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Integrable operators and the squares of Hankel operators

Gordon Blower

Department of Mathematics and Statistics, Lancaster University

Lancaster, LA1 4YF, England, UK. E-mail: g.blower@lancaster.ac.uk

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Abstract Integrable operators arise in random matrix theory, where they describe the asymptotic eigenvalue distribution of large self-adjoint random matrices from the generalized unitary ensembles. This paper gives sufficient conditions for an integrable operator to be the square of a Hankel operator, and applies the condition to the Airy, associated Laguerre, modified Bessel and Whittaker functions.

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1. Introduction

Integrable operators with kernels of the form

$$W(x,y)=\frac{f(x)g(y)-f(y)g(x)}{x-y} \tag{1.1}$$

have applications in quantum field theory and random matrix theory, where they are used to describe the asymptotic distribution of large random matrices; see [5, 17, 18]. Tracy and Widom [19] observed that many important distributions in random matrix theory can be defined using solutions of systems

$$m(x)\frac{d}{dx}\begin{bmatrix} f(x)\\ g(x)\end{bmatrix}=\begin{bmatrix}\alpha(x) & \beta(x)\\ -\gamma(x) & -\alpha(x)\end{bmatrix}\begin{bmatrix} f(x)\\ g(x)\end{bmatrix} \tag{1.2}$$

where $m(x),\alpha(x),\beta(x)$ and $\gamma(x)$ are polynomials. In [17, 18], Tracy and Widom considered the Airy and Bessel kernels which describe the soft and hard edges of generalized unitary ensembles, and proved the apparently miraculous identities that the operators with these kernels were squares of self-adjoint Hankel operators; then they used this property to compute their eigenfunctions and eigenvalues.

Let K be a separable Hilbert space, and $L^2((0,\infty);dx;K)$ be the Bochner–Lebesgue space of strongly measurable functions $\phi:(0,\infty)\to K$ such that $\int_0^\infty \|\phi(x)\|^2_K dx<\infty$.

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In [2] we considered a general class of differential equations which gives rise to integrable operators that have the form $W = \Gamma^*\Gamma$, where $\Gamma : L^2(0,\infty) \to L^2((0,\infty); K)$ is a continuous Hankel operator. The general theorem of [2] specialised to the Airy and Bessel kernels, and in this paper we prove related results which deal with other integrable operators.

After recalling some definitions, we state and prove the main Theorem 1.2, and then in section 2 give some applications. In section 3 we discuss the scope of Theorem 1.2 as it applies to (1.2) in reduced forms, and then in section 4 prove another theorem which encompasses other applications, as in section 5.

Definition *(Integrable operators)* Let I be a subinterval of $\mathbf{R}$. An integrable operator on $L^2(I; dx)$ is a continuous linear operator W with kernel

$$W(x,y) = 2\sum_{j=1}^{n} \frac{f_j(x)g_j(y)}{x-y} \qquad (x, y \in I; x \neq y) \tag{1.3}$$

where f_j, g_j are bounded and measurable functions such that $\sum_{j=1}^{n} f_j(x)g_j(x) = 0$ almost everywhere on I.

Lemma 1.1. *Suppose further that the f_j and g_j are real-valued. Then W is self-adjoint if and only if*

$$W(x,y) = \frac{\langle Jv(x), v(y)\rangle}{x-y} \qquad (x, y \in I; x \neq y) \tag{1.4}$$

where $v(x) = \mathrm{col}[f_1(x), \dots, f_n(x); g_1(x), \dots, g_n(x)]$ and

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{1.5}$$

with identity matrix $I_n \in M_n(\mathbf{R})$ and the usual inner product on $\mathbf{R}^{2n}$.

Proof. Clearly the kernel of W is symmetric if and only if the numerator of $W(x,y)$ is skew-symmetric, in which case we can write

$$2\sum_{j=1}^{n} f_j(x)g_j(y) = \sum_{j=1}^{n} f_j(x)g_j(y) - \sum_{j=1}^{n} f_j(y)g_j(x), \tag{1.6}$$

and the matrix expression follows directly.

