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MARK L. MacDONALD

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Cohomological invariants of odd degree Jordan algebras

BY MARK L. MACDONALD
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB.
e-mail: M.MacDonald@dpmms.cam.ac.uk

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Abstract

In this paper we determine all possible cohomological invariants of $\mathbf{Aut}(J)$-torsors in Galois cohomology with mod 2 coefficients (characteristic of the base field not 2), for J a split central simple Jordan algebra of odd degree $n \geqslant 3$. This has already been done for J of orthogonal and exceptional type, and we extend these results to unitary and symplectic type. We will use our results to compute the essential dimensions of some groups, for example we show that $\mathrm{ed}(\mathbf{PSp}_{2n}) = n + 1$ for n odd.

1. *Introduction*

The Stiefel–Whitney classes of quadratic forms over k define invariants in Galois cohomology $H^*(k, \mathbb{Z}/2\mathbb{Z})$ up to isometry [**3**], [**14**]. It is shown in [**5**, chapter VI] that the even Stiefel–Whitney classes form a basis of all cohomological invariants of $\mathbf{SO}_n$-torsors for $n \geqslant 3$ odd. This is done by identifying the $\mathbf{SO}_n$-torsors with isomorphism classes of determinant 1 quadratic forms of dimension n. These torsors may be further identified with isomorphism classes of algebras with orthogonal involution of degree n, by sending q to its adjoint involution ad_q (see [**11**, theorem 4·2, p. 42]). These classes may further be identified with isomorphism classes of central simple Jordan algebras of degree n whose associated composition algebra C is one dimensional (see [**9**, p. 210]). We wish to extend these results to $\dim(C) = 2$ or 4, which is to say, odd degree algebras with unitary and symplectic involutions. In fact, in the octonion case $\dim(C) = 8$, we only have a Jordan algebra when the degree is 3, and then it is called an Albert algebra. The mod 2 cohomological invariants of Albert algebras have been determined in [**5**, chapter VI]. Nevertheless, we include this case here for completeness.

For any $n \geqslant 3$ we will define $J_n^r := (M_n(C), -)_+$ over k, as the split Jordan algebra of hermitian elements, where C is the split composition algebra of $\dim(C) = 2^r$. If $r = 3$ then we insist that $n = 3$. Here $-$ denotes the conjugate transpose involution. The following table summarizes, for $n = 2m + 1$, the (split) automorphism groups $\mathbf{Aut}(J_n^r)$, together with their mod 2 cohomological invariants (see Theorem 4·7). We list the degrees of invariants which form an $H^*(k_0)$-basis of all invariants.

r	$\mathbf{Aut}(J^r_{2m+1})$	Degrees of H^*-basis of $\mathrm{Inv}(\mathbf{Aut}(J^r_n))$
0	$\mathbf{SO}_{2m+1}$	$0, 2, 4, \dots, 2m$
1	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathbf{PGL}_{2m+1}$	$0, 1, 3, \dots, 2m+1$
2	$\mathbf{PSp}_{2(2m+1)}$	$0, 2, 4, \dots, 2m+2$
3	F_4	$0, 3, 5$

Here $m \geqslant 1$, and F_4 denotes the split simple group of type F_4. To specify the semi-direct product in $r = 1$, we just need to describe how the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{PGL}_{2m+1}$. This action is defined by sending any $[a] \in \mathbf{PGL}_{2m+1}$ to its inverse transpose $[(a^t)^{-1}]$ (see [**11**, 29·20]).

We will show (Theorem 4·7) that for $r = 1, 2$ and 3 and $n \geqslant 3$ odd, the normalized invariants of $\mathbf{Aut}(J^r_n)$ in degree greater than zero are in one-to-one correspondence with the even Stiefel-Whitney classes of n-dimensional quadratic forms (which are the invariants of $\mathbf{Aut}(J^0_n)$). Under this bijection the degree zero Stiefel–Whitney class corresponds to the degree r invariant which classifies the associated composition algebra.

