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Tullock's contest with reimbursements

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Abstract We consider Tullock's contest with reimbursements. It turns out that the winner-reimbursed contest maximizes net total spending while the loser-reimbursed contest minimizes net total spending. We investigate properties of contests with reimbursements and compare them with Tullock's classic contest. Applications for R&D, government contracts, and elections are discussed.

Keywords Contests · Reimbursement

JEL Classification D72 · D74

1 Introduction

Contest literature has greatly expanded since Tullock (1980) presented his simple yet powerful rent-seeking model.[1] However, the contest literature is almost silent about contests with reimbursements. Kaplan et al. (2002) provide two examples of contests with reimbursements in politics and economics. In politics, in the primary election, candidates raise and spend money to be the party's choice for the general election.[2] All losers pay the costs and are not able to advance, while the winner advances to the general election and receives increased funding to compete in this election. In this sense, the winner in the primary wins the

[1] See surveys by Nitzan (1994), Szymanski (2003), Konrad (2007), and Congleton et al. (2008).

[2] Here we consider US Presidential elections as there have been cases where a candidate for the US Senate or the US House has lost a primary and went on to compete and win in the general election (for example, see Lieberman's senatorial victory in Connecticut, http://www.usatoday.com/news/politicselections/vote2006/CT/CT.htm?csp=34).

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prize of the primary election and is monetarily reimbursed the cost of the primary election, so she can be competitive in the general election. The second example, Kaplan et al. (2002) provide, is jet contracts. Boeing and Lockheed Martin were competing for a Joint Strike Fighter (JSF) contract. In order to win the bid for the government contract, they had to design a prototype to test in flight. Under the rules of this type of contest, both companies built this prototype up-front to win this JSF government contract. This contract would enable the winning company to make more JSFs for the government purchase. Thus, the government gives both the prize, the contract, to the winner and reimburses the winner its costs of entry through the purchase of the later completed jets. Moreover, the JSF contract was also a loser-reimbursement mechanism because both Boeing and Lockheed Martin were given a total of 2.2 billion USD to demonstrate their concepts for the final competition.[3] Another example of a reimbursement is Californian JSF Income Tax Credit which is given to any industry in the area which has won subcontracts to construct parts for the JSF.[4] This tax credit is a winner-reimbursement mechanism.

In this paper we consider n-player Tullock contests where the winner and the losers can be reimbursed. We are interested to know how different reimbursement schemes affect (net) total spending. First, we find the reimbursement schemes which maximize net total spending. It turns out that there is a continuum of such designer-optimal reimbursement schemes. In all of them, the winner has to be fully compensated for her effort. The easiest mechanism to implement and the one with the lowest total spending is the reimbursement scheme where only the winner gets her effort reimbursed. In fact, this is probably one of the most popular contest designs in real life. Then, we find reimbursement schemes which minimize net total spending. In all of them, losers have to be reimbursed. Therefore, the winner-reimbursed and loser-reimbursed contests define the boundaries for net total spending in all contests with reimbursements.

We consider properties of contests with reimbursements in Sect. 3. The unique symmetric pure-strategy Nash equilibrium is considered. The equilibrium effort depends on the prize value, V, the number of players, n, and the marginal return, r. We demonstrate the standard results that (1) individual effort, the expected individual payoff, and (net) total effort are increasing functions of the prize value, V, and (2) individual effort and (net) total effort are increasing functions of the parameter r, while the expected individual payoff is a decreasing function of the parameter r.

We discover surprising properties if the number of players changes. In particular, total effort is a *decreasing* function of the number of players in a winner-reimbursed contest, and individual effort is an *increasing* function of the number of players in a loser-reimbursed contest. Note that the winner gets the value of the prize, V, and is compensated for her effort in a winner-reimbursed contest. If the number of players increases, the chance of winning the contest decreases and each player decreases her spending (like in the Tullock's contest). However, individual spending reduction is so strong that this effect prevails in the aggregate (unlike the Tullock's case). The opposite situation occurs in a loser-reimbursed contest. Each player pays only if she wins (she is reimbursed otherwise). Therefore, if the number of player increases, the chance of winning the contest decreases and each player increases her spending in order to improve this chance.

Our previous observation leads to the following intriguing consequences. Even though net total spending is higher in a winner-reimbursement contest, total spending can be higher

[3] See an unclassified military document on http://www.dtic.mil/descriptivesum/Y2000/Navy/0603800N.pdf, page 6.

