^{e,\pm}(\phi) = \left(\phi_{(s,t]}^{\pm}(x) : t \geq s\right).$$

Let $E = (e_k : k \in \mathbb{N})$ be any countable dense subset of $\mathbb{R}^2$. Write $Z^{E,\pm}$ for the maps $D^\circ(\mathbb{R}, \mathcal{D}) \to D_E$ given by $Z^{E,\pm} = (Z^{e_k,\pm} : k \in \mathbb{N})$. Write $Z^E = Z^{E,+}$. The following result is a criterion for weak convergence on $D^\circ(\mathbb{R}, \mathcal{D})$. If we restrict to measures supported on $C^\circ(\mathbb{R}, \mathcal{D})$, this is directly analogous to [6], Theorem 4.1.

THEOREM 6.1. *Let* $(\mu_n : n \in \mathbb{N})$ *and* μ *be Borel probability measures on* $D^\circ(\mathbb{R}, \mathcal{D})$*. Assume that* $Z^{E,-} = Z^{E,+}$ *holds* μ_n*-almost surely for all* n *and* μ*-almost surely. Assume further that* $\mu_n \circ (Z^E)^{-1} \to \mu \circ (Z^E)^{-1}$ *weakly on* D_E*. Then* $\mu_n \to \mu$ *weakly on* $D^\circ(\mathbb{R}, \mathcal{D})$*.*

---

[8] As in the case of the standard Skorokhod topology, localization in time sits awkwardly with the stretching of time introduced via the homeomorphisms λ. There is no fundamental obstacle, just some messiness at the edges. Note that, when $I \cup \lambda(I) \subseteq [-n, n]$, we have

$$\|\chi_n(I)\phi_I^\times - \chi_n(\lambda(I))\psi_{\lambda(I)}^\times\| = d_{\mathcal{D}}(\phi_I, \psi_{\lambda(I)}).$$

Also, for all intervals I, we have $|\chi_n(\lambda(I)) - \chi_n(I)| \leq \gamma(\lambda)$ and

$$\|\chi_n(I)\phi_I^\times - \chi_n(\lambda(I))\psi_{\lambda(I)}^\times\| \leq \chi_n(I) d_{\mathcal{D}}(\phi_I, \psi_{\lambda(I)}) + |\chi_n(\lambda(I)) - \chi_n(I)| \|\psi_{\lambda(I)}^\times\|.$$

PROOF. Set

$$D^\circ(E) = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : Z^{E,+}(\phi) = Z^{E,-}(\phi)\},$$
$$D^\circ_E = \{Z^E(\phi) : \phi \in D^\circ(E)\}.$$

Let Φ_n and Φ be random variables in $D^\circ(\mathbb{R}, \mathcal{D})$ having distributions μ_n and μ, respectively. Then $Z^E(\Phi_n) \to Z^E(\Phi)$ weakly on D_E. Also $\Phi_n, \Phi \in D^\circ(E)$ almost surely, so $Z^E(\Phi_n), Z^E(\Phi) \in D^\circ_E$ almost surely. Now D°_E *is measurable and* Z^E *maps* $D^\circ(E)$ *bijectively to* D°_E. Denote the inverse bijection by Φ^E. Then $\Phi^E : D^\circ_E \to D^\circ(E)$ *is measurable and continuous*. Hence, $\Phi_n = \Phi^E(Z^E(\Phi_n)) \to \Phi^E(Z^E(\Phi)) = \Phi$ weakly on $D^\circ(\mathbb{R}, \mathcal{D})$. □

The Poisson disturbance flow with disturbance f and the lattice disturbance flow with disturbance f were defined in Section 2. Properties (4), (5) and (6) hold in both cases and imply that the flow $\Phi = (\Phi_I : I \subseteq \mathbb{R})$ may be considered as a Borel random variable in $D^\circ(\mathbb{R}, \mathcal{D})$. Moreover, as we noted in (7), for either of these flows Φ, for all $e \in \mathbb{R}^2$, we have $Z^{e,-}(\Phi) = Z^{e,+}(\Phi)$ almost surely. The same is true when Φ is a coalescing Brownian flow, as shown in Theorem 3.1. Our main result now follows directly from Proposition 2.3 and Theorem 6.1.

THEOREM 6.2. *The Poisson disturbance flow with disturbance f and the lattice disturbance flow with disturbance f both converge weakly to the coalescing Brownian flow on the circle on $D^\circ(\mathbb{R}, \mathcal{D})$, uniformly in $f \in \mathcal{D}^*$ as f becomes small and localized, that is, as $\rho(f) \to \infty$ and $\lambda(f) \to 0$.*

7. Time reversal. Time reversal acts as an isometry on our metric spaces of weak flows. The time reversal of a disturbance flow with disturbance f is the disturbance flow with disturbance f^{-1}. We use these facts to give a new proof of the time-reversibility of the coalescing Brownian flow, and to obtain a weak limit for the joint law of forward and backward trajectories for disturbance flows.

For $f^+ \in \mathcal{R}$ and $f^- \in \mathcal{L}$, we define a *left-continuous inverse* $(f^+)^{-1} \in \mathcal{L}$ and a *right-continuous inverse* $(f^-)^{-1} \in \mathcal{R}$ by

$$(f^+)^{-1}(y) = \inf\{x \in \mathbb{R} : f^+(x) > y\},$$
$$(f^-)^{-1}(y) = \sup\{x \in \mathbb{R} : f^-(x) < y\}.$$

The map $f^+ \mapsto (f^+)^{-1} : \mathcal{R} \to \mathcal{L}$ is a bijection, with $((f^+)^{-1})^{-1} = f^+$ and

$$(f_1^+ \circ f_2^+)^{-1} = (f_2^+)^{-1} \circ (f_1^+)^{-1}, \qquad f_1, f_2 \in \mathcal{R}.$$

We have $f^+ \circ (f^+)^{-1} = \mathrm{id}$ if and only if f^+ is a homeomorphism. Define for $f = \{f^-, f^+\} \in \mathcal{D}$ the *inverse* $f^{-1} = \{(f^+)^{-1}, (f^-)^{-1}\} \in \mathcal{D}$. Note that $(f^{-1})^\times =$

$-f^\times$, so the map $f \mapsto f^{-1} : \mathcal{D} \to \mathcal{D}$ is an isometry. Define the *time-reversal map* $\wedge : D^\circ(\mathbb{R}, \mathcal{D}) \to D^\circ(\mathbb{R}, \mathcal{D})$ by

$$\hat{\phi}_I = \phi_{-I}^{-1},$$

where $-I = \{-x : x \in I\}$. It is straightforward to check that this is a well-defined isometry of $D^\circ(\mathbb{R}, \mathcal{D})$, which restricts to an isometry of $C^\circ(\mathbb{R}, \mathcal{D})$.

PROPOSITION 7.1. *The time-reversal of a disturbance flow with disturbance f is a disturbance flow with disturbance f^{-1}.*

PROOF. Fix $f \in \mathcal{D}^*$. Set $g = f^{-1}$ and

$$\Delta = \{(x, y) \in \mathbb{R}^2 : y < f(x)\} = \{(x, y) \in \mathbb{R}^2 : x > g(y)\},$$
$$\Delta_0 = \{(x, y) \in \mathbb{R}^2 : y < x\}.$$

Then, by Fubini's theorem,

$$\int_0^1 \tilde{f}(x)\,dx = \int_0^1 \int_{\mathbb{R}} (1_\Delta - 1_{\Delta_0})(x, y)\,dx\,dy = -\int_0^1 \tilde{g}(y)\,dy \tag{23}$$

and

$$\int_0^1 \tilde{f}(x)^2\,dx = \int_0^1 \int_{\mathbb{R}} 2(y - x)(1_\Delta - 1_{\Delta_0})(x, y)\,dx\,dy = \int_0^1 \tilde{g}(y)^2\,dy. \tag{24}$$

So $g \in \mathcal{D}^*$ and $\rho(g) = \rho(f)$. We may construct a lattice disturbance flow Φ with disturbance f from a sequence $(\Theta_n : n \in \mathbb{Z})$ of independent random variables, uniformly distributed on $(0, 1]$, by

$$\Phi_I^\pm = f_{\Theta_n}^\pm \circ \cdots \circ f_{\Theta_m}^\pm,$$

where m and n are respectively the minimal and maximal integers in ρI. Then

$$\hat{\Phi}_I^\pm = g_{\Theta_{-n}}^\pm \circ \cdots \circ g_{\Theta_{-m}}^\pm.$$

Since $(\Theta_n : n \in \mathbb{Z})$ and $(\Theta_{-n} : n \in \mathbb{Z})$ have the same distribution, it follows that $\hat{\Phi}$ is a lattice disturbance flow with disturbance g. The Poisson case is similar. □

We were surprised by the calculations (23) and (24) which, though elementary, we did not suspect until we realized they were forced by the known reversibility of the universal scaling limit. On the other hand, we can now deduce the reversibility of the limit, as already known for other formulations of the coalescing Brownian flow. See, for example, [1, 6, 16, 19] and the references therein.

COROLLARY 7.2. *The law μ_A of the coalescing Brownian flow on the circle is invariant under time-reversal.*

PROOF. Fix $r \in (0, 1/2]$ and define $f = f_r \in \mathcal{D}^*$ by

$$f^+(n+x) = n + \big(r \vee x \wedge (1-r)\big), \qquad n \in \mathbb{Z}, x \in [0,1).$$

Then $\tilde{f}^+(x) = ((r-x) \vee 0) + ((1-r-x) \wedge 0)$ for $x \in [0,1)$, so $\rho(f) = 3/(2r^3)$ and

$$\int_0^1 \tilde{f}(x)\tilde{f}(x+a)\,dx = 0, \qquad 2r \le a \le 1-2r,$$

so $\lambda(f) \le 2r$. Moreover, $\rho(f^{-1}) = \rho(f)$ and $\lambda(f^{-1}) \le 2r$.

Write μ_A^f for the law of a lattice disturbance flow with disturbance f. Set $\hat{\mu}_A = \mu_A \circ \wedge^{-1}$ and $\hat{\mu}_A^f = \mu_A^f \circ \wedge^{-1}$. Consider the limit $r \to 0$. By Theorem 6.2, we know that $\mu_A^f \to \mu_A$ and $\mu_A^{f^{-1}} \to \mu_A$, weakly on $D^\circ(\mathbb{R}, \mathcal{D})$. Since the time-reversal map $\phi \mapsto \hat{\phi}$ is an isometry, it follows, using the preceding proposition, that $\mu_A^{f^{-1}} = \hat{\mu}_A^f \to \hat{\mu}_A$, weakly on $D^\circ(\mathbb{R}, \mathcal{D})$. Hence, $\mu_A = \hat{\mu}_A$. □

The same argument may be used to prove time reversibility of the coalescing Brownian flow on the line, as introduced in the next section. In fact, Theorem 8.5 below applies to show that the $\sqrt{r}$-scale disturbance flow (defined below) with disturbance f_r (as above) converges weakly as $r \to 0$ to the coalescing Brownian flow on the line. Then reversibility follows by the argument of Corollary 7.2.