The $\mathbf{Aut}(J^1_3)$-invariants are discussed in detail in [**11**, Section 19·B, §30·C] or [**8**], including a mod 3 invariant of degree 2. These invariants determine an $\mathbf{Aut}(J^1_3)$-torsor up to isomorphism (considered there as a degree 3 algebra with unitary involution).

Recently in [**6**] Garibaldi, Parimala and Tignol have classified mod 2 invariants of degree $\leqslant 3$ for $\mathbf{Aut}(J^2_n)$-torsors for n even.

In the final section we determine the essential dimension at 2 for our groups $\mathbf{Aut}(J^r_n)$ for $n \geqslant 3$ odd. In each case it is equal to the lower bound given by [**2**, theorem 1]. In particular, we find that $\mathrm{ed}(\mathbf{PSp}_{2n}) = n + 1$ for $n \geqslant 3$ odd, where previously the best upper bound was given by $2n^2 - 3n - 1$ in [**12**].

2. *Preliminaries*

Throughout we will let k be a field extension of a fixed base field k_0 of characteristic not 2, and k_s will be a separable closure of k.

For any k-algebra A (by which we will mean finite dimensional, not necessarily associative, with identity), we will use the usual notions from Galois cohomology [**11**, section 29] to identify $\mathbf{Aut}_{alg}(A)$-torsors over k with k-isomorphism classes of k-algebras B such that $A_{k_s} \cong B_{k_s}$. Similarly for algebras with involution.

In the Introduction we defined the split Jordan algebras $J^r_n := (M_n(C), -)_+$ for $r = 0, 1, 2$ when $n \geqslant 3$, and also for $r = 3$ when $n = 3$. Here, and throughout this paper we will write $\dim(C) = 2^r$. These Jordan algebras are pairwise non-isomorphic, and over k_s they represent nearly all simple Jordan algebras by a theorem of Albert (see [**9**, Ch. V.6, p. 204]). We will say that a simple Jordan algebra is of degree n (for some $n \geqslant 3$), if it becomes isomorphic to J^r_n over k_s. These are the only kind of Jordan algebras that we will consider.

Furthermore, for $r = 0, 1, 2$ we have that $(M_n(C), -)$ is a central simple algebra with orthogonal (resp. unitary, symplectic) involution, where C is again the split composition algebra of dimension 2^r. They are pairwise non-isomorphic, and over k_s they form all k_s-isomorphism classes of central simple algebras with involution. We say that a central simple algebra with involution has degree n if it becomes isomorphic to $(M_n(C), -)$ over k_s.

One must be careful with the potentially confusing terminology here. A central simple algebra with involution over k is defined to be central and simple as an algebra-with-involution, and might not be central or simple as an algebra over k (see [**9**, p. 208] or [**11**, section 2] for precise definitions).

For $r = 0, 1, 2$ we have a 1-to-1 correspondence between isomorphism classes of these two types of objects given by $(A, \sigma) \leftrightarrow (A, \sigma)_+$. So we can view $\mathbf{Aut}(J_n^r)$-torsors as either isomorphism classes of algebras with involution or as isomorphism classes of Jordan algebras (see [**9**, chapter V·7, p. 210]). An advantage of the Jordan algebra point of view is that it includes the exceptional $r = 3$ case.

2·1. *Invariants of quadratic forms*

We will follow the notation of [**5**] and write $H^*(k)$ or even H^* for the Galois cohomology ring $H^*(\mathrm{Gal}(k_s/k), \mathbb{Z}/2\mathbb{Z})$. For $a \in k^*/(k^*)^2$, we will denote the corresponding element $(a) \in H^1(k)$ so that we have $(a \cdot b) = (a) + (b)$. We let $\mathsf{Quad}_{n,1}(k)$ be the pointed set of k-isometry classes of n-dimensional quadratic forms of determinant 1. And we let $\mathsf{Pf_r}(k)$ be the pointed set of k-isometry classes of r-Pfister forms. We will write $\mathrm{Inv}(G) = \mathrm{Inv}_{k_0}(H^1(-, G))$ for the group of cohomological invariants in H^* of G-torsors.