[4] See http://www.laedc.com/businessassistance/incentives/incentives-ca.html.

in a loser-reimbursement contest. Therefore, the loser-reimbursed contests deserve serious attention in R&D because they stimulate high total spending which might be one of the designer's goals. This may explain why the government continues to fund universities despite the fact that private industry receives a larger number of patents than universities.

We are aware of two papers where the authors consider a particular type of contest with reimbursements: a lottery ($r = 1$) where the winner gets her effort reimbursed. Cohen and Sela (2005) show that in a two-player case there exists a unique internal equilibrium where the weak contestant wins with higher probability than the stronger one. Matros (2008) analyzes the n-player case. He finds all (multiple) equilibria in pure strategies and discusses their properties.

The idea of including reimbursement possibilities into a contest mechanism is not new. It has already been used in the auction literature. Riley and Samuelson (1981) introduced the *Sad Loser Auction*—a two-player all-pay auction where the winner (the highest bidder) gets her bid back and wins the prize. Recently, Goeree and Offerman (2004) analyze the Amsterdam auction in which the highest losing bidder obtains a premium which depends on her own bid. It is important to note that Sad Loser or Amsterdam auctions *cannot* produce higher expected revenue than the optimal auction. At the same time, the winner-reimbursed contest provides the highest expected net total effort, but Tullock's contest does not.

Baye et al. (2005) is an important complement to our paper. These authors use an all-pay auction framework in order to compare different litigation systems. There are three main differences between our models. First, they consider a game with incomplete information: their players have private values. Second, they examine an all-pay auction instead of a contest. Finally, the reimbursements are made by players in their model.

2 n-player contest

Consider an n-player Tullock contest with one prize where the contest designer can reimburse players' contributions. We assume that the winner/loser reimbursements are additively separable in the winner's and loser's spending. Formally, player i exerts effort x_i in order to maximize the following function

$$\max_{x_i} \frac{x_i^r}{\sum_{j=1}^n x_j^r}\left(V_i + \pi^W(x_i)\right) + \left(1 - \frac{x_i^r}{\sum_{j=1}^n x_j^r}\right)\pi^L(x_i) - x_i, \tag{1}$$

where the first term in (1) is the probability of winning the contest times the contest prize for player i, V_i, and the winner's reimbursement, $\pi^W(x_i)$; the second term is the probability of losing the contest times the loser's reimbursement, $\pi^L(x_i)$; and the last term is the cost of effort.[5] We assume that individual reimbursements are linear functions only of individual effort[6]

$$\pi^W(x) = \alpha x, \quad \pi^L(x) = \gamma x, \tag{2}$$

where

$$0 \le \alpha \le 1, \quad 0 \le \gamma \le 1, \quad \alpha + \gamma < 2. \tag{3}$$

[5]We assume that if $x_1 = \cdots = x_n = 0$, then nobody wins the prize.

[6]Baye et al. (2005) also examine linear reimbursements.

Assumption (3) guarantees that individual reimbursement does not exceed individual effort and total reimbursement is strictly less than total effort.[7]

2.1 Tullock's contest with reimbursements

Note that there are three parameters in our model: n—the number of players in the contest, r—the marginal return on individual effort, and $(V_1, \dots, V_n)$—the vector of individual prize values. As is standard in the contest literature, in order to find a closed form solution and analyze its properties, we have to put a restriction on one of these parameters. In this paper, we assume that all prize values are the same, $V_1 = \cdots = V_n = V$, and will analyze a unique symmetric equilibrium. This approach is in line with Tullock's classic paper. In a complementary paper, Matros (2008), case $r = 1$ is considered.[8] Finally, case $n = 2$ is left for the future research.[9]

The first-order condition of the maximization problem (1)–(2) when $V_1 = \cdots = V_n = V$ is

$$\frac{r x_i^{r-1}[\sum_{j\neq i} x_j^r]}{[\sum_{j=1}^n x_j^r]^2}(V + (\alpha - \gamma)x_i) + \frac{x_i^r}{\sum_{j=1}^n x_j^r}(\alpha - \gamma) + \gamma - 1 = 0. \tag{4}$$

In a symmetric equilibrium $x_1 = \cdots = x_n = x^*$, so from the first-order condition (4) we get

$$x^*(\alpha, \gamma) = \frac{(n-1)rV}{(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r}. \tag{5}$$

Expression (5) gives individual equilibrium spending for contests with reimbursements if

$$0 < r \le 1. \tag{6}$$

Note that condition (6) corresponds to the standard assumption in the literature that the contest success function is concave. We will assume that condition (6) holds for the rest of the paper. More details on the second-order conditions can be found in Appendix A.