From the flow-level result Theorem 6.2, we can deduce weak convergence also for paths running forward and backward in time from a given sequence of points $E = (e_k : k \in \mathbb{N})$ in $\mathbb{R}^2$. For $e = (s, x) \in \mathbb{R}^2$, define $\check{D}_e = \{\xi \in D(\mathbb{R}, \mathbb{R}) : \xi_s = x\}$ and set $\check{D}_E = \prod_{k=1}^\infty \check{D}_{e_k}$. For $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$, define

$$\check{Z}_t^{e,\pm}(\phi) = \begin{cases} \phi_{(s,t]}^{\pm}(x), & t \ge s, \\ (\phi^{-1})_{(t,s]}^{\pm}(x), & t < s. \end{cases} \tag{25}$$

Then $\check{Z}^{e,\pm}(\phi) \in \check{D}_e$ and extends $Z^{e,\pm}(\phi)$, as defined in Section 5, from $[s, \infty)$ to the whole of $\mathbb{R}$. For all $e \in \mathbb{R}^2$, we have $\check{Z}^{e,+} = \check{Z}^{e,-}$ almost everywhere on $D^\circ(\mathbb{R}, \mathcal{D})$ for both μ_A and μ_A^f, for any disturbance f. So, we drop the $\pm$. Denote by $\check{\mu}_E^f$ the law of $(\check{Z}^{e_k} : k \in \mathbb{N})$ on $\check{D}_E$ under μ_A^f and by $\check{\mu}_E$ the corresponding law under μ_A.

COROLLARY 7.3. *We have $\check{\mu}_E^f \to \check{\mu}_E$ weakly on $\check{D}_E$, uniformly in $f \in \mathcal{D}^*$, as $\rho(f) \to \infty$ and $\lambda(f) \to 0$.*

PROOF. We can check that $\check{Z}^{(s,x),+}$ is continuous as a map $D^\circ(\mathbb{R}, \mathcal{D}) \to \check{D}_{(s,x)}$ at $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$ provided

$$\check{Z}^{(s,x\pm\delta),+}(\phi) \to \check{Z}^{(s,x),+}(\phi)$$

uniformly on $\mathbb{R}$ as $\delta \to 0$. Since this property holds for μ_A almost all ϕ, the claimed limit follows from Theorem 6.2 by a standard property of weak convergence. $\square$

Weak convergence of the forward paths to coalescing Brownian motions was shown in Proposition 2.3. The corresponding backward property is immediate from the fact that the time reversal of a disturbance flow is another such flow. What is new in the result just proved is the identification of the limit of the joint law of these backward and forward paths—which has the property that the bi-infinite paths never cross.

8. Local limits. We now prove local weak convergence of disturbance flows, for a scale $\varepsilon \in (0, 1]$ intermediate between the scale of the disturbance f and the unit scale of the circle. Some variations of our set-up will be needed, as we rescale in a way which does not preserve the degree 1 property (1), and the limit object is the coalescing Brownian flow on the line. Write $\bar{\mathcal{D}}$ for the set of all pairs $\{f^-, f^+\}$ where $f^+ : \mathbb{R} \to \mathbb{R}$ is nondecreasing and right-continuous and where f^- is the left-continuous modification of f^+. For $\varepsilon \in (0, 1]$, define the scaling map $\sigma_\varepsilon : \bar{\mathcal{D}} \to \bar{\mathcal{D}}$ by

$$\sigma_\varepsilon f(x) = \varepsilon^{-1} f(\varepsilon x).$$

This map can be thought of as zooming in on the neighborhood around the origin. We associate to a disturbance flow $\Phi = (\Phi_I : I \subseteq \mathbb{R})$ the *ε-scale disturbance flow* $\Phi^\varepsilon = (\Phi^\varepsilon_I : I \subseteq \mathbb{R})$, given by

$$\Phi^\varepsilon_I = \sigma_\varepsilon(\Phi_{\varepsilon^2 I}).$$

For $e \in \mathbb{R}^2$, we write $X^{e,\varepsilon}$ for the trajectory of Φ^ε starting from e. By the estimate (8), the jumps of $X^{e,\varepsilon}$ are bounded in absolute value by $\varepsilon^{-1}(3/\rho)^{1/3}$. A small variation of the proof of Proposition 2.1 then leads to the following result.

PROPOSITION 8.1. *The trajectory $X^{e,\varepsilon}$ of the ε-scale Poisson disturbance flow with disturbance f converges weakly to Brownian motion on D_e, uniformly in $f \in \mathcal{D}^*$ and $\varepsilon \in (0, 1]$ as $\varepsilon^3 \rho(f) \to \infty$.*

Fix a sequence $E = (e_k : k \in \mathbb{N})$ in $\mathbb{R}^2$ and write D_E and C_E for the spaces of cadlag and continuous paths starting from E, as in Section 2. Write $e_k = (s_k, x_k)$ and recall the coordinate processes Z^k and their filtration $(\mathcal{Z}_t)_{t \in \mathbb{R}}$, defined in Section 2. Define on C_E the collision times

$$\bar{T}^{jk} = \inf\{t \ge s_j \vee s_k : Z^j_t = Z^k_t\}.$$

The law $\bar{\mu}_E$ on C_E of coalescing Brownian motions *on the line* then has the following martingale characterization: for all j, k, the processes $(Z^k_t)_{t \ge s_k}$ and

$(Z_t^j Z_t^k - (t - \bar{T}^{jk})^+)_{t \geq s_j \vee s_k}$ are both continuous local martingales in the filtration $(\mathcal{Z}_t)_{t \in \mathbb{R}}$.

For small ε, we shall need to quantify the localization of a disturbance in terms of the smallest constant $\lambda = \lambda(f, \varepsilon) \in (0, 1]$ such that

$$\rho \int_0^1 |\tilde{f}(x+a)\tilde{f}(x)|\, dx \leq \lambda, \qquad a \in [\varepsilon\lambda, 1 - \varepsilon\lambda].$$

PROPOSITION 8.2. *The joint distribution $\mu_E^{f,\varepsilon}$ of the family of trajectories $(X^{e,\varepsilon} : e \in E)$ in the ε-scale Poisson disturbance flow with disturbance f converges weakly to the coalescing Brownian law $\bar{\mu}_E$ on D_E, uniformly in $f \in \mathcal{D}^*$, as $\varepsilon \to 0$ with $\varepsilon^3 \rho(f) \to \infty$ and $\lambda(f, \varepsilon) \to 0$.*

PROOF. Write X^k for $X^{e_k,\varepsilon}$ within the proof. The family of laws $\{\mu_E^{f,\varepsilon} : f \in \mathcal{D}^*, \varepsilon \in (0, 1]\}$ is tight on D_E. Let μ be a weak limit law of this family for the limit $\varepsilon \to 0$ with $\varepsilon^3 \rho(f) \to \infty$ and $\lambda = \lambda(f, \varepsilon) \to 0$. Then, as in Proposition 2.3, under μ, for all j, the processes $(Z_t^j : t \geq s_j)$ and $((Z_t^j)^2 - t : t \geq s_j)$ are continuous local martingales. For all j, k, the process

$$X_t^j X_t^k - \int_{s_j \vee s_k}^t b(\varepsilon X_s^j, \varepsilon X_s^k)\, ds, \qquad t \geq s_j \vee s_k,$$

is a martingale. Note that $|b(\varepsilon X_s^j, \varepsilon X_s^k)| \leq \lambda$ until $|X_t^j - X_t^k|$ leaves $[\lambda, \varepsilon^{-1} - \lambda]$. Define for $R \geq 1$

$$\bar{T}^{jk,R} = \inf\{t \geq s_j \vee s_k : |Z_t^j - Z_t^k| \notin [1/R, R]\}$$

then, $\bar{T}^{jk,R} \uparrow \bar{T}^{jk}$ everywhere on C_E as $R \to \infty$. Under μ, the process $(Z_t^j Z_t^k : s_j \vee s_k \leq t < \bar{T}^{jk,R})$ is a local martingale for all R, so $(Z_t^j Z_t^k : s_j \vee s_k \leq t < \bar{T}^{jk})$ is also a local martingale. Now μ inherits from the laws $\mu_E^{f,\varepsilon}$ the property that, almost surely, the process $(Z_t^j - Z_t^k : t \geq s_j \vee s_k)$ does not change sign. Hence, $Z_t^j - Z_t^k$ is constant for $t \geq \bar{T}^{jk}$. It follows that $(Z_t^j Z_t^k - (t - \bar{T}^{jk})^+)_{t \geq s_j \vee s_k}$ is a continuous local martingale. Hence, $\mu = \bar{\mu}_E$. □

We obtain state-spaces for flows on the line by replacing $\mathcal{D}$ by $\bar{\mathcal{D}}$ in the definitions made in Sections 3 and 5, and replacing the metric $d_{\mathcal{D}}$ by

$$d_{\bar{\mathcal{D}}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \sup_{t \in [-n,n]} \big(|f^\times(t) - g^\times(t)| \wedge 1\big). \tag{26}$$

Denote by $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$ the set of continuous weak flows with values in $\bar{\mathcal{D}}$. Define the coordinate processes $Z^e = Z^{e,+}$ and $Z^{e,-}$ and their filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ on $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$

just as for $C^\circ(\mathbb{R},\mathcal{D})$ in Section 3. The collision time $\bar{T}^{ee'}: C^\circ(\mathbb{R},\bar{\mathcal{D}}) \to [0,\infty]$, for $e=(s,x)$ and $e'=(s',x')$, is now given by

$$\bar{T}^{ee'}(\phi) = \inf\{t \geq s \vee s' : Z_t^e(\phi) = Z_t^{e'}(\phi)\}.$$

The following result is proved in [5], Section 9 and, analogously to Theorem 3.1, shows that there exists a unique probability measure on $C^\circ(\mathbb{R},\bar{\mathcal{D}})$ under which the left and right coordinate processes $Z^{e,\pm}$ agree almost surely for all $e \in \mathbb{R}^2$ and behave as coalescing Brownian motions.