□

Definition *(Hankel operator)* Let $\phi \in L^2((0,\infty); K)$. The Hankel operator with symbol ϕ is the integral operator

$$\Gamma_\phi f(s) = \int_0^\infty \phi(s+t)f(t)\,dt. \tag{1.7}$$

Nehari's theorem [13] gives a sufficient condition for $\Gamma_\phi : L^2(0,\infty) \to L^2((0,\infty);K)$ to be continuous and gives an expression for the operator norm $\|\Gamma\|$; while the Hilbert–Schmidt norm of Γ_ϕ satisfies

$$\|\Gamma_\phi\|^2_{HS} = \int_0^\infty s\|\phi(s)\|^2_K \, ds \tag{1.8}$$

when $\sqrt{s}\phi(s) \in L^2((0,\infty);K)$. Clearly Γ_ϕ is self-adjoint when $K = \mathbf{R}$.

Given an integrable operator, it is often valuable to identify a Hankel operator Γ_ϕ such that $\Gamma_\phi^*\Gamma_\phi = W$ and to determine whether W is of trace class. In particular, when $K = \mathbf{R}$ and $W = \Gamma_\phi^2$, the spectral resolution of the self-adjoint operator Γ_ϕ determines the spectral resolution of W. This is the basis of the successful calculations in [17, 18, 19], which also exploited the fact the eigenvectors of Γ_ϕ can be relatively easy to analyze. In [12], Megretskiĭ, Peller and Treil characterized the spectral multiplicity function of a self-adjoint Hankel operator. Further, in applications to determinantal point fields as in [15], one often wishes to show that W is of trace class and satisfies $0 \le W \le I$.

We consider first the case associated with the differential equation

$$\frac{dv}{dx} = J(\Omega_1 x + \Omega_0 + \Omega_{-1}x^{-1})v. \tag{1.9}$$

Theorem 1.2. *Suppose that Ω_1, Ω_0 and Ω_{-1} are real symmetric $(2n) \times (2n)$ constant matrices such that $\Omega_1 \ge 0$ and $-\Omega_{-1} \ge 0$. Suppose further that v satisfies (1.9), and that $v(x)$ and $v(x)/x$ are bounded functions in $L^2((0,\infty);dx;\mathbf{R}^{4n})$. Then there exists a real linear subspace K of $\mathbf{R}^{4n}$ with $\dim(K) \le \mathrm{rank}(\Omega_1) + \mathrm{rank}(\Omega_{-1})$ and $\phi \in L^2((0,\infty);dx;K)$ such that $\Gamma_\phi : L^2(0,\infty) \to L^2((0,\infty);dx;K)$ is continuous and the kernel*

$$W(x,y) = \frac{\langle Jv(x), v(y)\rangle}{x-y} \tag{1.10}$$

factors as

$$W = \Gamma_\phi^*\Gamma_\phi. \tag{1.11}$$

In particular, if $\mathrm{rank}(\Omega_1) + \mathrm{rank}(\Omega_{-1}) = 1$, *then* $W = \Gamma_\psi^2$ for some $\psi \in L^2(0,\infty)$.

Proof. By Lemma 2.1 of [2], which essentially depends upon the continuity of the Hilbert transform on $L^2(\mathbf{R})$, we know that W is also a continuous linear operator on $L^2(0,\infty)$.

We have

$$\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)W(x,y) = \frac{1}{x-y}\Big(\Big\langle J\frac{dv}{dx}, v(y)\Big\rangle + \Big\langle Jv(x), \frac{dv}{dy}\Big\rangle\Big) \qquad (x \ne y), \tag{1.12}$$

where dv/dx and dv/dy satisfy (1.9). Now $J^2 = -I$ and $J^* = -J$, so the matrices involved in the differential equation satisfy

$$J^2(\Omega_1 x + \Omega_0 + \Omega_{-1}x^{-1}) + (\Omega_1 y + \Omega_0 + \Omega_{-1}y^{-1})J^*J$$
$$= -\Omega_1 x + \Omega_1 y - \Omega_{-1}x^{-1} + \Omega_{-1}y^{-1}; \tag{1.13}$$

so by dividing by $x - y$, we obtain

$$\Bigl(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Bigr)W(x,y) = -\bigl\langle \Omega_1 v(x), v(y)\bigr\rangle + \Bigl\langle \Omega_{-1}\frac{v(x)}{x}, \frac{v(y)}{y}\Bigr\rangle. \tag{1.14}$$