2.2 (Net) total spending

Define total equilibrium spending

$$TS = nx^*$$

and net total spending as total spending without the winner's and losers' reimbursements

$$NT = nx^* - \pi^W\left(x^*\right) - (n-1)\pi^L\left(x^*\right) = (n - \alpha - (n-1)\gamma)x^*. \tag{7}$$

Note that, first, the players spend TS, then the winner is determined and she or/and the losers are reimbursed. Finally, the contest designer receives NT. It is important to emphasize that the reimbursement scheme is a key difference between our model and that of Baye et al. (2005). In their model, the players reimburse the costs of the winner (or loser). In our case, the contest designer reimburses the players' costs.

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[7] We consider general contests with reimbursements in Appendix B.

[8] Hillman and Riley (1989) consider case $r = 1$ for the standard contest and characterize a unique equilibrium.

[9] Nti (1999) considers this case for the standard contests and finds a unique solution.

We want to analyze the impact of reimbursements on (net) total spending. Typically, the contest designer either wants to maximize net total spending (the optimal contest), minimize net total spending (minimize rent dissipation), or maximize total spending (if the designer's goal is the intensity of research, for example in R&D contests). It is easy to see in (5) that total spending is increasing in both reimbursement parameters α and γ. Therefore, Tullock's original contest, $\alpha=\gamma=0$, gives *the lowest total spending*. Obviously, if players are completely reimbursed whether they win or lose, $\alpha=\gamma=1$, players will want to spend without any bounds. Since we have condition (3), this contest is not included in our consideration. However, by continuity, for any $T_0\in\Re$, $T_0<+\infty$, it is always possible to find a reimbursement scheme such that $\alpha+\gamma<2$ and $TS>T_0$.

2.3 Reimbursement schemes

It is clear from (7) that, for a fixed x^*, reimbursements (α,γ) have a negative direct effect on net spending because they are subtracted from the total. However, they also have a positive strategic effect because they increase x^*. Thanks to closed form solutions, (5), we are able to determine the net effects: NT increases in α and decreases in γ. The former follows because the designer covers only one player's (namely the winner's) cost, but all players increase their efforts, whereas the latter follows because all $(n-1)$ losers are reimbursed, which overcomes their cost increase. Formally, note that using (5), net total spending can be rewritten as

$$NT=(n-\alpha-(n-1)\gamma)\frac{(n-1)rV}{(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r}. \tag{8}$$

Net total spending depends on variable α in the following way

$$\frac{\partial(NT)}{\partial\alpha}=-(n-1)rV \times\frac{(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r-(n-\alpha-(n-1)\gamma)[n+(n-1)r]}{[(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r]^2}.$$

It is easy to see that

$$\begin{aligned}&(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r-(n-\alpha-(n-1)\gamma)[n+(n-1)r]\\&\quad=-(1-\gamma)(n-1)nr<0,\quad \text{if } \gamma<1.\end{aligned}$$

Therefore, net total spending increases monotonically in α

$$\frac{\partial(NT)}{\partial\alpha}>0\quad\text{for } \gamma<1. \tag{9}$$

It means that for a fixed $\gamma<1$

- in a designer-optimal reimbursement scheme (which *maximizes* net total spending) the contest designer should *always* return the winner's spending, $\alpha=1$;
- in a reimbursement scheme which *minimizes* net total spending the contest designer should *never* return the winner's spending, $\alpha=0$.[10]

[10]We use condition (3) here because otherwise the designer would want to impose a tax on the winner to make α as small as possible.