THEOREM 8.3. *There exists a unique Borel probability measure $\bar{\mu}_A$ on $C^\circ(\mathbb{R},\bar{\mathcal{D}})$ under which, for all $e, e' \in \mathbb{R}^2$, the processes $(Z_t^e)_{t \geq s(e)}$ and $(Z_t^e Z_t^{e'} - (t-\bar{T}^{ee'})^+)_{t \geq s(e)\vee s(e')}$ are continuous local martingales for $(\mathcal{F}_t)_{t\in\mathbb{R}}$. Moreover, for all $e \in \mathbb{R}^2$, we have $Z^{e,+} = Z^{e,-}$ $\bar{\mu}_A$-almost surely.*

We call any $C^\circ(\mathbb{R},\bar{\mathcal{D}})$-valued random variable with law $\bar{\mu}_A$ a *coalescing Brownian flow*. The space $D^\circ(\mathbb{R},\bar{\mathcal{D}})$ of cadlag weak flows $(\phi_I : I \subseteq \mathbb{R})$ with $\phi_I \in \bar{\mathcal{D}}$ for all I is defined analogously to $D^\circ(\mathbb{R},\mathcal{D})$. The Skorokhod-type metric on $D^\circ(\mathbb{R},\bar{\mathcal{D}})$ is defined just as for $D^\circ(\mathbb{R},\mathcal{D})$, except that the metric of the uniform norm on $\mathcal{S}$ is replaced by a metric of uniform convergence on compacts on the space $\bar{\mathcal{S}}$ of contractions on $\mathbb{R}$. The following result follows from [5], Lemma 14.1. It extends [6], Theorem 4.1, in allowing processes with jumps in time. Note that the additional noncrossing criterion needed in [6] holds automatically in the space of weak flows.

THEOREM 8.4. *Let $(\mu_n : n \in \mathbb{N})$ and μ be Borel probability measures on $D^\circ(\mathbb{R},\bar{\mathcal{D}})$. Assume that $Z^{E,-} = Z^{E,+}$ holds μ_n-almost surely for all n and μ-almost surely. Assume further that, for any finite sequence E in $\mathbb{R}^2$, we have $\mu_n \circ (Z^E)^{-1} \to \mu \circ (Z^E)^{-1}$ weakly on D_E. Then $\mu_n \to \mu$ weakly on $D^\circ(\mathbb{R},\bar{\mathcal{D}})$.*

The ε-scale Poisson disturbance flow Φ^ε with disturbance f may be considered as a Borel random variable in $D^\circ(\mathbb{R},\bar{\mathcal{D}})$. Moreover, for all $e \in \mathbb{R}^2$, we have $Z^{e,-}(\Phi^\varepsilon) = Z^{e,+}(\Phi^\varepsilon)$ almost surely. The same is true in the lattice case. Hence, Proposition 8.2 and Theorems 8.3 and 8.4 imply the following local limit theorem.

THEOREM 8.5. *The ε-scale Poisson disturbance flow with disturbance f and the ε-scale lattice disturbance flow with disturbance f both converge weakly to the coalescing Brownian flow on the line on $D^\circ(\mathbb{R},\bar{\mathcal{D}})$, uniformly in $f \in \mathcal{D}^*$, as $\varepsilon \to 0$ with $\varepsilon^3\rho(f) \to \infty$ and $\lambda(f,\varepsilon) \to 0$.*

APPENDIX

A.1. Some properties of the space $\mathcal{D}$ of nondecreasing functions of degree 1. We give proofs in this subsection of a number of assertions made in Section 3.

PROPOSITION A.1. *The map $f \mapsto f^\times : \mathcal{D} \to \mathcal{S}$ is a well-defined bijection, with inverse given by*

$$f^-(x) = \inf\{t + f^\times(t) : t \in \mathbb{R}, x = t - f^\times(t)\},$$
$$f^+(x) = \sup\{t + f^\times(t) : t \in \mathbb{R}, x = t - f^\times(t)\}.$$

PROOF. Recall that $f^\times(t) = t - x$, where x is the unique point such that $f^-(x) \le 2t - x \le f^+(x)$. The periodicity of $f^\times$ is an easy consequence of the degree 1 condition. We now show that $f^\times$ is a contraction. Fix $s, t \in \mathbb{R}$ and suppose that $f^\times(s) = s - y$. Switching the roles of s and t if necessary, we may assume without loss that $x \ge y$. If $x = y$, then $f^\times(s) - f^\times(t) = s - t$. On the other hand, if $x > y$, then $2s - y \le f^+(y) \le f^-(x) \le 2t - x$, so

$$\begin{aligned} -(t-s) &\le -(t-s) + (2t - x) - (2s - y) = f^\times(t) - f^\times(s) \\ &= (t-s) - (x-y) < t - s. \end{aligned}$$

In both cases, we see that $|f^\times(t) - f^\times(s)| \le |t - s|$. Hence, $f^\times \in \mathcal{S}$.

Suppose now that $g \in \mathcal{S}$. Consider, for each $x \in \mathbb{R}$, the set

$$I_x = \{t + g(t) : t \in \mathbb{R}, x = t - g(t)\}.$$

Since g is a contraction, these sets are all intervals, and, since g is bounded, they cover $\mathbb{R}$. For $x, y \in \mathbb{R}$ with $x > y$, and for $s, t \in \mathbb{R}$ with $x = t - g(t)$, $y = s - g(s)$, we have $t - s - (g(t) - g(s)) = x - y > 0$, so $s \le t$, and so

$$t + g(t) - (s + g(s)) = t - s + (g(t) - g(s)) \ge 0.$$

Define $h^+(y) = \sup I_y$ and $h^-(x) = \inf I_x$. We have shown that $h^+(y) \le h^-(x)$. Moreover, since the intervals I_x cover $\mathbb{R}$, the functions $h^\pm$ must be the left-continuous and right-continuous versions of a nondecreasing function h, which then has the degree 1 property, because g is periodic. Thus, $h \in \mathcal{D}$.

For each $t \in \mathbb{R}$, we have $h^\times(t) = t - x$, where $2t - x \in I_x$, and so $2t - x = s + g(s)$ for some $s \in \mathbb{R}$ with $x = s + g(s)$. Then $s = t$ and so $h^\times(t) = g(t)$. Hence, $h^\times = g$. On the other hand, if we take $g = f^\times$ and if x is a point of continuity of f, then we find $I_x = \{f(x)\}$, so $h^+(x) = h^-(x) = f(x)$. Hence, $h = f$. We have now shown that $f \mapsto f^\times : \mathcal{D} \to \mathcal{S}$ is a bijection, and that its inverse has the claimed form. $\square$

PROPOSITION A.2. *For $f, g \in \mathcal{D}$ and $\varepsilon > 0$,*

$$d_\mathcal{D}(f, g) \le \varepsilon \iff f^-(x - \varepsilon) - \varepsilon \le g^-(x) \le g^+(x) \le f^+(x + \varepsilon) + \varepsilon \quad \textit{for all } x \in \mathbb{R}.$$

Moreover, for any sequence $(f_n : n \in \mathbb{N})$ in $\mathcal{D}$,

$$f_n \to f \text{ in } \mathcal{D} \iff f_n^+(x) \to f(x) \quad \textit{at all points } x \in \mathbb{R} \textit{ where } f \textit{ is continuous}.$$

PROOF. Suppose that $d_{\mathcal{D}}(f, g) \le \varepsilon$ and that x is a continuity point of g. Then $g(x) = t + g^{\times}(t)$ for some $t \in \mathbb{R}$ with $x = t - g^{\times}(t)$. We must have $x + \varepsilon \ge t - f^{\times}(t)$ and $g(x) \le t + f^{\times}(t) + \varepsilon$, so $f^{+}(x+\varepsilon) + \varepsilon \ge t + f^{\times}(t) + \varepsilon \ge g(x)$. Similarly $f^{-}(x-\varepsilon) - \varepsilon \le g(x)$. These inequalities extend to all $x \in \mathbb{R}$ by taking left and right limits along continuity points.

Conversely, suppose that $t \in \mathbb{R}$ is such that $|f^{\times}(t) - g^{\times}(t)| = d_{\mathcal{D}}(f, g)$ and let $x = t - g^{\times}(t)$ and $y = t - f^{\times}(t)$. Then x is the unique point with $g^{-}(x) + x \le 2t \le g^{+}(x) + x$ and y is the unique point such that $f^{-}(y) + y \le 2t \le f^{+}(y) + y$. Hence, $f^{-}(x-\varepsilon) - \varepsilon \le g^{-}(x) \le g^{+}(x) \le f^{+}(x+\varepsilon) + \varepsilon$ implies $y \in [x-\varepsilon, x+\varepsilon]$ and so $d_{\mathcal{D}}(f, g) = |y - x| \le \varepsilon$.

It follows directly that for any sequence $(f_n : n \in \mathbb{N})$ in $\mathcal{D}$, if $d_{\mathcal{D}}(f_n, f) \to 0$ as $n \to \infty$, then $f_n^{+}(x) \to f(x)$ at all points $x \in \mathbb{R}$ where f is continuous.

Now suppose $f_n^{+}(x) \to f(x)$ at all points $x \in \mathbb{R}$ where f is continuous. By equicontinuity, it will suffice to show that $f_n^{\times}(t) \to f^{\times}(t)$ for each $t \in \mathbb{R}$. Set $x = t - f^{\times}(t)$ and $x_n = t - f_n^{\times}(t)$. Given $\varepsilon > 0$, choose $y_1 \in (x-\varepsilon, x)$ and $y_2 \in (x, x+\varepsilon)$, both points of continuity of f. Now $f(y_1) + y_1 < 2t < f(y_2) + y_2$, so there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have $f_n^{+}(y_1) + y_1 < 2t < f_n^{+}(y_2) + y_2$, which implies $x_n \in [y_1, y_2]$, and hence $|f_n^{\times}(t) - f^{\times}(t)| < \varepsilon$, as required. □

PROPOSITION A.3. *Suppose $f_n \to f$, $g_n \to g$, $h_n \to h$ in $\mathcal{D}$ with $h_n^{+} \le f_n^{+} \circ g_n^{+}$ for all n. Then $h^{+} \le f^{+} \circ g^{+}$.*

PROOF. It will suffice to establish the inequality at all points x where g and h are both continuous. Given $\varepsilon > 0$, since f^{+} is right-continuous, there exists a point $y > g(x)$ where f is continuous and such that $f(y) < f^{+}(g(x)) + \varepsilon$. Then $f_n^{+}(y) < f^{+}(g(x)) + \varepsilon$ and $g_n^{+}(x) \le y$ eventually, so

$$h_n^{+}(x) \le f_n^{+}\big(g_n^{+}(x)\big) \le f_n^{+}(y) < f^{+}\big(g(x)\big) + \varepsilon$$

eventually. Hence, $h^{+}(x) = \lim_{n\to\infty} h_n^{+}(x) \le f^{+}(g^{+}(x))$, as required. □

A.2. Some properties of the continuous flow-space $C^{\circ}(\mathbb{R}, \mathcal{D})$ and cadlag flow-space $D^{\circ}(\mathbb{R}, \mathcal{D})$. We give proofs in this subsection of a number of assertions made in Sections 3 and 5.

PROPOSITION A.4. *For $(s, x) \in \mathbb{R}^2$ and $\phi \in D^{\circ}(\mathbb{R}, \mathcal{D})$, the map*

$$t \mapsto \phi_{(s,t]}^{+}(x) : [s, \infty) \to \mathbb{R}$$

is cadlag, and is moreover continuous whenever $\phi \in C^{\circ}(\mathbb{R}, \mathcal{D})$.

