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ITÔ *vs.* STRATONOVICH: THIRTY YEARS LATER

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The Itô *vs.* Stratonovich controversy, about the "correct" calculus to use for integration of Langevin equations, was settled to general satisfaction some thirty years ago. Recently, however, it has started to re-emerge, following the advent of new experimental techniques. We briefly review the historical background and discuss critically some of the most recent contributions. We show that some of the new findings are not well-based.

Keywords: Itô stochastic calculus; Stratonovich stochastic calculus; anticipating stochastic differential equations; Brownian motion; Langevin equations; nanoscale forces.

1. Introduction

In the macroscopic world, physical systems are usually nonlinear and subject to noise (random fluctuations). The nonlinearity introduces subtleties into how noise influences a system, and seemingly vexing conundrums arise where the noise is quasi-white and enters multiplicatively in one of the parameters of a model equation. In the presence of multiplicative noise, it turns out that a choice must be made as to which is the appropriate stochastic calculus to be used: this choice appears to be somehow arbitrary, which sparked a widespread debate, usually referred to as the it Itô *vs.* Stratonovich controversy. It attracted considerable attention in the physics community for almost a decade. The controversy was eventually settled to general satisfaction but, as so often happens in such cases, a few years later it has started to re-emerge. One of the reasons has been the introduction of new experimental techniques that allow thermodynamic properties to be probed on the nanometre scale. In this paper we briefly review the controversy and the basic mathematics that underlay it, and recall the main conclusions reached in the earlier debate. We then consider some of the most recent papers, and point out where some of their conclusions appear not to be well-founded.

2. Background and historical notes

The Itô *vs.* Stratonovich controversy took place in the physics literature (mainly) from the late 1970s to the early 1980s, in the burgeoning field of nonlinear stochastic physics. To appreciate the reasons behind the controversy, we need first to recall briefly the basic ideas behind stochastic processes.

2.1. *Stochastic calculus*

The root of the controversy lay with the counter-intuitive nature of stochastic calculus in the presence of non-linearities, when considered from a physicist's point of view. Here we will provide intuitive arguments; a more formal derivation can be found in [1]. We start from the stochastic differential equation (SDE)

$$dx = f(x)\, dt + g(x)\, dW \tag{1}$$

representing the increment of an observable x (for example, the position of a Brownian particle). Here dW is the increment of a Wiener process $W(t)$ defined in probabilistic terms (we assume $W(0) = 0$ for compactness of notation, without loss of generality) via

$$P(W(t)) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{W(t)^2}{2t}}.$$

Amongst others quantities, we need to deal with those of form $\int g(x)\, dW$. A possible approach to the integration of Eq. (1) is through a Taylor expansion where, in the simplest non-trivial case ($f(x) = 0$, $g(x) \neq 0$), we can write

$$x(t) - x(0) = \int_0^t g(x)\, dW \approx \int_0^t dW \left[g(x(0)) + g'(x(0))(x(s) - x(0))\right]. \tag{2}$$

To lowest order, $x(t) - x(0) = g(x(0))W(t)$. If $g'(x) \neq 0$, at the next order an integral of the form $\int W(t)\, dW$ appears. To compute it, the standard approach is through a discretization, in the mean square limit,

$$\int_0^t W\, dW = \text{m.s.} \lim_{n\to\infty} \sum_{i=1}^{n} W(t_i^*) \left[W(t_i) - W(t_{i-1})\right] \tag{3}$$

using a suitable partition $0 = t_0 < t_i < t_n = t$, and where $t_{i-1} \leq t_i^* \leq t_i$: in what follows, we take $t_i^* = t_{i-1} + \alpha(t_i - t_{i-1})$, with $0 \leq \alpha \leq 1$. The mean square limit is defined as

$$\text{m.s.} \lim_{n\to\infty} X_n = X \leftrightarrow \lim_{n\to\infty} \left\langle (X_n - X)^2 \right\rangle = 0 \tag{4}$$

where the average $\langle ... \rangle$ is taken over the realizations of the Wiener process. In the evaluation of the mean square limit appearing in Eq. (3), we need to estimate terms like $\langle W(t')W(s')\rangle$. Assuming that $t' < s'$, we have that $\langle W(t')W(s')\rangle = \langle W(t')[W(s') - W(t')]\rangle + \langle W(t')W(t')\rangle$. Recalling that the increment $[W(s') - W(t')]$

is independent of $W(t')$, it follows that $\langle W(t')W(s')\rangle = \langle W(t')W(t')\rangle = t'$ and in general $\langle W(t')W(s')\rangle = \min(t', s')$. We have that

$$\text{m.s.} \lim_{n\to\infty} \langle \sum_{i=1}^{n} W(t_i^*)\,[W(t_i) - W(t_{i-1})]\rangle = \sum t_{i-1} + \alpha(t_i - t_{i-1}) - t_{i-1} = \alpha t.$$