We introduce the positive roots of the positive semidefinite matrices Ω_1 and Ω_{-1}, and the column vector $\phi(x) = \mathrm{col}[\sqrt{\Omega_1}v(x), \sqrt{-\Omega_{-1}}v(x)/x]$; so that, $\phi \in L^2((0,\infty); dx; \mathbf{R}^{4n})$ and

$$\Bigl(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Bigr)W(x,y) = -\langle \phi(x), \phi(y)\rangle \qquad (x, y > 0). \tag{1.15}$$

Integrating this equation, we obtain

$$W(x,y) = \int_0^\infty \langle \phi(x+t), \phi(y+t)\rangle\, dt + g(x-y) \tag{1.16}$$

for some differentiable function g; but $W(x,y)$ and the integral converge to 0 as $x \to \infty$ or $y \to \infty$; so $g = 0$. Hence $W = \Gamma_\phi^* \Gamma_\phi$, and we deduce that Γ_ϕ is a continuous Hankel operator from $L^2(0,\infty) \to L^2((0,\infty); dx; \mathbf{R}^{4n})$.

Finally, we observe that ϕ takes values in a linear subspace K of $\mathrm{range}(\Omega_1) \oplus \mathrm{range}(\Omega_{-1})$ which has dimension less than or equal to $\mathrm{rank}(\Omega_1) + \mathrm{rank}(\Omega_{-1})$.

If the sum of the ranks of Ω_1 and Ω_{-1} equals one, then ϕ takes values in a one-dimensional real linear subspace of $\mathbf{R}^{4n}$, so $\phi(x) = e\psi(x)$ for some unit vector $e \in \mathbf{R}^{4n}$ and some $\psi \in L^2(0,\infty)$; hence Γ_ψ is self-adjoint and $W = \Gamma_\psi^2$.

□

Remarks 1.3. (i) In their applications to random matrix theory, Tracy and Widom considered integrable operators on unions of intervals such as $\cup_{j=1}^m [a_{2j-1}, a_{2j}]$. Many analytical problems reduce to considering one interval at a time, and so are addressed by the current paper. In a subsequent article [20], they generalized their results to kernels of the form

$$\frac{\langle C\varphi(x), \varphi(y)\rangle}{x-y}$$

where C is an antisymmetric $n \times n$ matrix.

(ii) Theorem 1.2 has an analogue for discrete kernels on $\mathbf{Z}_+$, as in [11].

2. Factorization for some differential equations with simple poles

In this section we consider how Theorem 1.2 applies to some differential equations that are satisfied by familiar special functions.

2.1. The Airy equation

The Airy function Ai gives rise to a solution $u(x) = \mathrm{Ai}(x+s)$ of the differential equation

$$u''(x) = (s+x)u(x) \qquad (x > 0). \tag{2.1}$$

Since the standard asymptotic formula for the Airy function [16, p. 18] gives

$$\mathrm{Ai}(x) = \frac{1}{2x^{1/4}\sqrt{\pi}}\Bigl(1 + O(x^{-3/2}\Bigr)\exp\Bigl(-\frac{2}{3}x^{3/2}\Bigr) \qquad (x \to \infty) \tag{2.2}$$

the hypotheses of Theorem 1.2 are satisfied by $v(x) = \mathrm{col}[\phi(x), \phi'(x)]$ where $\phi(s)$ $= \mathrm{Ai}(x+s)$; so $W = \Gamma_\phi^2$ where Γ_ϕ is a Hankel operator in the Hilbert–Schmidt class. See [2, 17] for more details.

2.2. The Laguerre equation

The Laguerre equation [14] may be expressed as

$$u''(x) + \Bigl(-\frac{1}{4} + \frac{n+1}{x}\Bigr)u(x) = 0, \tag{2.3}$$

with solution $u(x) = xe^{-x/2}L_n^{(1)}(x)$ where

$$L_n^{(1)}(x) = \frac{x^{-1}e^x}{n!}\frac{d^n}{dx^n}\bigl(x^{n+1}e^{-x}\bigr) \qquad (x > 0) \tag{2.4}$$

is the Laguerre polynomial of degree n and parameter $\alpha = 1$. The Laplace transform of u is the rational function

$$\mathcal{L}(u;\lambda) = (n+1)\frac{(\lambda - \frac{1}{2})^n}{(\lambda + \frac{1}{2})^{n+2}} \qquad (\Re\lambda > -1/2).$$

Theorem 1.2 applies directly to the system

$$\frac{d}{dx}\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/4 - (n+1)/x & 0 \end{bmatrix}\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} \tag{2.5}$$

and gives the formula

$$\frac{u(x)u'(y) - u'(x)u(y)}{x-y} = (n+1)\int_0^\infty \frac{u(x+t)u(y+t)}{(x+t)(y+t)}\,dt \tag{2.6}$$

where $\phi(x) = u(x)/x$ gives a Hankel operator Γ_ϕ of Hilbert–Schmidt type.