PROOF. Given $t \ge s$ and $\varepsilon > 0$, we can choose $\delta > 0$ so that for all $u \in (t, t+\delta]$, $d_{\mathcal{D}}(\phi_{(t,u]}, \mathrm{id}) < \varepsilon/2$. For such u and for x a point of continuity of $\phi_{(s,t]}$, we have

$$\begin{aligned}\phi^+_{(s,t]}(x)-\varepsilon &= \phi^-_{(s,t]}(x)-\varepsilon\\ &\le \phi^-_{(t,u]}\circ\phi^-_{(s,t]}(x)\\ &\le \phi^-_{(s,u]}(x)\\ &\le \phi^+_{(s,u]}(x)\\ &\le \phi^+_{(t,u]}\circ\phi^+_{(s,t]}(x)\\ &\le \phi^+_{(s,t]}(x)+\varepsilon,\end{aligned}$$

so $|\phi^+_{(s,u]}(x)-\phi^+_{(s,t]}(x)|\le\varepsilon$. The final estimate extends to all x by right-continuity. Hence, the map is right continuous. A similar argument shows that, for $u\in(s,t)$, we have $|\phi^+_{(s,u]}(x)-\phi^+_{(s,t)}(x)|\to 0$ as $u\to t$, so that the map has a left limit at t given by $\phi^+_{(s,t)}(x)$. Finally, if $\phi\in C^\circ(\mathbb{R},\mathcal{D})$, then $\phi_{(s,t)}=\phi_{(s,t]}$, so the map is continuous. □

PROPOSITION A.5. *For all $\phi\in C^\circ(\mathbb{R},\mathcal{D})$, the map $(s,t)\mapsto\phi_{ts}:\{(s,t):s\le t\}\to\mathcal{D}$ is continuous. Moreover, for all $\phi\in D^\circ(\mathbb{R},\mathcal{D})$ and for any sequence of bounded intervals $I_n\to I$, we have $\phi_{I_n}\to\phi_I$.*

PROOF. The first assertion follows from the second: given $\phi\in C^\circ(\mathbb{R},\mathcal{D})$ and sequences $s_n\to s$ and $t_n\to t$, then, passing to a subsequence if necessary, we can assume that $(s_n,t_n]\to I$ for some interval I with $\inf I=s$ and $\sup I=t$. Then, by the second assertion, we have $\phi_{t_ns_n}\to\phi_I=\phi_{ts}$, as required.

So, let us fix $\phi\in D^\circ(\mathbb{R},\mathcal{D})$ and a sequence of bounded intervals $I_n\to I$. By combining the cadlag and weak flow properties, we can show the following variant of the cadlag property: for all $t\in\mathbb{R}$, we have

$$\phi_{[s,t)}\to\mathrm{id}\quad\text{as } s\uparrow t,\qquad \phi_{(t,u]}\to\mathrm{id}\quad\text{as } u\downarrow t. \tag{27}$$

For each n, there exist two disjoint intervals J_n and J'_n, possibly empty, such that $I\triangle I_n=J_n\cup J'_n$. For any such J_n and J'_n, using the weak flow property, we obtain

$$d_{\mathcal{D}}(\phi_I,\phi_{I_n})\le\|\phi_{J_n}-\mathrm{id}\,\|+\|\phi_{J'_n}-\mathrm{id}\,\|.$$

Set $s=\inf I$, $s_n=\inf I_n$, $t=\sup I$ and $t_n=\sup I_n$. Then $s_n\to s$, $t_n\to t$, and

$$\begin{aligned}&\text{if } s\in I \text{ then } s\in I_n \text{ eventually},\qquad &&\text{if } s\notin I \text{ then } s\notin I_n \text{ eventually},\\ &\text{if } t\in I \text{ then } t\in I_n \text{ eventually},\qquad &&\text{if } t\notin I \text{ then } t\notin I_n \text{ eventually}.\end{aligned}$$

Hence, using the cadlag property or (27), or both, we find that $\phi_{J_n}\to\mathrm{id}$ and $\phi_{J'_n}\to\mathrm{id}$, which proves the proposition. □

PROPOSITION A.6. *The metrics d_C and d_D generate the same topology on $C^\circ(\mathbb{R}, \mathcal{D})$.*

PROOF. On comparing the definitions of $d_C^{(n)}$ and $d_D^{(n)}$ for each $n \in \mathbb{N}$, and considering the choice $\lambda = \mathrm{id}$, we see that $d_D \le d_C$. Hence, it will suffice to show, given $\phi \in C^\circ(\mathbb{R}, \mathcal{D})$, $n \in \mathbb{N}$ and $\varepsilon > 0$, that there exists $\varepsilon' > 0$ such that, for all $\psi \in C^\circ(\mathbb{R}, \mathcal{D})$, we have $d_C^{(n)}(\phi, \psi) < \varepsilon$ whenever $d_D^{(n+1)}(\phi, \psi) < \varepsilon'$. By the preceding proposition, there exists a $\delta \in (0, 1]$ such that $d_{\mathcal{D}}(\phi_{ts}, \phi_{t's'}) < \varepsilon/2$ whenever $|s - s'|, |t - t'| \le \delta$ and $s, t \in (-n, n)$. Set $\varepsilon' = \delta \wedge (\varepsilon/2)$ and suppose that $d_D^{(n+1)}(\phi, \psi) < \varepsilon'$. Then there exists an increasing homeomorphism λ of $\mathbb{R}$, with $|\lambda(t) - t| \le \delta$ for all t, such that, for all intervals I, we have $\|\chi_{n+1}(I)\psi_I^\times - \chi_{n+1}(\lambda(I))\phi_{\lambda(I)}^\times\| < \varepsilon/2$. Given $s, t \in (-n, n)$ with $s < t$, take $I = (s, t]$. Then $\chi_{n+1}(I) = \chi_{n+1}(\lambda(I)) = 1$, so $d_{\mathcal{D}}(\phi_{\lambda(t)\lambda(s)}, \psi_{ts}) = \|\psi_I^\times - \phi_{\lambda(I)}^\times\| < \varepsilon/2$. But then, for all such s, t, we have

$$d_{\mathcal{D}}(\phi_{ts}, \psi_{ts}) \le d_{\mathcal{D}}(\phi_{ts}, \phi_{\lambda(t)\lambda(s)}) + d_{\mathcal{D}}(\phi_{\lambda(t)\lambda(s)}, \psi_{ts}) < \varepsilon,$$

so $d_C^{(n)}(\phi, \psi) < \varepsilon$, as required. □

PROPOSITION A.7. *The metric spaces $(C^\circ(\mathbb{R}, \mathcal{D}), d_C)$ and $(D^\circ(\mathbb{R}, \mathcal{D}), d_D)$ are complete and separable.*

PROOF. The argument for completeness is a variant of the corresponding argument for the usual Skorokhod space $D(\mathbb{R}, S)$ of cadlag paths in complete separable metric space S, as found, for example, in [2]. Suppose then that $(\psi^n)_{n\ge1}$ is a Cauchy sequence in $D^\circ(\mathbb{R}, \mathcal{D})$. There exists a subsequence $\phi^k = \psi^{n_k}$ such that $d_D^{(n)}(\phi^n, \phi^{n+1}) < 2^{-n}$ for all $n \ge 1$. It will suffice to find a limit in $D^\circ(\mathbb{R}, \mathcal{D})$ for $(\phi^n)_{n\ge1}$. Recall the definition of γ from (21). There exist increasing homeomorphisms κ_n of $\mathbb{R}$ for which $\gamma(\kappa_n) < 2^{-n}$ and

$$d_{\mathcal{D}}(\phi_I^n, \phi_{\kappa_n(I)}^{n+1}) < 2^{-n}, \qquad I \cup \kappa_n(I) \subseteq (-n, n).$$

For each $n \ge 1$, the sequence $(\kappa_{n+m} \circ \cdots \circ \kappa_n)_{m\ge1}$ converges uniformly on $\mathbb{R}$ to an increasing homeomorphism, λ_n say, with $\gamma(\lambda_n) < 2^{-n+1}$. Then $\kappa_n \circ \lambda_n^{-1} = \lambda_{n+1}^{-1}$, so

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_{\lambda_{n+1}^{-1}(I)}^{n+1}) < 2^{-n}, \qquad I \subseteq (-n+1, n-1).$$

So, for all $m \ge n$,

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_{\lambda_{n+m}^{-1}(I)}^{n+m}) < 2^{-n+1}, \qquad I \subseteq (-n+1, n-1). \tag{28}$$

Hence, for all bounded intervals $I \subseteq \mathbb{R}$, $(\phi_{\lambda_n^{-1}(I)}^n)_{n\ge1}$ is a Cauchy sequence in $\mathcal{D}$, which, since $\mathcal{D}$ is complete, has a limit $\phi_I \in \mathcal{D}$. On letting $m \to \infty$ in (28), we obtain

$$d_{\mathcal{D}}(\phi_{\lambda_n^{-1}(I)}^n, \phi_I) < 2^{-n+1}, \qquad I \subseteq (-n+1, n-1).$$

By Proposition A.3, $\phi = (\phi_I : I \subseteq \mathbb{R})$ has the weak flow property. To see that ϕ is cadlag, suppose given $\varepsilon > 0$ and $t \in \mathbb{R}$. Choose n such that $2^{-n+1} \le \varepsilon/3$ and $|t| \le n-2$. Then choose $\delta \in (0, 1]$ such that

$$d_{\mathcal{D}}\big(\phi^n_{\lambda_n^{-1}(s,t)}, \mathrm{id}\big) < \varepsilon/3, \qquad d_{\mathcal{D}}\big(\phi^n_{\lambda_n^{-1}(t,u)}, \mathrm{id}\big) < \varepsilon/3$$

whenever $s \in (t-\delta, t)$ and $u \in (t, t+\delta)$. For such s and u, we then have

$$d_{\mathcal{D}}(\phi_{(s,t)}, \mathrm{id}) < \varepsilon, \qquad d_{\mathcal{D}}(\phi_{(t,u)}, \mathrm{id}) < \varepsilon.$$

Hence, $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$. For $m \le n-3$, we have

$$\begin{aligned} d_D^{(m)}(\phi^n, \phi) &\le \gamma(\lambda_n) \vee \sup_{I \subseteq (-m-2, m+2)} \big\| \chi_m\big(\lambda_n^{-1}(I)\big)\phi^{n\times}_{\lambda_n^{-1}(I)} - \chi_m(I)\phi_I^\times \big\| \\ &\le \gamma(\lambda_n) \vee \sup_{I \subseteq (-m-2, m+2)} \big\{ d_{\mathcal{D}}\big(\phi^n_{\lambda_n^{-1}(I)}, \phi_I\big) + \gamma(\lambda_n)\|\phi_I^\times\| \big\} \\ &\le 2^{-n+1}\Big(1 + \sup_{I \subseteq (-m-2, m+2)} \|\phi_I^\times\|\Big). \end{aligned}$$

Hence, $d_D(\phi^n, \phi) \to 0$ as $n \to \infty$. We have shown that $D^\circ(\mathbb{R}, \mathcal{D})$ is complete. If the sequence $(\phi^n)_{n\ge 1}$ in fact lies in $C^\circ(\mathbb{R}, \mathcal{D})$, then by an obvious variation of the argument for the cadlag property, the limit ϕ also lies in $C^\circ(\mathbb{R}, \mathcal{D})$. Hence, $C^\circ(\mathbb{R}, \mathcal{D})$ is also complete. In particular, $C^\circ(\mathbb{R}, \mathcal{D})$ is a closed subspace in $D^\circ(\mathbb{R}, \mathcal{D})$.