Note that the value of the integral *depends on the point* within the interval $[t_i, t_{i-1}]$ where the process $W(t)$ is evaluated: in other words, it depends on α. Although in principle any value of α in the range $[0, 1]$ is possible, in the literature only two values are commonly found: $\alpha = 0$ (Itô calculus [2]) and $\alpha = 1/2$ (Stratonovich calculus [3]). If $W(t)$ were a smooth function, clearly $\langle \int W\, dW\rangle = \langle \frac{1}{2}W(t)^2\rangle = \frac{1}{2}t$, which means that the "standard" result (holding for Riemann-Stieltjes integrals) is recovered by setting $\alpha = 1/2$.

From Eq. (1), it is possible to write the Fokker-Planck equation driving the probability distribution $P(x, t)$

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{-f(x) - \alpha g(x)g'(x) + \frac{1}{2}\frac{\partial}{\partial x}g^2(x)\right\} P(x,t) \qquad (5)$$

which obviously depends on α and where $g'(x) = \partial g(x)/\partial x$; the ensuing equilibrium distribution will depend on α too, and on the functions $f(x)$ and $g(x)$.

In a nutshell, the controversy centred on what is the "correct" choice of α for the description of natural phenomena. We note that it is possible to have sets of *different* $f(x)$ and α which lead algebraically to the *same* $f(x) + \alpha g(x)g'(x)$: this implies that there could be systems characterized by different $f(x)$ and α but which have the same equilibrium distribution.

Rewriting $\int g(x)\ dW = \sum g(x(t_i^*)[dW(t_i) - dW(t_{i-1})]$ we note that $x(t)$ is given by the solution of the SDE (1). Intuitively, this means that, assuming we integrated the SDE up to $x(t_{i-1})$, we get the value of $x(t_i)$ using, in general, values of $x(t)$ for times in the range $[t_{i-1}, t_i]$ even though these are not yet known. This is not a difficulty for continuous functions because, in the limit $t_{i-1} \to t_i$, $x(t)$ is well behaved ($x(t_i) - x(t_{i-1}) \propto t_i - t_{i-1}$). But it poses a problem for stochastic processes ($W(t_i){-}W(t_{i-1}) \propto \sqrt{t_i - t_{i-1}}$). Itô calculus elegantly solves this problem, by evaluating $g(x)$ at time t_{i-1}, where it is known. Hence Itô prescription is termed *non-anticipating*, whereas all other prescriptions are called *anticipating*.

2.2. *History*

It is impossible to mention here all of the papers that tackled the controversy: we review briefly those that we feel were particularly helpful in shaping the growing understanding on the part of the nonlinear and stochastic physics community.

Perhaps one of the earliest papers to question what is the applicable stochastic calculus in nature was [4], where Stratonovich calculus was used in models of population growth. In [5], following some theoretical works on the correspondence between stochastic calculus and ordinary calculus [6, 7], coupled stochastic differential equations were simulated on a hybrid computer. It was shown that any calculus could be achieved in practice through the tuning of parameters. In [8] it was argued that in theoretical biology Itô calculus should be preferred. In two companion papers [9, 10], non-additive stochastic processes were considered, and the Itô-Stratonovich controversy discussed, looking at the rôle of noise correlation in some physical and chemical systems. In [11] a different approach was followed: starting from a stochastic system with inertia, via contraction an SDE of the form of Eq. (1) was obtained, and in the process it turned out that the "correct" stochastic calculus was Stratonovich. A similar approach to derive the "correct" calculus via contraction from a system with inertia was used in [12]. In [13] a criterion was presented which purposely allowed selection of the correct stochastic calculus. From a theoretical point of view, [14] eventually settled the argument. The consensus that had emerged was that –