2.3. The Bessel equation

The differential equation

$$u''(x) + \frac{1}{x}u(x) = 0 \tag{2.7}$$

has solution $u(x) = \sqrt{x}J_1(2\sqrt{x})$, where J_1 is the Bessel function of the first kind of order one. The Laplace transform of u satisfies $\mathcal{L}(u, \lambda) = \lambda^{-2}\exp(-1/\lambda)$. Now the standard asymptotic formula for the Bessel function [10, p. 171] shows that

$$\phi(x) = \frac{u(x)}{x} \asymp \frac{2^{1/4}}{\sqrt{\pi}}x^{-3/4}\cos\Big(2\sqrt{x} - \frac{3\pi}{4}\Big) \qquad (x \to \infty); \tag{2.8}$$

so ϕ belongs to $L^2(0, \infty)$; hence one can follow the proof of Theorem 1.2 and derive the formula

$$\frac{u(x)u'(y) - u'(x)u(y)}{x - y} = \int_0^\infty \frac{u(x+t)u(y+t)}{(x+t)(y+t)}\,dt \qquad (x, y > 0). \tag{2.9}$$

Here the Hankel operator Γ_ϕ is not Hilbert–Schmidt.

2.4. The Carleman operator with multiple spectrum

The system

$$\frac{d}{dx}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & 1/x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} \tag{2.10}$$

has the form considered in Theorem 1.2 and evidently has solution $f(x) = \log x$ and $g(x) = 1$; further

$$W(x, y) = \frac{\log x - \log y}{x - y} = \int_0^\infty \frac{dt}{(x+t)(y+t)} \qquad (x, y > 0) \tag{2.11}$$

has a similar form to an integrable kernel, except that $\log x$ is unbounded. Power showed that Carleman's operator Γ, where

$$\Gamma h(x) = \int_0^\infty \frac{h(y)dy}{x+y} \qquad (h \in L^2(0, \infty)), \tag{2.12}$$

is continuous on $L^2(0, \infty)$ and has spectrum $[0, \pi]$ with spectral multiplicity two; see [13]. Hence W is a continuous linear operator on $L^2(0, \infty)$ with spectrum $[0, \pi^2]$ with multiplicity two. This example illustrates that simple differential equations can give positive definite Hankel operators with multiple spectra.

2.5. Parabolic cylinder functions: non factorization

For $\Re p > -1$, let D_p be the parabolic cylinder function, which satisfies

$$D_p''(x) + \Big(p + \frac{1}{2} - \frac{x^2}{4}\Big) D_p = 0,$$

and let

$$H(x,y) = \frac{D_p(x)D_p'(y) - D_p'(x)D_p(y)}{x-y}. \tag{2.13}$$

Then $\pm H$ is not the square of a self-adjoint Hankel operator Γ_ϕ. By following the proof of Theorem 1.2, we obtain

$$\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) H(x,y) = -\frac{1}{2}(x+y)D_p(x)D_p(y), \tag{2.14}$$

where $(x+y)D_p(x)D_p(y)/2$ cannot equal $\pm\phi(x)\phi(y)$; indeed, for suitable $x_1, x_2 > 0$, the 2×2 matrix $[x_j + x_k]_{j,k=1,2}$ has both positive and negative eigenvalues. When n is a nonnegative integer, D_n is known as a Hermite function, and may be written

$$\phi_n(x) = (n!)^{-1/2}(2\pi)^{-1/4}(-1)^n e^{x^2/4}\frac{d^n}{dx^n}e^{-x^2/2}. \tag{2.15}$$