We turn to the question of separability. Let us write D_N for the set of those $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ such that:

(i) for some $n \in \mathbb{N}$ and some rationals $t_1 < \cdots < t_n$, we have $\phi_J = \mathrm{id}$ for all time intervals J, which do not intersect the set $\{t_1, \dots, t_n\}$;

(ii) for all other time intervals I, the maps ϕ_I and ϕ_I^{-1} on $\mathbb{R}$ are constant on all space intervals which do not intersect $2^{-N}\mathbb{Z}$.

Note that each $\phi \in D_N$ is determined by the maps $\phi_{(t_k, t_m]}$, for integers $0 \le k < m \le n$, where $t_0 < t_1$, and for each of these maps there are only countably many possibilities (finitely many if we insist that $\phi(0) \in [0, 1)$). Hence, D_N is countable and so is $D_* = \bigcup_{N\ge 1} D_N$. We shall show that D_* is also dense in $D^\circ(\mathbb{R}, \mathcal{D})$.

Fix $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$ and $n_0 \ge 1$. It will suffice to find, for a given $\varepsilon > 0$, a $\psi \in D_*$ with $d_D^{(n_0)}(\phi, \psi) < \varepsilon$. By the cadlag property and compactness, there exist $n \in \mathbb{N}$ and reals $s_1 < \cdots < s_n$ in $I_0 = (-n_0 - 1, n_0 + 1)$ such that $d_{\mathcal{D}}(\phi_I, \mathrm{id}) < \varepsilon/4$ for every subinterval I of I_0, which does not intersect $\{s_1, \dots, s_n\}$. To see this, let

$$A = \big\{t \in I_0 : d_{\mathcal{D}}(\phi_{(s,t]}, \phi_{(s,t)}) \ge \varepsilon/4 \text{ for some } s < t\big\}.$$

If A contains infinitely many points, then there exists a sequence $(u_m)_{m\in\mathbb{N}}$ in A and $u \in \mathbb{R}$ such that $u_m \to u$ strictly monotonically. Suppose that $u_m \uparrow u$. Then, as in the proof of Proposition A.5, $\|\phi_{(u_m,u)} - \mathrm{id}\| < \varepsilon/8$ and $\|\phi_{[u_m,u)} - \mathrm{id}\| < \varepsilon/8$ for m is sufficiently large. But then, for all $s < u_m$,

$$\begin{aligned} d_{\mathcal{D}}(\phi_{(s,u_m]}, \phi_{(s,u_m)}) &\le d_{\mathcal{D}}(\phi_{(s,u_m]}, \phi_{(s,u)}) + d_{\mathcal{D}}(\phi_{(s,u)}, \phi_{(s,u_m)}) \\ &\le \|\phi_{(u_m,u)} - \mathrm{id}\| + \|\phi_{[u_m,u)} - \mathrm{id}\| \\ &< \varepsilon/4, \end{aligned}$$

contradicting $u_m \in A$. A similar contradiction arises if $u_m \downarrow u$, so A contains finitely many points. Therefore, $I_0 \setminus A$ consists of the disjoint union of finitely many open intervals. It remains to show that if J is one of these intervals, there exists some $\eta > 0$ such that if an interval $I \subseteq J$ and $\sup I - \inf I < \eta$, then $d_{\mathcal{D}}(\phi_I, \mathrm{id}) < \varepsilon/4$. If not then, there exists a sequence of intervals $I_m \subseteq J$ with $\sup I_m - \inf I_m < m^{-1}$ and $d_{\mathcal{D}}(\phi_{I_m}, \mathrm{id}) \ge \varepsilon/4$. By restricting to a subsequence if necessary $I_m \to I$ where $I = \varnothing$ or $\{t\}$ for some $t \in J$. Therefore, $\phi_{I_m} \to \phi_I$. But $\phi_\varnothing = \mathrm{id}$ and $d_{\mathcal{D}}(\phi_{\{t\}}, \mathrm{id}) < \varepsilon/4$ for all $t \notin A$, which contradicts $d_{\mathcal{D}}(\phi_{I_m}, \mathrm{id}) \ge \varepsilon/4$ for all m.

Next we can find rationals $t_1 < \cdots < t_n$ in I_0 and an increasing homeomorphism λ of $\mathbb{R}$, with $\lambda(t) = t$ for $t \notin I_0$, with $\gamma(\lambda) \sup_{I \subseteq I_0} \|\phi_I^\times\| < \varepsilon/4$, and such that $\lambda(t_m) = s_m$ for all m. Set $s_0 = t_0 = -n_0 - 1$.

For $f \in \mathcal{D}$, write $\Delta(f)$ for the set of points where f is not continuous. Define, for $m = 0, 1, \dots, n$,

$$\Delta_m = \bigcup_{k=0}^{m-1} \Delta\big(\phi^{-1}_{(s_k,s_m]}\big) \cup \bigcup_{k=m+1}^{n} \Delta(\phi_{(s_m,s_k]}).$$

Then Δ_m is countable, so we can choose $N \ge 1$ with $16 \cdot 2^{-N} \le \varepsilon$ and choose $\varepsilon_m \in \mathbb{R}$ with $|\varepsilon_m| \le 2^{-N}$ such that

$$\tau_m(\Delta_m) \cap 2^{-N}\mathbb{Z} = \varnothing, \qquad m = 0, 1, \dots, n,$$

where $\tau_m(x) = x + \varepsilon_m$. Set

$$\delta^-(x) = 2^N \lceil 2^{-N} x \rceil, \qquad \delta^+(x) = 2^N \lfloor 2^{-N} x \rfloor + 1.$$

Note that $\delta = \{\delta^-, \delta^+\} \in \mathcal{D}$. Define for $0 \le k < m \le n$

$$\begin{aligned} \psi^-_{(t_k,t_m]} &= (\delta^{-1})^- \circ (\tau_m)^{-1} \circ \phi^-_{(s_k,s_m]} \circ \tau_k \circ \delta^-, \\ \psi^+_{(t_k,t_m]} &= (\delta^{-1})^+ \circ (\tau_m)^{-1} \circ \phi^+_{(s_k,s_m]} \circ \tau_k \circ \delta^+. \end{aligned}$$

Then $\psi_{(t_k,t_m]} = \{\psi^-_{(t_k,t_m]}, \psi^+_{(t_k,t_m]}\} \in \mathcal{D}$ by our choice of ε_k and ε_m. Moreover, $\delta^+ \circ (\delta^{-1})^+ \ge \mathrm{id}$ and $\delta^- \circ (\delta^{-1})^- \le \mathrm{id}$ so, for $0 \le m < m' < m'' \le n$, we obtain the inequalities

$$\psi^-_{(t_{m'},t_{m''}]} \circ \psi^-_{(t_m,t_{m'}]} \le \psi^-_{(t_m,t_{m''}]} \le \psi^+_{(t_m,t_{m''}]} \le \psi^+_{(t_{m'},t_{m''}]} \circ \psi^+_{(t_m,t_{m'}]}$$

from the corresponding inequalities for ϕ. We use the equations $\|\delta - \mathrm{id}\| = 2^{-N}$ and $\|\tau_m - \mathrm{id}\| = |\varepsilon_m|$ to see that

$$d_{\mathcal{D}}(\phi_{(s_k,s_m]}, \psi_{(t_k,t_m]}) \le 4 \cdot 2^{-N}, \qquad 0 \le k < m \le n.$$

For all intervals J such that $J \cap \{t_1, \dots, t_n\} = \{t_{k+1}, \dots, t_m\}$, define $\psi_J = \psi_{(t_k,t_m]}$. For such intervals J, with $J \subseteq I_0$, we have $d_{\mathcal{D}}(\phi_{(s_k,s_m]\setminus\lambda(J)}, \mathrm{id}) < \varepsilon/4$ and $d_{\mathcal{D}}(\phi_{\lambda(J)\setminus(s_k,s_m]}, \mathrm{id}) < \varepsilon/4$; so, using the weak flow property for ϕ,

$$\begin{aligned} d_{\mathcal{D}}(\psi_J, \phi_{\lambda(J)}) &\le d_{\mathcal{D}}(\psi_{(t_k,t_m]}, \phi_{(s_k,s_m]}) + d_{\mathcal{D}}(\phi_{(s_k,s_m]}, \phi_{\lambda(J)}) \\ &\le 4 \cdot 2^{-N} + 2\varepsilon/4 \\ &< 3\varepsilon/4. \end{aligned}$$

Define $\psi_J = \mathrm{id}$ for all intervals J which do not intersect $\{t_1, \dots, t_n\}$. For such intervals J with $J \subseteq I_0$, we have $d_{\mathcal{D}}(\psi_J, \phi_{\lambda(J)}) \le d_{\mathcal{D}}(\mathrm{id}, \phi_{\lambda(J)}) \le \varepsilon/4$. Now $\psi \in D_N$ and

$$d_D^{(n_0)}(\phi, \psi) \le \gamma(\lambda) \vee \sup_{J \subseteq I_0} \{d_{\mathcal{D}}(\psi_J, \phi_{\lambda(J)}) + \gamma(\lambda)\|\phi_J^{\times}\|\} < \varepsilon$$

as required. This proves that $D^\circ(\mathbb{R}, \mathcal{D})$ is separable and, since $C^\circ(\mathbb{R}, \mathcal{D})$ is a closed subspace of $D^\circ(\mathbb{R}, \mathcal{D})$, it follows that $C^\circ(\mathbb{R}, \mathcal{D})$ is also separable. □

PROPOSITION A.8. *For all $s, t \in \mathbb{R}$ with $s < t$, and all $x \in \mathbb{R}$, the map $\phi \mapsto \phi_{ts}^+(x)$ on $C^\circ(\mathbb{R}, \mathcal{D})$ is Borel measurable. Moreover, the Borel σ-algebra on $C^\circ(\mathbb{R}, \mathcal{D})$ is generated by the set of all such maps with s, t and x rational.*

For all bounded intervals $I \subseteq \mathbb{R}$ and all $x \in \mathbb{R}$, the map $\phi \mapsto \phi_I^+(x)$ on $D^\circ(\mathbb{R}, \mathcal{D})$ is Borel measurable. Moreover, the Borel σ-algebra on $D^\circ(\mathbb{R}, \mathcal{D})$ is generated by the set of all such maps with $I = (s, t]$ and with s, t and x rational.