Net total spending depends on variable γ in the following way

$$\frac{\partial(NT)}{\partial\gamma} = -(n-1)^2 rV \times \frac{[(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r] - (n-\alpha-(n-1)\gamma)[n-r]}{[(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r]^2}.$$

It is easy to see that

$$[(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r] - (n-\alpha-(n-1)\gamma)[n-r] = nr(1-\alpha) > 0, \quad \text{if } \alpha < 1.$$

Therefore, net total spending decreases monotonically in γ

$$\frac{\partial(NT)}{\partial\gamma} < 0 \quad \text{for } \alpha < 1. \tag{10}$$

It means that for a fixed $\alpha < 1$

- in a designer-optimal reimbursement scheme the contest designer should *never* return loser's spending, $\gamma = 0$;[11]
- in a reimbursement scheme which *minimizes* net total spending the contest designer should *always* return the loser's spending, $\gamma = 1$.

The following proposition summarizes our findings in this section.

Proposition 1

- *There is a continuum of optimal reimbursement schemes. In each such scheme, the winner has to be reimbursed*:

$$\alpha = 1 \quad \textit{and} \quad 0 \le \gamma < 1.$$

The maximal net total spending is equal to

$$NT^{\text{Max}} = NT^{W} = \frac{(n-1)rV}{(n-r)}.$$

- *There is a continuum of reimbursement schemes which minimize net total spending. These reimbursement schemes require that each loser gets her spending back*:

$$0 \le \alpha < 1 \quad \textit{and} \quad \gamma = 1.$$

The minimal net total spending is equal to

$$NT^{\text{Min}} = NT^{L} = \frac{(n-1)rV}{n+(n-1)r}.$$

[11] We use condition (3) here because otherwise the designer would want to impose a tax on the loser to make γ as small as possible.

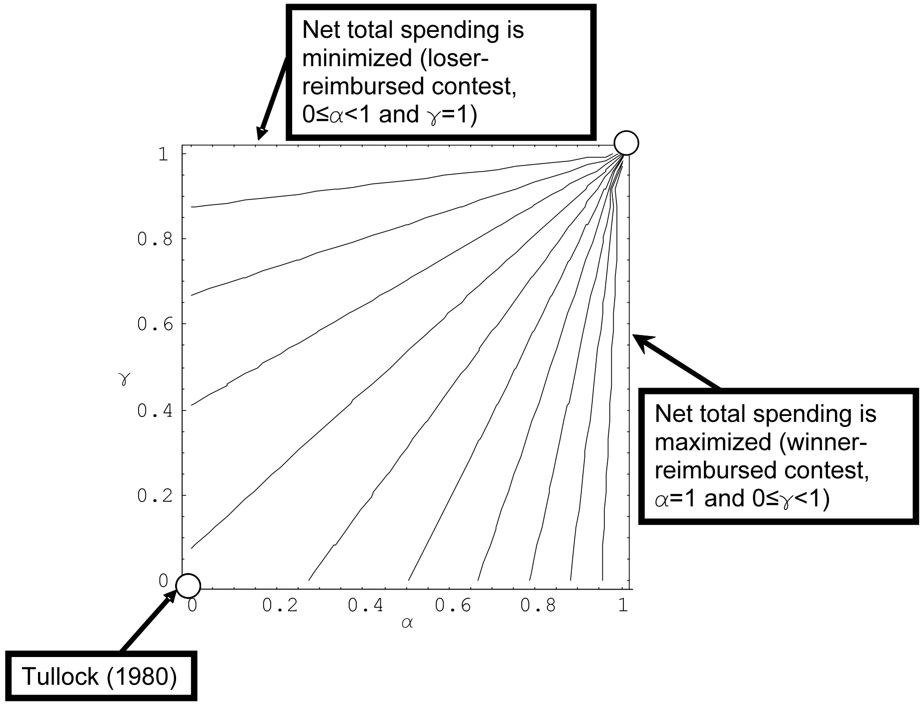

Fig. 1 Contour map for net total spending

Intuitively, if the winner gets reimbursed, this increases the actual prize, and, as a result, this increases the competition in the contest and net total spending (designer's revenue). Proposition 1 shows that net total spending is *independent of*

- the losers' reimbursement in the designer-optimal contest;
- the winner's reimbursement in a reimbursement contest which *minimizes* net total spending.

The loser-reimbursement scheme provides incentives to spend more (positive effect), because each loser gets her money back; as a result net total spending is equal to the winning spending only (negative effect: only one contributor). As Proposition 1 shows, the negative effect is stronger.