Aubrun [1] considers self-adjoint Hankel operators Γ_{ϕ_n} and $\Gamma_{\phi_{n+1}}$ such that $H = (1/2)(\Gamma_{\phi_n}\Gamma_{\phi_{n+1}} + \Gamma_{\phi_{n+1}}\Gamma_{\phi_n})$; this gives information about the singular numbers of H. For Hankel squares one has more precise information about the eigenvalues. Borodin and Okounkov [4] have considered the discrete Hermite kernel, and derived the formula

$$\frac{\phi_{m+1}(s)\phi_n(s) - \phi_m(s)\phi_{n+1}(s)}{m-n} = \int_s^\infty \phi_m(t)\phi_n(t)\,dt \qquad (m,n = 0,1,\dots, m\neq n). \tag{2.16}$$

Here the variable n is the degree of the Hermite polynomial factor in ϕ_n, and (2.16) is essentially different from (1.11).

3. Reducing to standard form

Definition (*Operator monotone*) Let I be an interval in $\mathbf{R}$. A continuous function $\omega : I \to \mathbf{R}$ is operator monotone increasing if, whenever S and T are continuous and self-adjoint linear operators on Hilbert space that have spectra in I,

$$S \leq T \Rightarrow \omega(S) \leq \omega(T). \tag{3.1}$$

Further, ω is operator monotone if and only if the matrices

$$\Big[\frac{\omega(x_j) - \omega(x_k)}{x_j - x_k}\Big]_{j,k=1,\dots,m}, \tag{3.2}$$

with diagonal entries $\omega'(x_j)$, are positive semidefinite for all $m = 2, 3, \dots$ and $x_j \in I$ with $j = 1, \dots, m$. In [2] we used Loewner's characterization of operator monotone functions, which shows in particular that an operator monotone function on $(0, \infty)$ extends to an analytic function on a domain U containing $(0, \infty)$ as in [9, p. 541].

Theorem 1.2 shows that, under mild technical conditions, W admits of a factorization $W = \Gamma_\phi^* \Gamma_\phi$ whenever $\omega(x) = \langle \Omega(x)\xi, \xi \rangle$ is operator monotone on $(0, \infty)$, for all $\xi \in \mathbf{R}^{4n}$ where $\Omega(x) = \Omega_1 x + \Omega_0 + \Omega_{-1} x^{-1}$.

Suppose that I has 0 as an endpoint, and let U be a domain that contains I. Suppose that $A(z)$ is a matrix function into $M_{2n}(\mathbf{C})$ that is analytic on U, except for an isolated singularity at $z = 0$, and that

$$\frac{dv}{dz} = A(z)v \quad (z \in U). \tag{3.3}$$

By a standard change of variable, we mean $w(z) = T(z)v(z)$, where the analytic function $T : U \to M_n(\mathbf{C})$ has $T(z)$ invertible as a matrix for each $z \in U$. The following result is commonly known as Birkhoff's normal form, although the first correct statement and proof is due to Turrittin [21]. Gantmacher considered some related examples which resemble (2.10) in [6, p. 146].

Proposition 3.1. *Suppose that $A(z) = \sum_{k=-\infty}^{-1} A_k z^k$ is a Laurent expansion that converges for all $z \neq 0$. Then there exists a standard change of variable that reduces (3.3) to*

$$\frac{dw}{dz} = \Big(\frac{A_{-1}}{z} + \frac{A_{-2}}{z^2} \Big) w. \tag{3.4}$$

Further, if the eigenvalues λ_j of A_{-1} have differences $\lambda_j - \lambda_k$ that are never equal to a natural integer, then one can remove the term in A_{-2}.

The appearance of the term A_{-2} in (3.4) is important, since $-1/x^2$ is not operator monotone on $(0, \infty)$ by [9, p. 554]. So we cannot simply adapt the proofs in section 1 to deal with the case in which A_{-2} appears. However, if ω is operator monotone on $(0, \infty)$, then $\omega(\sqrt{t})$ is likewise. This suggests the change of independent variable $x = \sqrt{t}$, which we exploit in the examples in section 5. Further, we adjust the definition of the kernel and the Hankel operators so that we can obtain a factorization theorem in the next section.