PROOF. The assertions for $C^\circ(\mathbb{R}, \mathcal{D})$ can be proved more simply than those for $D^\circ(\mathbb{R}, \mathcal{D})$. We omit details of the former, but note that these follow also from the latter, by general measure theoretic arguments, given what we already know about the two spaces.

The proof for $D^\circ(\mathbb{R}, \mathcal{D})$ is an adaptation of the analogous result for the classical Skorokhod space; see, for example, [10], page 335. We prove first the Borel measurability of the evaluation maps. Given a bounded interval I and $x \in \mathbb{R}$, we can find $s_n, t_n \in \mathbb{R}$ such that $(s_n, t_n] \to I$ as $n \to \infty$. Then $\phi_I^+(x) = \lim_{m\to\infty} \lim_{n\to\infty} \phi_{(s_n,t_n]}^+(x + 1/m)$, by Proposition A.5. Hence, it will suffice to consider intervals I of the form $(s, t]$. Fix s, t and x and define for each $m, n \in \mathbb{N}$ a function $F_{m,n}$ on $D^\circ(\mathbb{R}, \mathcal{D})$ by

$$F_{m,n}(\phi) = \int_s^{s+1/n} \int_t^{t+1/n} \int_x^{x+1/m} \phi_{(s',t']}^+(x')\, dx'\, dt'\, ds'.$$

Suppose $\phi^k \to \phi$ in $D^\circ(\mathbb{R}, \mathcal{D})$. We can choose increasing homeomorphisms λ_k of $\mathbb{R}$ such that, $\gamma(\lambda_k) \to 0$ and, uniformly in $r \in [s-1, s+1]$ and $u \in [t-1, t+1]$, we have

$$d_{\mathcal{D}}\big(\phi^k_{\lambda_k(r,u]}, \phi_{(r,u]}\big) \to 0.$$

Define

$$f(r,u) = \int_x^{x+1/m} \phi_{\lambda(r,u]}(x')\,dx', \qquad f_k(r,u) = \int_x^{x+1/m} \phi^k_{\lambda_k(r,u]}(x')\,dx'.$$

Then $f_k(r,u) \to f(r,u)$, uniformly in $r \in [s-1, s+1]$ and $u \in [t-1, t+1]$. Set $\mu_k = \lambda_k^{-1}$. Then

$$\begin{aligned} F_{m,n}(\phi^k) &= \int_{\mu_k(s)}^{\mu_k(s+1/n)} \int_{\mu_k(t)}^{\mu_k(t+1/n)} f_k(r,u)\,d\lambda_k(u)\,d\lambda_k(r) \\ &\to \int_s^{s+1/n} \int_t^{t+1/n} f(r,u)\,du\,dr = F_{m,n}(\phi), \end{aligned}$$

so $F_{m,n}$ is continuous on $D^\circ(\mathbb{R}, \mathcal{D})$. By Proposition A.5, we have

$$\phi^+_{(s,t]}(x) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn^2} F_{m,n}(\phi).$$

Hence, $\phi \mapsto \phi^+_{(s,t]}(x)$ is Borel measurable, as required.

Write now $\mathcal{E}$ for the σ-algebra on $D^\circ(\mathbb{R}, \mathcal{D})$ generated by all maps of this form with s, t and x rational. It remains to show that $\mathcal{E}$ contains the Borel σ-algebra of $D^\circ(\mathbb{R}, \mathcal{D})$. Write $\{(I_k, z_k) : k \in \mathbb{N}\}$ for an enumeration of the set $\{(s,t] : s, t \in \mathbb{Q}, s < t\} \times \mathbb{Q}$. It is straightforward to show that, for all k, the map $\phi \mapsto \phi^\times_{I_k}(z_k)$ is $\mathcal{E}$-measurable. Fix $n \in \mathbb{N}$, $\phi^0 \in D^\circ(\mathbb{R}, \mathcal{D})$, $r \in (0, \infty)$ and $k \in \mathbb{N}$, and consider the set

$$A(k,r) = \big\{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : \big(\chi_n(I_1)\phi^\times_{I_1}(z_1), \dots, \chi_n(I_k)\phi^\times_{I_k}(z_k)\big) \in B(k,r)\big\},$$

where

$$B(k,r) = \bigcup_\lambda \Big\{(y_1, \dots, y_k) \in \mathbb{R}^k : \max_{j\le k} \big|y_j - \chi_n\big(\lambda(I_j)\big)\phi^{0\times}_{\lambda(I_j)}(z_j)\big| < r\Big\},$$

where the union is taken over all increasing homeomorphisms λ of $\mathbb{R}$ with $\gamma(\lambda) < r$. Note that $B(k,r)$ is an open set in $\mathbb{R}^k$, so $A(k,r) \in \mathcal{E}$, so $A = \bigcup_{m\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} A(k, r - 1/m) \in \mathcal{E}$.

Consider the set

$$C = \big\{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : d_D^{(n)}\big(\phi, \phi^0\big) < r\big\}.$$

It is straightforward to check from the definition of $d_D^{(n)}$, that $C \subseteq A$. Suppose that $\phi \in A$. We shall show that $\phi \in C$. Then $C = A$, so $C \in \mathcal{E}$, and since sets of this form generate the Borel σ-algebra, we are done.

We can find an $m \in \mathbb{N}$ and, for each $k \in \mathbb{N}$, a λ_k with $\gamma(\lambda_k) < r - 1/m$ such that

$$\left|\chi_n(I_j)\phi^{\times}_{I_j}(z_j) - \chi_n\big(\lambda_k(I_j)\big)\phi^{0\times}_{\lambda_k(I_j)}(z_j)\right| < r - 1/m, \qquad j = 1, \dots, k.$$

Without loss of generality, we may assume that the sequence $(\lambda_k : k \in \mathbb{N})$ converges uniformly on compacts, and that its limit, λ say, satisfies $\gamma(\lambda) \le r - 1/m$. By Proposition A.5, for each j, there is an interval $\hat{I}_j$, having the same endpoints as I_j such that $\phi_{\lambda(\hat{I}_j)}$ is a limit point in $\mathcal{D}$ of the sequence $(\phi_{\lambda_k(I_j)} : k \in \mathbb{N})$, so $\phi^{\times}_{\lambda(\hat{I}_j)}$ is a limit point in $\mathcal{S}$ of the sequence $(\phi^{\times}_{\lambda_k(I_j)} : k \in \mathbb{N})$. Then

$$\left|\chi_n(I_j)\phi^{\times}_{I_j}(z_j) - \chi_n\big(\lambda(\hat{I}_j)\big)\phi^{0\times}_{\lambda(\hat{I}_j)}(z_j)\right| \le r - 1/m$$

for all j. For all bounded intervals I and all $z \in \mathbb{R}$, we can find a sequence $(j_p : p \in \mathbb{N})$ such that $I_{j_p} \to I$, $\hat{I}_{j_p} \to I$ and $z_{j_p} \to z$. So, we obtain

$$\left|\chi_n(I)\phi^{\times}_{I}(z) - \chi_n\big(\lambda(I)\big)\phi^{0\times}_{\lambda(I)}(z)\right| \le r - 1/m.$$

Hence, $d^{(n)}_D(\phi, \phi^0) \le r - 1/m$ and $\phi \in C$, as we claimed. □

Recall that, for $e = (s, x) \in \mathbb{R}^2$ and $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$, we set

$$Z^{e,\pm}(\phi) = \big(\phi^{\pm}_{(s,t]}(x) : t \ge s\big)$$

and for sequences $E = (e_k : k \in \mathbb{N})$ in $\mathbb{R}^2$, we set $Z^{E,\pm} = (Z^{e_k,\pm} : k \in \mathbb{N})$. Also

$$C^{\circ,\pm}_E = \{Z^{E,\pm}(\phi) : \phi \in C^\circ(\mathbb{R}, \mathcal{D})\}, \qquad D^{\circ,\pm}_E = \{Z^{E,\pm}(\phi) : \phi \in D^\circ(\mathbb{R}, \mathcal{D})\}$$

and

$$D^\circ(E) = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : Z^{E,+}(\phi) = Z^{E,-}(\phi)\},$$
$$D^\circ_E = \{Z^E(\phi) : \phi \in D^\circ(E)\}.$$

PROPOSITION A.9. *Let E be a countable subset of $\mathbb{R}^2$ containing*[9] *$\mathbb{Q}^2$. Then $Z^{E,+} : C^\circ(\mathbb{R}, \mathcal{D}) \to C^{\circ,+}_E$ is a bijection, $C^{\circ,+}_E$ is a measurable subset of C_E, and the inverse bijection $\Phi^{E,+} : C^{\circ,+}_E \to C^\circ(\mathbb{R}, \mathcal{D})$ is a measurable map. Moreover, $Z^{E,+} : D^\circ(\mathbb{R}, \mathcal{D}) \to D^{\circ,+}_E$ is also a bijection, $D^{\circ,+}_E$ is a measurable subset of D_E and the inverse bijection $\Phi^{E,+} : D^{\circ,+}_E \to D^\circ(\mathbb{R}, \mathcal{D})$ is also a measurable map. Moreover, the same statements hold with $+$ replaced by $-$, we have $D^\circ_E = D^{\circ,+}_E \cap D^{\circ,-}_E$ and $\Phi^{E,+} = \Phi^{E,-}$ on D°_E.*

[9]The role of $\mathbb{Q}$ here could be played by any countable dense subset of $\mathbb{R}$. The same comment applies to Propositions A.10 and A.11.