To define the Stiefel–Whitney classes of a quadratic form q over k, take a diagonalization $q \cong \langle a_1, \ldots, a_n \rangle$. Then define $w_i(q)$ to be the ith degree part of the product

$$w(q) = \prod_{j=1}^{n} (1 + (a_j)) \in H^*(k).$$

This product is called the *total Stiefel–Whitney class*. It is independent of the diagonalization, which can be shown by a chain equivalence argument (see [**14**]).

For an r-Pfister form,

$$q = \langle\langle a_1, \ldots, a_r \rangle\rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_r \rangle,$$

define an invariant by $e_r(q) = (a_1) \cdots (a_r) \in H^r(k)$. It is shown in [**5**, VI] that the H^*-module of invariants of r-Pfister forms, $\mathrm{Inv}(\mathsf{Pf}_r)$, has an $H^*(k_0)$-basis consisting of $\{1, e_r\}$.

3. *An upper bound for the invariants*

In this section we show that $\mathrm{Inv}(\mathbf{Aut}(J_n^r))$ injectively embeds into the tensor product of two groups of invariants that we understand well (see Corollary 3·6).

For any associative composition algebra C over k, and hermitian form h on a free n-dimensional C-module V, the *trace form*, q_h is the quadratic form over k on V such that $q_h(x) = h(x, x)$. If ϕ is the norm of C, and $V \cong C \otimes V_0$ as k-vector spaces, then $q_h \cong \phi \otimes q_0$ for some n-dimensional quadratic form q_0 on V_0.

LEMMA 3·1. *Let C be an associative composition algebra over k with conjugation involution, and let h, h' be hermitian forms over C. Then $h \cong h'$ iff $q_h \cong q_{h'}$.*

Proof. This is shown in [**18**, 10·1·1,10·1·7].

PROPOSITION 3·2. *Let C be an associative composition algebra over k of dimension* 1 (*resp.* 2, 4). *Then isomorphism classes of involutions on $M_n(C)$ of orthogonal (resp. unitary, symplectic) type correspond to similarity classes of n-dimensional hermitian forms over C, under $ad_h \leftrightarrow h$.*

Proof. This is proved in [**11**, proposition 4·2, p. 43].

We will call a Jordan algebra over k *reduced* if it is isomorphic to one of the form $(M_n \otimes C, ad_q \otimes -)_+$ for some composition k-algebra C, and n-dimensional quadratic form q over k.

Notice this implies Jacobson's [**9**] definition of reduced in terms of orthogonal idempotents, and his definition implies this one, by his Coordinatization theorem for $n \geqslant 3$.

LEMMA 3·3. *Let J be a central simple Jordan algebra of odd degree n and of type $r = 0$ or 2. Then J is reduced.*

Proof. So $J \cong (A, \sigma)_+$, where (A, σ) is a central simple algebra with an involution of the first kind. So its index is a power of 2 (see [**11**, p. 18, 2·8]).

Case $r = 0$. The degree of A as a central simple algebra is n, so the index divides n, and hence A is split. So J is reduced by Proposition 3·2 (or [**11**, p. 1]).

Case $r = 2$. The degree of A as a central simple algebra is $2n$, so the index divides 2, and hence J is reduced by proposition 3·2 and the remarks at the beginning of this section.

LEMMA 3·4. *Let J be a central simple Jordan algebra of odd degree. Then J becomes reduced after extending scalars by a field extension of odd degree.*

Proof. The only non-reduced algebras are when $r = 3$ or $r = 1$ by Lemma 3·3. For the $r = 3$ case, as stated in the introduction, we must have $n = 3$. By [**20**, 6·1] any non-reduced Albert algebra becomes reduced after a cubic extension.

For $r = 1$ there are two cases, as follows.

Case $J \cong (B \times B^{op}, \sigma)_+$. Here σ is the exchange involution, and B is a central simple algebra over k of degree n odd. Then any maximal subfield L is a splitting field for B of degree dividing n. Then J_L is reduced by Propositon 3·2.

Case $J \cong (A, \sigma)_+$. Here A is a central simple algebra over K of degree n odd, where K is a quadratic extension of k, and σ is an involution of the second kind (i.e. unitary) over k. Since the Brauer group of a finite field is trivial, we may assume that k is infinite.