- The parameter α is part of the model: it must be chosen on physical grounds and it cannot be inferred through algebraic manipulations.
- In an experiment, a probability distribution is measured: knowledge of this distribution function is not enough to infer α, but additional information is needed, e.g. knowledge of $f(x)$). Eq. (5) is more fundamental than Eq. (1).
- In many real cases, Eq. (1) is an effective (mesoscopic) equation. One should be careful in using some known microscopic force as the term $f(x)$: in principle, in the passage from the microscopic to the mesoscopic level, the "deterministic" microscopic force might not coincide with the mesoscopic force which, inserted in Eq. (1), reproduces the observed dynamics.
- In a typical, continuous, real physical system, we expect Stratonovich calculus to apply; whereas in a system which is intrinsically discontinuous, e.g. in the stock exchange or in the evolution of biological populations, we expect Itô calculus to apply.

Analogue simulations [15, 16] confirmed that continuous physical systems indeed obey Stratonovich calculus: the same SDE's numerically integrated on a digital computer enforcing Itô calculus clearly reproduced the dynamics theoretically expected of Itô calculus. In [17] it was reported that the equilibrium distribution of a stochastic system driven by two weakly autocorrelated additive and multiplicative noises behaved more Itô (Stratonovich) like when the additive noise was faster (slower) than the multiplicative noise (the physical interpretation being that the faster the additive noise, the less continuous the system would appear on the timescale of the multiplicative noise).

3. Itô *vs.* Stratonovich again

Following a few years which saw little further interest in the problem, since the beginning of the nineties a few papers started to appear in the literature which again focused on the stochastic calculus realised in nature. In [18] the rôle and use of an anticipating SDE is discussed. The different integral calculi in quantum mechanical SDEs are discussed in [19,20]. The stochastic calculus to use in bacterial interactions is discussed in [21]. Different calculi in stochastic partial differential equations are discussed in [22]. Mixed stochastic calculi in systems with different time scales are discussed in [23], whereas [24] tackles a similar problem, i.e. the appropriate calculus to use for an overdamped system obtained as a contraction of the dynamics of an underdamped system in the presence of correlated noise. In [25] the problem of the appropriate stochastic calculus in different coordinate systems is discussed. The different calculi are considered in [26], with application to magnetic systems. Itô and Stratonovich calculi have also been studied recently with applications in oceanography [27], population growth [28], and optimal filtering [29]. A recent paper relates the Itô *vs.* Stratonovich problem to thermodynamics [30].

3.1. *A recent case study: anticipating SDE's*

A number of theoretical papers [31–33] have also appeared advocating the possibility of $\alpha = 1$. As counter-intuitive as this may seem, some experiments [34, 35] have nonetheless claimed to have found empirical evidence that there could indeed be physical systems where $\alpha = 1$. We now focus on a discussion of [34, 35], noting that [34] has been the subject of a comment [36] and a reply [37]: we will show that some of the arguments in [34] are not well-founded.

The work of [34] reports experiments on a colloidal particle near a wall in the presence of a gravitational field, electrostatic repulsion from the wall and random scattering from the solvent, the latter being modelled as space-dependent noise. In [34] two different approaches are suggested to derive from the experiments the force acting on the colloidal particle: one approach is based on drift measurements; the other uses the equilibrium distribution (in space) of the colloidal particle. The central result of [34] is a striking difference between these two forces. From this discrepancy the authors of [34] infer the stochastic calculus realised in the system.

Let us first recast our SDE (1) in the form used in [34]:

$$dx = f(x)dt + g(x)dW = \frac{F(x)}{\gamma(x)}dt + \sqrt{2D_\perp(x)}dW \tag{6}$$

where $F(x)$ in the SDE, following [34], is *assumed* to be equal to the deterministic force and $\gamma(x)$ is some damping, assumed to be very large, so that an overdamped SDE can be considered. A drift measurement [34] looks at the distance Δx travelled during a short time Δt. From (5) we can relate $\Delta x/\Delta t$ to our model quantities

$$\langle \Delta x/\Delta t \rangle \equiv \overline{v}_d = \left\langle \frac{dx}{dt} \right\rangle = f(x) + \alpha g(x) g'(x) = \frac{F(x)}{\gamma(x)} + \alpha \frac{dD_\perp(x)}{dx} \tag{7}$$

Note that $\overline{v}_d$ is *not* proportional to $F(x)$, i.e. to the deterministic force: this means that, unlike the case when the diffusion $g(x)$ is constant, a drift measurement cannot, in general, be used to infer the microscopic deterministic force, contrary to the assumption of [34] that $F(x) = \gamma(x)\overline{v}_d$.