PROOF. We discuss only the cadlag case. The same comments apply as in the preceding proof about the relationship of the cadlag and continuous cases. It is straightforward to see from the density of E in $\mathbb{R}^2$ and the continuity properties of cadlag weak flows that $Z^{E,+}$ and $Z^{E,-}$ are both injective on $D^\circ(\mathbb{R},\mathcal{D})$. We shall instead give an explicit description of the ranges $D_E^{\circ,\pm}$ and explicit constructions of inverse maps $\Phi^{E,+}$ and $\Phi^{E,-}$, which agree on $D_E^{\circ,+}\cap D_E^{\circ,-}$, allowing us to establish measurability (as well as injectivity). Consider for $z\in D_E$ the conditions

$$z_t^{(s,x+n)} = z_t^{(s,x)} + n, \qquad s,t,x\in\mathbb{Q}, s<t, n\in\mathbb{Z} \tag{29}$$

and

$$z_t^{(s,x)} = \inf_{y\in\mathbb{Q}, y>x} z_t^{(s,y)}, \qquad (s,x)\in E, t\in\mathbb{Q}, t>s. \tag{30}$$

Under these conditions, define for $s,t\in\mathbb{Q}$ with $s<t$ and for $x\in\mathbb{R}$,

$$\Phi^-_{(s,t]}(x) = \sup_{y\in\mathbb{Q}, y<x} z_t^{(s,y)}, \qquad \Phi^+_{(s,t]}(x) = \inf_{y\in\mathbb{Q}, y>x} z_t^{(s,y)}.$$

Then $\Phi_{(s,t]} = \{\Phi^-_{(s,t]}, \Phi^+_{(s,t]}\}\in\mathcal{D}$ and

$$\Phi^+_{(s,t]}(x) = z_t^{(s,x)}, \qquad s,t,x\in\mathbb{Q}, s<t.$$

Now consider the following additional conditions on z:

$$\Phi^-_{(t,u]}\circ\Phi^-_{(s,t]} \le \Phi^-_{(s,u]} \le \Phi^+_{(s,u]} \le \Phi^+_{(t,u]}\circ\Phi^+_{(s,t]}, \qquad s,t,u\in\mathbb{Q}, s<t<u \tag{31}$$

and for all $\varepsilon>0$ and all $n\in\mathbb{N}$, there exist $\delta>0$, $m\in\mathbb{Z}^+$ and $u_1,\dots,u_m\in(-n,n)$ such that

$$\|\Phi_{(s,t]} - \mathrm{id}\| < \varepsilon \tag{32}$$

whenever $s,t\in\mathbb{Q}\cap(-n,n)$ with $0<t-s<\delta$ and $(s,t]\cap\{u_1,\dots,u_m\}=\varnothing$.

Note that the inequalities between functions required in (31) hold whenever the same inequalities hold between their restrictions to $\mathbb{Q}$, by left and right continuity. Note also that condition (32) is equivalent to the following condition involving quantifiers only over countable sets:

> for all rationals $\varepsilon>0$ and all $n\in\mathbb{N}$, there exist a rational $\delta>0$ and an $m\in\mathbb{Z}^+$ such that, for all rationals $\eta>0$, there exist rationals $s_1,t_1,\dots,s_m,t_m\in(-n,n)$, with $s_i<t_i$ for all i and with $\sum_{i=1}^m(t_i-s_i)<\eta$, such that
>
> $$\|\Phi_{(s,t]} - \mathrm{id}\| < \varepsilon$$
>
> whenever $s,t\in\mathbb{Q}\cap(-n,n)$ with $0<t-s<\delta$ and $(s,t]\cap((s_1,t_1]\cup\cdots\cup(s_m,t_m])=\varnothing$.

Denote by $D_E^{*,+}$ the set of those $z \in D_E$ where conditions (29), (30), (31) and (32) all hold. Then $D_E^{*,+}$ is a measurable subset of D_E. Fix $z \in D_E^{*,+}$. Given a bounded interval I, we can find sequences of rationals s_n and t_n such that $(s_n, t_n] \to I$ as $n \to \infty$. Then, by conditions (31) and (32),

$$d_{\mathcal{D}}(\Phi_{(s_n,t_n]}, \Phi_{(s_m,t_m]}) \le \|\Phi_{(s_n,s_m]} - \mathrm{id}\| + \|\Phi_{(t_n,t_m]} - \mathrm{id}\| \to 0$$

as $n, m \to \infty$. So the sequence $\Phi_{(s_n,t_n]}$ converges in $\mathcal{D}$, with limit Φ_I, say, and Φ_I does not depend on the approximating sequences of rationals. In the case where $I = I_1 \oplus I_2$, there exists another sequence of rationals u_n such that $(s_n, u_n] \to I_1$ and $(u_n, t_n] \to I_2$ as $n \to \infty$. Hence, $\Phi = (\Phi_I : I \subseteq \mathbb{R})$ has the weak flow property, by Proposition A.3. It is straightforward to deduce from (32) that Φ is moreover cadlag, so $\Phi = \Phi(z) \in D^\circ(\mathbb{R}, \mathcal{D})$. It follows from its construction and the preceding proposition that the map $z \mapsto \Phi(z) : D_E^{*,+} \to D^\circ(\mathbb{R}, \mathcal{D})$ is measurable.

Now, for all $z \in D_E^{*,+}$, we have $Z^{E,+}(\Phi(z)) = z$ and for all $\phi \in D^\circ(\mathbb{R}, \mathcal{D})$, we have $Z^{E,+}(\phi) \in D_E^{*,+}$ and $\Phi(Z^{E,+}(\phi)) = \phi$. Hence, $D_E^{\circ,+} = D_E^{*,+}$ and $Z^{E,+} : D^\circ(\mathbb{R}, \mathcal{D}) \to D_E^{\circ,+}$ is a bijection with inverse $\Phi^{E,+} = \Phi$.

Consider now for $z \in D_E$ the condition

$$z_t^{(s,x)} = \sup_{y \in \mathbb{Q}, y < x} z_t^{(s,y)}, \qquad (s,x) \in E, t \in \mathbb{Q}, t > s. \tag{33}$$

Denote by $D_E^{*,-}$ the set of those $z \in D_E$ where conditions (29), (31), (32) and (33) all hold, and define Φ on $D_E^{*,-}$ exactly as on $D_E^{*,+}$. Then, by a similar argument, $D_E^{\circ,-} = D_E^{*,-}$ and $Z^{E,-} : D^\circ(\mathbb{R}, \mathcal{D}) \to D_E^{\circ,-}$ is a bijection with inverse $\Phi^{E,-} = \Phi$. In particular, $\Phi^{E,+} = \Phi^{E,-}$ on $D_E^{\circ,-} \cap D_E^{\circ,+}$ and so $D_E^\circ = D_E^{\circ,-} \cap D_E^{\circ,+}$, as claimed. $\square$

PROPOSITION A.10. *Let E be a countable subset of $\mathbb{R}^2$ containing $\mathbb{Q}^2$. Then* $\mu_E(C_E^\circ) = 1$.

PROOF. We use an identification of C_E° analogous to that implied for D_E° by the preceding proof. The same five conditions (29), (30), (31), (32) and (33) characterize C_E° inside C_E, except that, in (32), only the case $m = 0$ is allowed. Recall that, under μ_E, for time–space starting points $e = (s, x)$ and $e' = (s', x')$, the coordinate processes Z^e and $Z^{e'}$ behave as independent Brownian motions up to

$$T^{ee'} = \inf\{t \ge s \vee s' : Z_t^e - Z_t^{e'} \in \mathbb{Z}\},$$

after which they continue to move as Brownian motions, but now with a constant separation. In particular, if $s = s'$ and $x' = x + n$ for some $n \in \mathbb{Z}$, then $T^{ee'} = 0$, so $Z_t^{e'} = Z_t^e + n$ for all $t \ge s$, so (29) holds almost surely.

Let $(s, x) \in E$ and $t, u \in \mathbb{Q}$, with $s \le t < u$. Consider the event

$$A = \Big\{ \sup_{y \in \mathbb{Q}, y < Z_t^{(s,x)}} Z_u^{(t,y)} = Z_u^{(s,x)} = \inf_{y' \in \mathbb{Q}, y' > Z_t^{(s,x)}} Z_u^{(t,y')} \Big\}.$$

Fix $n \in \mathbb{N}$ and set $Y = n^{-1}\lfloor nZ_t^{(s,x)} \rfloor$ and $Y' = Y + 1/n$. Then Y and Y' are $\mathcal{F}_t$-measurable, $\mathbb{Q}$-valued random variables. Now $\mathbb{P}(Y < Z_t^{(s,x)} < Y') = 1$ and

$$\{Y < Z_t^{(s,x)} < Y'\} \cap \{T^{(t,Y)(t,Y')} \leq u\} \subseteq A.$$

By the Markov property of Brownian motion, almost surely,

$$\mathbb{P}(T^{(t,Y)(t,Y')} \leq u | \mathcal{F}_t) \geq 2\Phi\left(\frac{1}{n\sqrt{2(u-t)}}\right),$$

and the right-hand side tends to 1 as $n \to \infty$. So, by bounded convergence, we obtain $\mathbb{P}(A) = 1$. On taking a countable intersection of such sets A over the possible values of s, x, t and u, we deduce that conditions (30), (31) and (33) hold almost surely.

It remains to establish the continuity condition (32). For a standard Brownian motion B starting from 0, we have, for $n \geq 4$,

$$\mathbb{P}\Big(\sup_{t \leq 1} |B_t| > n\Big) \leq e^{-n^2/2}.$$

Define, for $\delta > 0$ and $e = (s, x) \in E$,

$$V^e(\delta) = \sup_{s \leq t \leq s+\delta^2} |Z_t^e - x|.$$

Then, by scaling,

$$\mathbb{P}(V^e(\delta) > n\delta) \leq e^{-n^2/2}.$$

Consider, for each $n \in \mathbb{N}$ the set

$$E_n = \{(j2^{-2n}, k2^{-n}) : j \in \tfrac{1}{2}\mathbb{Z} \cap [-2^{2n}, 2^{2n}), k = 0, 1, \ldots, 2^n - 1\}$$

and the event

$$A_n = \bigcup_{e \in E_n} \{V^e(2^{-n}) > n2^{-n}\}.$$

Then $\mathbb{P}(A_n) \leq |E_n| e^{-n^2/2}$, so $\sum_n \mathbb{P}(A_n) < \infty$, so by Borel–Cantelli, almost surely, there is a random $N < \infty$ such that $V^e(2^{-n}) \leq n2^{-n}$ for all $e \in E_n$, for all $n \geq N$.

Given $\varepsilon > 0$, choose $n \geq N$ such that $(4n + 2)2^{-n} \leq \varepsilon$ and set $\delta = 2^{-2n-1}$. Then, for all rationals $s, t \in (-n, n)$ with $0 < t - s < \delta$ and all rationals $x \in [0, 1]$, there exist $e^{\pm} = (r, y^{\pm}) \in E_n$ such that

$$r \leq s < t \leq r + 2^{-2n},$$

$$x + n2^{-n} < y^+ \leq x + (n+1)2^{-n},$$

$$x - (n+1)2^{-n} \leq y^- < x - n2^{-n}.$$

Then, $Z_s^{e^-} < x < Z_s^{e^+}$, so

$$x - \varepsilon \le Z_t^{e^-} \le Z_t^{(s,x)} \le Z_t^{e^+} \le x + \varepsilon.$$

Hence, $\|\Phi_{(s,t]} - \mathrm{id}\| \le \varepsilon$, as required. □

Recall that Φ^E denotes the inverse of the evaluation map $Z^E : D^\circ(E) \to D_E^\circ$.