Definition (*Hankel operator*) For $I = (1, \infty)$, we use the Hankel operator

$$\Gamma_\psi g(x) = \int_1^\infty \psi(xy) g(y) \frac{dy}{y} \tag{3.5}$$

where $\psi \in L^2((1,\infty); dy/y; K)$; whereas for $I=(0,1)$ we use

$$\Gamma_\rho h(x) = \int_0^1 \rho(xy)h(y)\frac{dy}{y} \tag{3.6}$$

where $\rho \in L^2((0,1); dy/y; K)$. These definitions reduce to the case $I=(0,\infty)$ in section 1 by the changes of variables $y=e^t$ and $y=e^{-t}$ respectively.

4. Factorization theorem for differential equations with double poles

In this section we consider the differential equation

$$x\frac{dv}{dx} = J(\Omega_1 x + \Omega_0 + \alpha J + \Omega_{-1}x^{-1})v. \tag{4.1}$$

To accommodate forthcoming examples, we have introduced the skew-symmetric matrix αJ into the constant term for some $\alpha \in \mathbf{R}$.

Theorem 4.1. *Suppose that Ω_1, Ω_0 and Ω_{-1} are real symmetric $(2n)\times(2n)$ constant matrices such that $\Omega_1 \geq 0$ and $-\Omega_{-1} \geq 0$. Suppose further that v satisfies (4.1), and that $x^\alpha v(x)$ and $x^{\alpha-1}v(x)$ are bounded functions in $L^2((1,\infty); dx; \mathbf{R}^{4n})$. Then there exists a real linear subspace K of $\mathbf{R}^{4n}$ with $\dim(K) \leq \mathrm{rank}(\Omega_1) + \mathrm{rank}(\Omega_{-1})$ and $\phi \in L^2((1,\infty); dx/x; K)$ such that $\Gamma_\phi : L^2((1,\infty); dx/x) \to L^2((1,\infty); dx/x; K)$ is continuous and the kernel*

$$W(x,y) = \frac{(xy)^{(2\alpha+1)/2}}{x-y}\big\langle Jv(x), v(y)\big\rangle \tag{4.2}$$

factors as

$$W = \Gamma_\phi^*\Gamma_\phi. \tag{4.3}$$

In particular, if $\mathrm{rank}(\Omega_1) + \mathrm{rank}(\Omega_{-1}) = 1$, then $W = \Gamma_\psi^2$ for some $\psi \in L^2((1,\infty); dx/x)$.

Proof. We observe that by homogeneity

$$\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)\frac{(xy)^{(2\alpha+1)/2}}{x-y} = 2\alpha\frac{(xy)^{(2\alpha+1)/2}}{x-y} \qquad (x,y>0; x\neq y), \tag{4.4}$$

and hence

$$\begin{aligned}\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)W(x,y) &= 2\alpha\frac{(xy)^{(2\alpha+1)/2}}{x-y}\big\langle Jv(x), v(y)\big\rangle \\ &\quad + \frac{(xy)^{(2\alpha+1)/2}}{x-y}\Big(\Big\langle Jx\frac{dv}{dx}, v(y)\Big\rangle + \Big\langle Jv(x), y\frac{dv}{dy}\Big\rangle\Big),\end{aligned} \tag{4.5}$$

where the matrices involved in the final terms in (4.5) are

$$J^2(\Omega_1 x + \Omega_0 + \alpha J + \Omega_{-1}x^{-1}) + (\Omega_1 y + \Omega_0 - \alpha J + \Omega_{-1}y^{-1})J^*J$$

$$= -\Omega_1(x-y) + \Omega_{-1}(x-y)/(xy) - 2\alpha J.$$

By cancelling the terms that involve J, we obtain

$$\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)W(x,y) = -(xy)^{2\alpha+1)/2}\langle \Omega_1 v(x), v(y)\rangle + (xy)^{(2\alpha+1)/2}\langle \Omega_{-1} v(x), v(y)\rangle. \quad (4.6)$$

We introduce the column vector

$$\psi(x) = \begin{bmatrix} \sqrt{\Omega_1}x^{(2\alpha+1)/2}v(x) \\ \sqrt{-\Omega_{-1}}x^{(2\alpha-1)/2}v(x) \end{bmatrix} \quad (4.7)$$

which belongs to $L^2((1,\infty); dx/x; \mathbf{R}^{4n})$ and satisfies

$$\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)\int_1^\infty \langle \psi(tx), \psi(ty)\rangle\,\frac{dt}{t} = -\langle \psi(x), \psi(y)\rangle.$$

Hence

$$W(x,y) = \int_1^\infty \langle \psi(tx), \psi(ty)\rangle\,\frac{dt}{t} + h(x/y) \quad (4.8)$$

where $h(x/y) \to 0$ as $x \to \infty$ or $y \to \infty$; so $h = 0$. One can conclude the proof by arguing as in Theorem 1.2.