In [34] the force is also computed from the equilibrium distribution $P(x)$, which is assumed to exist (it is in fact measured in the experiments). This force is defined as $F_e(x) = -dU(x)/dx$ where $U(x) = -k_B T \ln(P(x))$. From (5) we obtain

$$U(x) = -k_B T \int \frac{f(x) + (\alpha - 1)g(x)g'(x)}{g^2(x)/2}\, dx$$

$$\frac{F_e(x)}{\gamma(x)} = -\frac{1}{\gamma(x)}\frac{dU(x)}{dx} = f(x) + (\alpha - 1)g(x)g'(x) \tag{8}$$

Eqs. (7) and (8) differ by $-g(x)g'(x) = -\frac{dD_\perp(x)}{dx}$, which is *exactly* the experimental discrepancy reported in [34] between the two "forces"; this difference is independent of α, i.e. independent of the stochastic calculus used to describe the physical system. However, the fact that the two "forces" differed by $-g'(x)g(x)$ was construed by the authors of [34] as evidence that $\alpha = 1$.

There is another reason why an experiment like the one described in [34, 35] cannot be used to infer the value of α, even if the force $f(x)$ were known [38]. Going back to Eq. 1, a possible discretization algorithm for numerical integration of $x(t)$ at first order in the integration time-step h is[a] [39]

$$x(h) = x(0) + W(h)g(x(0)) + h\, f(x(0)) + \alpha g'(x(0))g(x(0))\, h \tag{9}$$

The evolution of $x(t)$ depends *explicitly* on the stochastic calculus: hence, in principle we could infer α. But the experiments of [34,35] deal with a system where the overdamped limit has been taken: the SDE where this limit has not yet been taken (we assume the mass of the particle to be unity for compactness) reads:

$$\begin{aligned} dv &= [F(x) - \gamma(x)\, v]dt + \sqrt{2k_B T\gamma(x)}\, dW \\ dx &= v\, dt. \end{aligned} \tag{10}$$

The relationship between diffusion and the term multiplying v on the r.h.s. follows from the existence of an equilibrium distribution, i.e. detailed balance. To first order in h, Eq. (10) is integrated as

$$\begin{aligned} v(h) &= v(0) + W(h)\sqrt{2k_B T\gamma(x(0))} + h[F(x(0)) - \gamma(x(0))\, v(0)] \\ x(h) &= x(0) + h\, v(0) \end{aligned} \tag{11}$$

which does *not* depend on α, contrary to the scheme of Eq. (9)! Hence, it is not possible to guess the "correct" α from observations of the model given by Eq. (10). The quantity α appears when we go from Eq. (10) to Eq. (1), taking some limit. In doing so, however, one finds that the assumption that $F(x)$ in Eq. (1) equals $F(x)$ in Eq. (10) is *wrong* [38]. The correct procedure to obtain the model of Eq. (1) is:

[a] The expression for $x(h)$ which follows from the evaluation of the stochastic integrals is given by

$$x(h) = x(0) + W(h)g(x(0)) + h\, f(x(0)) + \frac{1}{2}g'(x(0))g(x(0))[W(h)^2 + (2\alpha - 1)\, h]$$

which coincides with Eq. (9) at order h, owing to the fact that $\langle W(h)^2 \rangle = h$.