PROPOSITION A.11. *Let E be a countable subset of $\mathbb{R}^2$ containing $\mathbb{Q}^2$. Then Φ^E is continuous.*

PROOF. Consider a sequence $(z_k : k \in \mathbb{N})$ in D_E° and suppose that $z_k \to z$ in D_E, with $z \in D_E^\circ$. Set $\phi^k = \Phi^E(z_k)$ and $\phi = \Phi^E(z)$. By analogy with the standard Skorohod topology, it will suffice to show that, for all $n_0 \in \mathbb{N}$ and all continuity points $-n_0 < t < n_0$, that is, $\phi_{\{t\}} = \mathrm{id}$, we have $\sup_{-n_0<s<t} d_{\mathcal{D}}(\phi^k_{(s,t]}, \phi_{(s,t]}) \to 0$ as $k \to \infty$. Given $\varepsilon > 0$, choose $0 < \eta < \varepsilon/3$. As in the proof of separability in Proposition A.7, there exist $m, n \in \mathbb{N}$ and discontinuity points $-n_0 = u_0 < u_1 < \cdots < u_n = n_0$ with $2/m + 3\eta < \varepsilon$ such that if $I \cap \{u_0, \dots, u_n\}$ with $\sup I - \inf I < 2/m$, then $\|\phi_I - \mathrm{id}\| < \eta$. Consider the finite set

$$F = \big(m^{-1}\mathbb{Z} \cap [-n_0, n_0]\big) \times \big(m^{-1}\mathbb{Z} \cap [0, 1)\big).$$

There exists a $K < \infty$ such that, for all $k \ge K$ and all $e_0 = (s_0, x_0) \in F$, $d_{e_0}(z_k^{e_0}, z^{e_0}) < 1/m$. Therefore, there exists some homomorphism of (s_0, ∞), $\lambda(= \lambda_{k,e_0})$, such that for all $t \in (s_0, n_0]|\lambda(t) - t| < 1/m$ and

$$\big|\phi^{k,+}_{(s_0,t]}(x_0) - \phi^+_{(s_0,\lambda(t)]}(x_0)\big| = \big|\phi^{k,-}_{(s_0,t]}(x_0) - \phi^-_{(s_0,\lambda(t)]}(x_0)\big| < 1/m.$$

For all $s \in [-n_0, n_0)$ and all $x \in [0, 1)$, there exists $(s_0, x_0) \in F$ such that

$$s_0 \le s < s_0 + 1/m, \qquad x_0 \le x + \eta + 2/m < x_0 + 1/m.$$

Then

$$\phi^{k,+}_{(s_0,s]}(x_0) \ge \phi^+_{(s_0,\lambda(s)]}(x_0) - 1/m \ge x_0 - \eta - 1/m > x,$$

so

$$\phi^{k,+}_{(s_0,t]}(x_0) \ge \phi^{k,+}_{(s,t]}(x), \qquad t \ge s.$$

Now, for all $t \in (s, n_0]$ with $|t - u_l| > 1/m$ for all $l \in \{0, \dots, n\}$, we have $d_{\mathcal{D}}(\phi_{(s_0,\lambda(t)]}, \phi_{(s,t]}) < 2\eta$, so

$$\phi^+_{(s_0,\lambda(t)]}(x_0) \le \phi^+_{(s,t]}(x_0 + 2\eta) + 2\eta.$$

So,

$$\phi^{k,+}_{(s_0,t]}(x_0) \le \phi^+_{(s_0,\lambda(t)]}(x_0) + 1/m \le \phi^+_{(s,t]}(x_0 + 2\eta) + 2\eta + 1/m,$$

and so

$$\phi^{k,+}_{(s,t]}(x) \le \phi^{+}_{(s,t]}(x+\varepsilon)+\varepsilon.$$

By a similar argument, for all $t \in (s, n_0]$ with $|t-u_l| > 1/m$ for all $l \in \{0,\dots,n\}$,

$$\phi^{k,-}_{(s,t]}(x) \ge \phi^{-}_{(s,t]}(x-\varepsilon)-\varepsilon,$$

so $d_{\mathcal{D}}(\phi^k_{(s,t]}, \phi_{(s,t]}) \le \varepsilon$. As $1/m$ can be chosen to be arbitrarily small, the result follows. □

A.3. List of notation. For ease of reference, we list below some of the notation that appears in the paper. In all definitions, $e=(s,x)\in\mathbb{R}^2$, $E=(e_k=(s_k,x_k):k\in\mathbb{N})$ in $\mathbb{R}^2$, $\varepsilon\in(0,1]$ and disturbance flows are with disturbance f.

	Disturbance flows:
$\Phi_{n,m}$:	The discrete disturbance flow in which disturbances are applied at integer times.
Φ:	The lattice disturbance flow, in which the disturbances are applied at times in the lattice $\mathbb{Z}/\rho$, or the Poisson disturbance flow, in which the disturbances are applied at the times of the atoms of a Poisson process with intensity ρ.
$\hat{\Phi}$:	The time reversed disturbance flow given by $\hat{\Phi}_I=\Phi^{-1}_{-I}$.
Φ^{ε}:	The ε-scale disturbance flow, that is, $\Phi^{\varepsilon}_I=\sigma_{\varepsilon}(\Phi_{\varepsilon^2 I})$.
$Z^{e,\pm}$:	The evaluation maps $Z^{e,\pm}: D^{\circ}(\mathbb{R},\mathcal{D})\to D_e$ given by $Z^{e,\pm}_t(\phi)=\phi^{\pm}_{(s,t]}(x)$.
$Z^{E,\pm}$:	The evaluation maps $Z^{E,\pm}: D^{\circ}(\mathbb{R},\mathcal{D})\to D_E$ given by $Z^{E,\pm}(\phi)=(Z^{e_k,\pm}(\phi):k\in\mathbb{N})$.
$\check{Z}^{e,\pm}(\phi)$:	The extension of the evaluation maps from $[s,\infty)$ to the whole of $\mathbb{R}$.
$\Phi^{E,\pm}$:	The inverse of $Z^{E,\pm}$ restricted to $D^{\circ,\pm}_E$.
Φ^{E}:	The inverse of $Z^{E,+}$ (or identically $Z^{E,-}$) restricted to D°_E.
	Metric spaces:
$(\mathcal{D}, d_{\mathcal{D}})$:	The set of disturbances on the circle together with the metric defined in (12).
$(\bar{\mathcal{D}}, d_{\bar{\mathcal{D}}})$:	The space of disturbances on the line together with the metric defined in (26).
$\mathcal{D}^*$:	$\mathcal{D}^*=\{f\in\mathcal{D}\setminus\{\mathrm{id}\}:\int_0^1(f(x)-x)\,dx=0\}$.
$(D(\mathbb{R},S),d)$:	The Skorohod space of cadlag paths in a metric space S, equipped with d, the Skorokhod metric on $D(\mathbb{R},S)$.
(D_e,d_e):	$D_e=D_x([s,\infty),\mathbb{R})$ is the Skorohod space of cadlag paths starting from x at time s, equipped with d_e, the Skorokhod metric on D_e.

(D_E, d_E): $D_E = \prod_{k=1}^{\infty} D_{e_k}$ and d_E is the metric on D_E defined in (9).

$\check{D}_e$: $\check{D}_e = \{\xi \in D(\mathbb{R}, \mathbb{R}) : \xi_s = x\}$.

$\check{D}_E$: $\check{D}_E = \prod_{k=1}^{\infty} \check{D}_{e_k}$.

C_e: The subspace of D_e consisting of continuous paths.

C_E: The subspace of D_E where each coordinate path is continuous, that is, $C_E = \prod_{k=1}^{\infty} C_{e_k}$.

$(C^\circ(\mathbb{R}, \mathcal{D}), d_C)$: The set of continuous weak flows on the circle with values in $\mathcal{D}$ together with the metric defined in (15).

$C^\circ(\mathbb{R}, \bar{\mathcal{D}})$: The space of continuous weak flows on the line.

$d_C^{(n)}$: The semimetric on $C^\circ(\mathbb{R}, \mathcal{D})$, restricted to time taking values in $(-n, n)$, as defined in (16).

$C_E^{\circ,\pm}$: The subspace of C_E given by $C_E^{\circ,\pm} = \{Z^{E,\pm}(\phi) : \phi \in C^\circ(\mathbb{R}, \mathcal{D})\}$.

$C^\circ(E)$: $C^\circ(E) = \{\phi \in C^\circ(\mathbb{R}, \mathcal{D}) : Z^{E,+}(\phi) = Z^{E,-}(\phi)\}$.

C_E°: $C_E^\circ = \{Z^E(\phi) : \phi \in C^\circ(E)\}$.

$(D^\circ(\mathbb{R}, \mathcal{D}), d_D)$: The set of cadlag weak flows with values in $\mathcal{D}$ together with the metric defined in (22).

$D^\circ(\mathbb{R}, \bar{\mathcal{D}})$: The space of cadlag weak flows on the line.

$d_D^{(n)}$: The semimetric on $D^\circ(\mathbb{R}, \mathcal{D})$, on a restricted time-interval, as defined in (20).

$D_E^{\circ,\pm}$: The subspace of D_E given by $D_E^{\circ,\pm} = \{Z^{E,\pm}(\phi) : \phi \in D^\circ(\mathbb{R}, \mathcal{D})\}$.

$D^\circ(E)$: $D^\circ(E) = \{\phi \in D^\circ(\mathbb{R}, \mathcal{D}) : Z^{E,+}(\phi) = Z^{E,-}(\phi)\}$.

D_E°: $D_E^\circ = \{Z^E(\phi) : \phi \in D^\circ(E)\}$.

Distributions:

μ_e: The distribution on the Skorohod space D_e of a standard Brownian motion starting from e.

μ_e^f: The distribution on D_e of the process $(\Phi_{(s,t]}(x))_{t \geq s}$.

$\mu_E, \bar{\mu}_E$: The distribution on D_E (or C_E) of a sequence of coalescing Brownian motions on the circle, respectively on the line, starting from E.

$\mu_E^f, \mu_E^{f,\varepsilon}$: The distributions on D_E of $(\Phi_{(s_k,\cdot]}(x_k) : k \in \mathbb{N})$, $(\Phi^{\varepsilon}_{(s_k,\cdot]}(x_k) : k \in \mathbb{N})$, respectively.

$\mu_A, \bar{\mu}_A$: The distribution on $C^\circ(\mathbb{R}, \mathcal{D})$, respectively on $C^\circ(\mathbb{R}, \bar{\mathcal{D}})$, of the coalescing Brownian flow on the circle, respectively on the line.

$\mu_A^f, \hat{\mu}_A^f$: The distributions on $D^\circ(\mathbb{R}, \mathcal{D})$ of Φ, $\hat{\Phi}$, respectively.

$\check{\mu}_E^f$: The law on $\check{D}_E$ of $(\check{Z}^{e_k} : k \in \mathbb{N})$ under μ_A^f.

$\check{\mu}_E$: The law on $\check{D}_E$ of $(\check{Z}^{e_k} : k \in \mathbb{N})$ under μ_A.

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