□

We consider later some examples in which $\Omega_{-1} = 0$. In this case, we can invoke the following existence theorem for solutions.

Proposition 4.2. *Suppose that the residue matrix A_{-1} has eigenvalues λ_j such that the differences $\lambda_j - \lambda_k$ are never equal to a natural integer. Then the differential equation*

$$z\frac{d}{dz}X = (A_0 z + A_{-1})X \quad (4.9)$$

with $X(z) \in M_{2n}(\mathbf{C})$ has a non-trivial solution of the form $X(z) = Y(z)z^{A_{-1}}$, where Y is an entire matrix function of order one.

For a proof see [8], where Hille also discusses the asymptotic form of the solutions in terms of the Laplace transform. Note that the Laplace transform of (4.9) has a similar form to (4.9) itself and in particular has the residue matrix $A_{-1} + I$.

5. Examples of factorization for differential equations with double poles

5.1. Modified Bessel functions

For $0 \leq \nu < 1$, MacDonald's function is defined by

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)\,dt \qquad (\Re z > 0) \quad (5.1)$$

and satisfies the modified Bessel equation $z^2K_\nu'' + zK_\nu' - (\nu^2 + z^2)K_\nu = 0$; hence $u(x) = \sqrt{x}K_\nu(2\sqrt{x})$ satisfies

$$u''(x) = \Big(\frac{1}{x} + \frac{\nu^2 - 1}{4x^2}\Big)u(x). \tag{5.2}$$

By [7, 8.451], $K_\nu(x)$ decays exponentially as $x \to \infty$. We can apply Theorem 4.1 to the system

$$x\frac{d}{dx}\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ x - 2 + \frac{1}{4}(\nu^2 - 1) & 2 \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix}, \tag{5.3}$$

so that, in terms of Theorem 4.1,

$$\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Omega_0 = \begin{bmatrix} -2 + \frac{1}{4}(\nu^2 - 1) & \frac{3}{2} \\ \frac{3}{2} & -1 \end{bmatrix}, \quad \alpha = -1/2, \quad \Omega_{-1} = 0, \tag{5.4}$$

where the residue matrix $A_{-1} = J\Omega_0 - \alpha I$ has eigenvalues $(1/2) \pm (\nu/2)$.

Thus one obtains

$$\frac{u(x)v(y) - u(y)v(x)}{x - y} = \int_1^\infty u(tx)u(ty)\frac{dt}{t}$$

and after some reduction, one deduces that

$$\frac{K_\nu(2\sqrt{x})\sqrt{y}K_\nu'(2\sqrt{y}) - \sqrt{x}K_\nu'(2\sqrt{x})K_\nu(2\sqrt{y})}{x - y} = \int_1^\infty K_\nu(2\sqrt{tx})K_\nu(2\sqrt{ty})dt \tag{5.5}$$

where the right-hand side is the square of a Hankel operator of Hilbert–Schmidt class.

5.2. Bessel functions

The Bessel function J_ν satisfies $x^2J_\nu'' + xJ_\nu' + (x^2 - \nu^2)J_\nu = 0$, and hence $u = \sqrt{x}J_\nu(2\sqrt{x})$ satisfies

$$u''(x) + \Big(\frac{1}{x} + \frac{1 - \nu^2}{4x^2}\Big)u(x) = 0. \tag{5.6}$$

One can apply Theorem 4.1 with some obvious sign changes to the system

$$x\frac{d}{dx}\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -x - 2 - \frac{1}{4}(1 - \nu^2) & 2 \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix}, \tag{5.7}$$

and after some reduction one obtains an identity from [18]

$$\frac{\sqrt{x}J_\nu'(2\sqrt{x})J_\nu(2\sqrt{y}) - J_\nu(2\sqrt{x})\sqrt{y}J_\nu'(2\sqrt{y})}{x - y} = \int_0^1 J_\nu(2\sqrt{tx})J_\nu(2\sqrt{ty})dt. \tag{5.8}$$

5.3. Whittaker's functions

The homogeneous confluent hypergeometric equation may be reduced to Whittaker's equation

$$w'' + \Big(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \nu^2}{x^2}\Big) w = 0, \tag{5.9}$$

and the solutions of this are known as Whittaker's functions. In [7, 9.227], the authors give a solution $w(z) = W_{\kappa,\nu}(z)$ such that $w(z) \asymp e^{-z/2} z^{\kappa}$ as $z \to \infty$ along $(0,\infty)$. Kernels involving Whittaker's function appear in [3].