Proposition 1 also demonstrates that net total spending in a contest with some reimbursement satisfies the following condition

$$NT^{\mathrm{Min}} = \frac{(n-1)rV}{n+(n-1)r} \leq NT \leq \frac{(n-1)rV}{(n-r)} = NT^{\mathrm{Max}}.$$

Figure 1 summarizes our findings for this section in a contour map of net total spending, NT, as a function of parameters α and γ. Note that Tullock's (1980) classic model corresponds to the case $\alpha = \gamma = 0$.

Figure 1 shows reimbursement schemes ($0 \leq \alpha < 1, \gamma = 1$) which minimize net total spending. From this line, net total spending increases along the γ-line, the loser gets back less and less, until $\alpha = \gamma = 0$, as in Tullock's model. Finally, net total spending increases along the α-line until its maximum is reached at $\alpha = 1$, $0 \le \gamma < 1$ (winner-reimbursed contest).

As we note above, total spending is increasing in both parameters, α and γ. Therefore, total spending is increasing from the bottom left corner (Tullock's contest) to the upper right corner in Fig. 1.

3 Properties of the unique symmetric equilibrium

In this section, we analyze how individual spending, (net) total spending, and the expected equilibrium payoff change if the number of players, the parameter r, or the prize value vary.

3.1 Number of players

It is a standard result in the contest literature that individual spending decreases and total spending increases as the number of players increases. These results do not hold in general in contests with reimbursements. From (5), we get

$$\begin{aligned}
&\frac{\partial x^*(\alpha,\gamma)}{\partial n}\\
&\quad= \frac{(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r - (n-1)(2(1-\gamma)n - (\alpha-\gamma) - (\alpha-\gamma)r)}{[(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r]^2} rV\\
&\quad= \frac{(1-\alpha) - (1-\gamma)(n-1)^2}{[(1-\gamma)n^2 - (\alpha-\gamma)n - (n-1)(\alpha-\gamma)r]^2} rV.
\end{aligned} \tag{11}$$

It is easy to see from (11) that if the number of players increases, then individual spending

- *increases* in the loser-reimbursed contest, $\gamma = 1$;
- *decreases* in the winner-reimbursed contest, $\alpha = 1$.

Therefore, we obtain the following result.

Proposition 2 *In the symmetric equilibrium, individual effort, (net) total spending, and the expected individual payoff are not monotonic functions of the number of players.*

We will consider the two extreme reimbursement schemes in detail now.

In the symmetric equilibrium, if the winner gets her effort reimbursed ($\alpha = 1$) and if losers get their effort reimbursed ($\gamma = 1$), equilibrium spending, total spending, net total spending, and the expected equilibrium payoff become

$$x^W = \frac{1}{(1-\gamma)}\frac{rV}{(n-r)}, \qquad x^L = \frac{1}{(1-\alpha)}\frac{(n-1)rV}{n+(n-1)r}. \tag{12}$$

$$TS^W = \frac{1}{(1-\gamma)}\frac{nrV}{(n-r)}, \qquad TS^L = \frac{1}{(1-\alpha)}\frac{(n-1)nrV}{n+(n-1)r}. \tag{13}$$

$$NT^W = \frac{(n-1)rV}{(n-r)}, \qquad NT^L = \frac{(n-1)rV}{n+(n-1)r}. \tag{14}$$

$$\pi^W = \frac{1-r}{(n-r)}V, \qquad \pi^L = \frac{V}{n+(n-1)r}. \tag{15}$$

The closed-form solution (12) helps to establish the following result.

Proposition 3 *In the symmetric equilibria*

- *the expected individual payoffs*, π^W *and* π^L, *are decreasing functions of the number of players*;
- *net total spending*, NT^W *and* NT^L, *are increasing functions of the number of players*;
- *individual effort*, x^W, *and total spending*, TS^W, *are decreasing functions of the number of players*;
- *individual effort*, x^L, *and total spending*, TS^L, *are increasing functions of the number of players*.

Proposition 3 shows not only the standard contest results about the expected individual payoffs and net total spending, but also intriguing observations about individual effort and total spending. In particular, in the winner-reimbursed contest, since individual effort decreases as the number of players increases, total spending is affected in two ways. Each player spends (much) less now, but there are more players. It turns out that the first effect is stronger. It is important to emphasize that the second effect is stronger in the contest without reimbursements. Proposition 3 also shows that individual spending *increases* as the number of players increases in the loser-reimbursed contest. If all losers get reimbursed, it means that only the winner has to pay. As a result, players become very aggressive, that is, if n increases players spend more in order to increase their winning chances (they try to compensate for the decrease in their probability of winning).