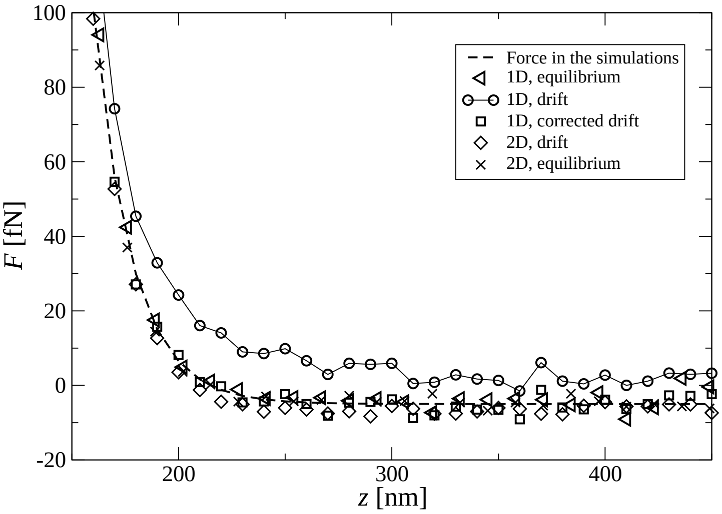

Fig. 1. Forces computed from a simulation of (10) and of the SDE obtained from contraction of (10) using Stratonovich calculus (see [38] for more details). To make contact with the experiment, we took $F(x) = Be^{-kx} + C$ with $B = 770$ pN, $C = -5$ fN, $k = (18\text{ nm})^{-1}$, and $D_\perp(x) = k_B T/\gamma(x)$, $\gamma(x) = 6\pi\eta R \frac{x+a}{x}$, with x in nm, $a = 700$ nm, $2R = 1.31$ nm, $T = 300$ K, $\eta = 8.5 \times 10^{-3}$ Pa s, mass $m = 6.3 \times 10^{-16}$ kg.

(a) write the Fokker-Planck equation corresponding to Eq. (10); (b) adiabatically eliminate v from the Fokker-Planck equation, which yields [12]

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\frac{1}{\gamma(x)}\left(-F(x) + k_B T \frac{\partial}{\partial x}\right) P(x,t); \tag{12}$$

and (c) write the SDE corresponding to Eq. (12). It is at this latter stage that α appears, as a choice that we must make, and it determines the correct relationship between the $F(x)$'s in Eqs. (10) and (1). In [34, 35], however, their equality was assumed [38], so that, in effect, a choice of stochastic calculus had already been made. Eq. (12) yields a force measured from the probability distribution, equal to $F(x)$ regardless of the stochastic calculus. Inspection of Eq. (8) shows that the stochastic calculus implicitly assumed in writing Eq. (1) was $\alpha = 1$. It is a legitimate choice, but should not be taken as a “proof” that $\alpha = 1$ when inferring the “correct” (in reality, “picked at a previous step of the derivation”) α.

Fig. 1 summarizes these arguments: integrating Eq. (10) [40, 41], the forces obtained from drift and equilibrium distribution measurements coincide; and they coincide, both with the deterministic force, and with the force obtained from equilibrium distribution measurements by integration of the correct overdamped 1D SDE obtained from contraction of Eq. (10) using Stratonovich calculus. The force from the drift measured in the 1D simulations differs from the corresponding force obtained from equilibrium measurements but, when the correction $g'(x)g(x)$ is applied, we again have coincidence with the deterministic force.

4. Conclusions

After being a major topic in nonlinear stochastic physics for an extended period, the Itô *vs.* Stratonovich controversy was finally settled some thirty years ago. The recent resuscitation of the debate has involved theoretical and experimental works whose claim to be able to determine experimentally what is the appropriate stochastic calculus in a system is not soundly based. Although these papers are very interesting, some of the conclusions are incorrect – primarily because the earlier debate seems to have been forgotten. Here we have reviewed this debate and discussed the new findings, showing that the "correct" calculus is still as elusive as ever, and that it can only be inferred from the chosen model. In particular, we discussed how the "force" derived from the drift of a Brownian particle need not necessarily coincide with the "force" obtained from the equilibrium distribution. We showed that the discrepancy reported in [34,35] has nothing to do with different stochastic calculi as the authors had inferred: it is simply a consequence of having two different definitions of force, neither of which corresponds to the true microscopic force, and which coincide only where the diffusion coefficient is constant. Furthermore, we recalled that some of the simplified models we use may be characterised by quantities that differ from the true microscopic quantities which appear in a full model.

It is evident that stochasticians of all kinds – mathematicians, physicists, engineers and others – need constant reminders that the Itô versus Stratonovich problem was *solved* long ago. The ideas expressed with such clarity and force by Van Kampen in the 1980s were amply validated by experiments (e.g. [15, 16]). His classic paper [14] has stood the test of time and is well worth re-reading.

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