Proposition 5.3. *Suppose that $u(x) = W_{\kappa,\nu}(2\sqrt{x})$ for some $\kappa \leq 0$. Then there exists a function $\Phi \in L^2((1,\infty); dx/x; K)$ for a separable Hilbert space K such that*

$$W(x,y) = (xy)^{1/4} \frac{u(x)u'(y) - u'(x)u(y)}{x - y} \tag{5.10}$$

factors as $W = \Gamma_\Phi^* \Gamma_\Phi$.

Proof. We can write, after a little reduction

$$x \frac{d}{dx} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 & 4 \\ x - 2\kappa\sqrt{x} - (\frac{1}{4} - \nu^2) - 6 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{5.11}$$

which gives

$$\Omega(x) = \frac{1}{4} \begin{bmatrix} x - 2\kappa\sqrt{x} - (\frac{1}{4} - \nu^2) - 6 & 5 \\ 5 & -4 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{5.12}$$

hence $\alpha = -1/4$. The function $\sqrt{x}$ is operator monotone increasing on $(0,\infty)$; indeed, $\phi_t(x) = t^{1/4}/(t+x)$ belongs to $L^2((0,\infty); dt)$ with $\|\phi_t\|_{L^2}^2 = \pi/(2\sqrt{x})$ and satisfies

$$\begin{aligned} \frac{\sqrt{x} - \sqrt{y}}{x - y} &= \frac{1}{\pi} \int_0^\infty \frac{\sqrt{t}dt}{(x+t)(y+t)} \\ &= \frac{1}{\pi} \int_0^\infty \phi_t(x)\phi_t(y)\, dt. \end{aligned} \tag{5.13}$$

We observe that $x^{1/4}u(x)\phi_t(x)$ belongs to $L^2((1,\infty); dx/x)$ and since

$$\begin{aligned} \Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big) W(x,y) &= -\frac{1}{4}(xy)^{1/4} u(x)u(y) \\ &\quad + \frac{\kappa}{2\pi}(xy)^{1/4} u(x)u(y) \int_0^\infty \phi_t(x)\phi_t(y)\, dt \end{aligned} \tag{5.14}$$

we have

$$W(x,y) = \frac{1}{4}\int_1^\infty u(xs)u(ys)(xy)^{1/4}\frac{ds}{s^{1/2}}$$
$$-\frac{\kappa}{2\pi}\int_1^\infty\int_0^\infty \phi_t(sx)\phi_t(sy)u(sx)u(sy)(xy)^{1/4}\frac{dtds}{s^{1/2}}. \tag{5.15}$$

Hence $W = \Gamma_\Phi^*\Gamma_\Phi$, where $K = \mathbf{R}\oplus L^2((0,\infty);dt)$ and $\Phi : (0,\infty)\to K$ is

$$\Phi(x) = 2^{-1}x^{1/4}u(x)\oplus(-\kappa/2\pi)^{1/2}x^{1/4}u(x)\phi_t(x). \tag{5.16}$$

□

Remarks 5.4. (i) The condition $\kappa \leq 0$ in Proposition 5.3 excludes the case of the associated Laguerre functions $w(x) = x^{(1+\alpha)/2}e^{-x/2}L_n^{(\alpha)}(x)$, where $L_n^{(\alpha)}$ with $(n = 0, 1, 2, \dots)$ are the associated Laguerre polynomials as in [14] and [7, 9.237].
(ii) The Laplace transforms of $\sqrt{x}K_\nu(2\sqrt{x})$ and $\sqrt{x}J_\nu(2\sqrt{x})$ may be expressed in terms of Whittaker's functions.
(iii) Bessel's equation may be transformed into the typical confluent hypergeometric equation as in [8, p. 228].

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