Corollary 1 *In the winner-reimbursed contest the highest total spending is achieved if* $n = 2$:

$$TS^W \leq \frac{1}{(1-\gamma)} \frac{2rV}{(2-r)}.$$

In the loser-reimbursed contest the lowest total spending is achieved if $n = 2$:

$$TS^L \geq \frac{1}{(1-\alpha)} \frac{2rV}{2+r}.$$

As Corollary 1 shows, the designer-optimal contest has the highest total spending if the contest is restricted to just two players. This finding is consistent with the tournament literature, see Fullerton and McAfee (1999) and Che and Gale (2003). Moreover, this result is relevant to many industries, see examples in Fullerton and McAfee (1999) and Che and Gale (2003).

It should be clear from Corollary 1 that total spending can be higher in the loser-reimbursed contest than in the winner-reimbursed contest. Suppose that parameters α, γ, and r are such that

$$\frac{1}{(1-\gamma)} \frac{2rV}{(2-r)} < \frac{1}{(1-\alpha)} \frac{2rV}{2+r}.$$

Then, $TS^W < TS^L$ for any $n \geq 2$. Moreover, if $\gamma = 0$ in the winner-reimbursed contest and $\alpha = 0$ in the loser-reimbursed contest, then total spending is higher in the loser-reimbursed contest than in the winner-reimbursed contest, if there are "enough contestants." Formally, if the following condition holds:

$$n^2 > 2(n + (n-1)r),$$

then $TS^W < TS^L$. If the contest is an R&D competition, the designer might care about not only net total spending, but also total spending (to increase the rate of innovations). Then, the loser-reimbursed contest might be a choice to consider. This may explain why the government continues to fund universities despite the fact that private industry receives a larger number of patents.

3.2 Marginal return, r

The following proposition describes how parameter r influences contests with reimbursements.

Proposition 4 *In the symmetric equilibrium, individual effort, and (net) total spending are increasing functions of the parameter r. The expected individual payoff is a decreasing function of the parameter r.*

Proof From (5), we get

$$\frac{\partial x^*(\alpha,\gamma)}{\partial r} = \frac{(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r+r(n-1)(\alpha-\gamma)}{[(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r]^2}(n-1)V$$
$$= \frac{(1-\alpha)+(n-1)(1-\gamma)}{[(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r]^2}(n-1)nV > 0.$$

Therefore,

$$\frac{\partial TS}{\partial V} = n\frac{\partial x^*(\alpha,\gamma)}{\partial r} > 0 \quad \text{and} \quad \frac{\partial NT}{\partial r} = [(1-\alpha)+(n-1)(1-\gamma)]\frac{\partial x^*(\alpha,\gamma)}{\partial r} > 0.$$

It is straightforward to see that

$$\frac{\partial \pi}{\partial V} = -\frac{1}{n}((1-\alpha)+(n-1)(1-\gamma))\frac{\partial x^*(\alpha,\gamma)}{\partial r} < 0. \qquad \square$$

Proposition 4 demonstrates that if parameter r increases, then individual effort increases too, because the probability of success is influenced more now. However, as a result, the expected individual payoff decreases, because each player is spending more, but has the same chance to win. Since players are spending more, the contest designer will also receive more; hence the increase in net total spending. Proposition 4 shows also that (net) total spending is the highest in the lottery, $r = 1$.

3.3 Prize V

Our final result in this section is standard. It shows that individual effort, the expected individual payoff, and (net) total spending are increasing in the prize value. The intuition is straightforward: if someone values something more, she spends more and expects higher payoff.

Proposition 5 *In the symmetric equilibrium, individual effort, the expected individual payoff, and (net) total spending are increasing functions of the prize value, V.*

Proof From (5), we get

$$\frac{\partial x^*(\alpha,\gamma)}{\partial V} = \frac{(n-1)r}{(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r} > 0.$$

Therefore,

$$\frac{\partial TS}{\partial V} = n\frac{\partial x^*(\alpha,\gamma)}{\partial V} > 0 \quad \text{and} \quad \frac{\partial NT}{\partial V} = [(1-\alpha)+(n-1)(1-\gamma)]\frac{\partial x^*(\alpha,\gamma)}{\partial V} > 0.$$

It is straightforward to see that

$$\begin{aligned}\frac{\partial \pi}{\partial V} &= \frac{1}{n}\left[1+(\alpha+(n-1)\gamma-n)\frac{\partial x^*(\alpha,\gamma)}{\partial V}\right]\\ &= \frac{(1-\alpha)+(1-\gamma)(n-1)(1-r)}{(1-\gamma)n^2-(\alpha-\gamma)n-(n-1)(\alpha-\gamma)r} \geq 0.\end{aligned}$$

□

As the proof shows, the expected payoff is *independent of* the prize value if and only if the contest with reimbursement is a winner-reimbursed contest ($\alpha = 1$) and $r = 1$. In fact, the expected payoff is zero in this case.[12]

Note that net total spending can never exceed the prize value

$$NT \leq NT^{\mathrm{Max}} = \frac{(n-1)rV}{(n-r)} \leq rV \leq V,$$

because of the assumption (3). However, total spending can exceed the prize value. For example, for $r = 1$,

$$TS^W = \frac{nV}{(n-1)} > V.$$

It is intriguing that the winner-reimbursed contest induces wasteful spending. This observation highlights the fact that contests are inefficient in general.

4 Conclusion

In this paper contests with reimbursements are considered. First, a unique symmetric equilibrium in such contests is found. Then, we describe reimbursement schemes which maximize and minimize net total spending. It turns out that there is continuum of such schemes. We describe all of them and discuss, in detail, their properties.

Our results have several applications. First, the winner-reimbursed schemes can be used by charities and casinos to increase net total spending. Second, the loser-reimbursed schemes can be used in R&D contests in order to stimulate participants. Moreover, total spending can be higher in the loser-reimbursed contest than in the winner-reimbursed contest. That might also be one of the designer's goals: to increase the rate of innovations in the contest.

Our work has many extensions. Further study would test our theory in the experimental laboratory. Another avenue for exploration would be to consider contests with reimbursements when the common value of the prize is unknown to players.

[12] Higgins et al. (1985) show that if a zero-profit condition is imposed, the number of players is independent of the value of the prize.

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Appendix A: Second-order conditions

The second-order condition of the maximization problem (1)–(2) when $V_1 = \cdots = V_n = V$ is

$$\frac{(r-1)[\sum_{j=1}^n x_j^r] - 2r x_i^r}{[\sum_{j=1}^n x_j^r]}(V + (\alpha-\gamma)x_i) + 2(\alpha-\gamma)x_i \leq 0. \tag{A1}$$

At the symmetric equilibrium it becomes

$$0 < r \leq \frac{(1-\gamma)(n-2)-\gamma+(2-\alpha)}{(1-\gamma)(n-2)-\gamma+\alpha}, \quad \text{if } (1-\gamma)(n-2)-\gamma+\alpha > 0. \tag{A2}$$

Note that if $(1-\gamma)(n-2)-\gamma+\alpha \leq 0$, then $r \geq 0$.

Expression (5) gives individual equilibrium spending for contests with reimbursements if the second-order condition (A2) holds and players obtain a non-negative expected payoff:

$$\frac{V}{n} + \left(\frac{1}{n}\alpha + \frac{n-1}{n}\gamma - 1\right)x^* \geq 0,$$

or

$$r \leq \frac{(1-\gamma)n-(\alpha-\gamma)}{(1-\gamma)n-(1-\gamma)}. \tag{A3}$$

Conditions (A2) and (A3) describe the range of parameter r for contests with reimbursements. Note that in Tullock's contest, $\alpha = \gamma = 0$, conditions (A2) and (A3) boil down to the standard expression $r = n/(n-1)$. It is easy to verify that if condition (6) holds, then both conditions (A2) and (A3) are satisfied.

Appendix B

Consider an n-player contest with one prize where the contest designer can reimburse players' contributions. We assume that the winner/loser reimbursements are additively separable in the winner's and losers' spending. Formally, player i exerts effort x_i in order to maximize the following function

$$\max_{x_i}\left[\frac{f(x_i)}{\sum_{j=1}^n f(x_j)}(V+\alpha x_i) + \left(1 - \frac{f(x_i)}{\sum_{j=1}^n f(x_j)}\right)\gamma x_i - x_i\right], \tag{B1}$$

where function f satisfies the standard assumptions[13]

$$f(x) \geq 0, \qquad f'(x) > 0, \qquad f''(x) \leq 0. \tag{B2}$$

[13] See Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) among others.

Maximization problem (B1) can be rewritten as

$$\max_{x_i}\left[\frac{f(x_i)}{\sum_{j=1}^{n} f(x_j)}(V+(\alpha-\gamma)x_i)-(1-\gamma)x_i\right]. \tag{B3}$$

There are two cases $\gamma < 1$ and $\gamma = 1$. Consider them in order.

Case 1: $\gamma = 1$

After change of variables

$$Z=\frac{V}{1-\alpha}, \tag{B4}$$

the maximization problem (B3) becomes

$$\max_{x_i}\left[\frac{f(x_i)}{\sum_{j=1}^{n} f(x_j)}(Z-x_i)\right]. \tag{B5}$$

The first-order condition for this loser-reimbursed contest is

$$\frac{f'(x_i)[\sum_{j\neq i} f(x_j)]}{[\sum_{j=1}^{n} f(x_j)]^2}(Z-x_i)-\frac{f(x_i)}{\sum_{j=1}^{n} f(x_j)}=0. \tag{B6}$$

Then, in the symmetric equilibrium, $x_1^*=\cdots=x_n^*=x^*$, from the first-order condition (B6), we get

$$(Z-x^*)=\frac{n}{n-1}\frac{f(x^*)}{f'(x^*)}. \tag{B7}$$

Condition (B7) describes how individual equilibrium spending depends on the prize value. Since the right-hand side of (B7) is an increasing function of x^* (from assumption (B2)) and the left-hand side of (B7) is a decreasing function of x^*, then individual equilibrium spending has to be an increasing function of the prize value Z. This situation is easy to imagine. We have decreasing and increasing functions which intersect at one point. If the prize value increases, the decreasing function moves up and the intersection point has to move to the right.

Since in the case $\gamma = 1$, individual equilibrium spending and net total equilibrium spending coincide, we conclude that individual equilibrium spending and net total spending are increasing functions of the parameter α (from (B4)).

Case 2: $\gamma < 1$

After change of variables

$$W=\frac{V}{1-\gamma} \quad \text{and} \quad \beta=\frac{\alpha-\gamma}{1-\gamma}, \tag{B8}$$

the maximization problem (B3) becomes

$$\max_{x_i}\frac{f(x_i)}{\sum_{j=1}^{n} f(x_j)}(W+\beta x_i)-x_i. \tag{B9}$$

The first-order condition for this winner-reimbursed contest is

$$\frac{f'(x_i)[\sum_{j\neq i} f(x_j)]}{[\sum_{j=1}^{n} f(x_j)]^2}(W+\beta x_i)+\frac{f(x_i)}{\sum_{j=1}^{n} f(x_j)}\beta-1=0. \tag{B10}$$

Then, in the symmetric equilibrium, $x_1^* = \cdots = x_n^* = x^*$, from the first-order condition (B10), we get

$$\frac{(n-1)f'(x^*)}{n^2 f(x^*)}(W + \beta x^*) + \frac{1}{n}\beta - 1 = 0,$$

or

$$(W + \beta x^*) = \frac{(n-\beta)n}{(n-1)} \frac{f(x^*)}{f'(x^*)}. \tag{B11}$$

Since the right and the left-hand sides of (B11) are increasing functions of x^* (from assumption (B2)), we cannot determine how individual equilibrium spending changes as a function of parameter β in general. Moreover, we are interested to know how net total spending changes if parameter β changes. We are able to find answers for the following class of functions

$$f(x) = x^r, \tag{B12}$$

where $0 < r \leq 1$.

Tullock's contest

Suppose that condition (12) holds. Then, (B11) becomes

$$(W + \beta x^*) = \frac{(n-\beta)n}{(n-1)} \frac{x^*}{r},$$

or

$$x^* = \frac{(n-1)r}{n^2 - \beta n - (n-1)\beta r} W.$$

It is easy to see now that individual equilibrium spending is an increasing function of parameter β. It means that individual equilibrium spending is an increasing function of parameter α and a decreasing function of parameter γ (from (B8)).

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