Let L/k be an odd degree extension such that $\mathrm{ind}(A_L) = d$ is minimal, where A_L is a simple associative algebra with centre $K_L = K \otimes_k L$. d is odd since $d|n$. Let D be the division K_L-algebra that is Brauer equivalent to A_L, and hence of degree d as a central simple algebra. Then D has an involution of the second kind τ that fixes L by [**11**, 3·1, p. 31].

We want to show that $d = 1$, so assume that $d > 1$. Then we can take a non-scalar element a in the degree d Jordan algebra $(D, \tau)_+$. Since k is infinite, we may choose a to be of maximal degree in the sense of [**9**, p. 224]. In other words, a is such that $\deg(m_a) = d$, where $m_a(\lambda) \in L[\lambda]$ is the minimal polynomial of a in $(D, \tau)_+$, for some indeterminate λ. Here we are using the fact from [**9**, p. 233] that the degree of the generic minimal polynomial of a generic element is equal to the degree of the Jordan algebra as defined in the Preliminaries. Also, the coefficients of m_a are in L by [**11**, 32·1·2, p. 452].

If we let α be a root of m_a in k_s, then the minimal polynomial of α is m_a. This is because D is a division algebra and hence m_a is irreducible. So $E = L(\alpha)$ is a field extension of degree d over L, and in particular, of odd degree over k. Then by considering the generic norm of $a' := \alpha 1 - a_E \in D_E$, we see that $n(a') = m_a(\alpha) = 0$ [**9**, p. 224]. So $a' \neq 0$ and is non-invertible, and hence D_E is not a division algebra. So $\mathrm{ind}(A_E) = \mathrm{ind}(D_E) < d$, which contradicts the minimality of d.

Therefore $d = 1$, and hence $A_L \cong M_n(K_L)$. So by Proposition 3·2 we see that J_L is reduced.

PROPOSITION 3·5. *For $n \geqslant 3$ odd, let J be an $\mathbf{Aut}(J_n^r)$-torsor over k. Then there is an odd-degree extension L/k such that J_L is in the image of*

$$\begin{aligned}\mathcal{H} : \mathsf{Pf}_r(L) \times \mathsf{Quad}_{n,1}(L) &\longrightarrow H^1(L, \mathbf{Aut}(J_n^r))\\ (\phi, q) &\longmapsto (M_n \otimes C_\phi, ad_q \otimes -)_+.\end{aligned}$$

Moreover, if $r = 2$ the map is a surjection, and for $r = 0$ it is a bijection.

Proof. From Lemma 3·3 and 3·4 we get L/k such that J_L is reduced, and since n is odd, we can scale q so that $det(q) = 1$. Lemma 3·3 gives the $r = 2$ surjection, and the $r = 0$ bijection is well–known.

COROLLARY 3·6. *We have an injective map of invariants*

$$\mathrm{Inv}(\mathbf{Aut}(J_n^r)) \hookrightarrow \mathrm{Inv}(\mathsf{Pf}_r) \otimes \mathrm{Inv}(\mathsf{Quad}_{n,1}).$$

Proof. By [**4**, lemma 5·3] we can use the surjectivities from Proposition 3·5 to induce an injective map on invariants. Then from [**5**, exercise 16·5], we can express the invariants of the direct product $\mathsf{Pf}_r \times \mathsf{Quad}_{n,1}$ as the tensor product of the invariants of each factor.

4. *Construction of the invariants*

Now it is a matter of deciding which of these invariants occur. In other words, we wish to determine the image of the injective map in Corollary 3·6. It turns out to be the constant invariants together with all multiples of e_r, the degree r invariant of Pf_r (Theorem 4·7).

THEOREM 4·1. *For n odd, $r = 0, 1, 2$ or 3, the invariants $e_r \otimes w_{2i} \in \mathrm{Inv}(\mathsf{Pf}_r) \otimes \mathrm{Inv}(\mathsf{Quad}_{n,1})$ extend uniquely to $\mathbf{Aut}(J_n^r)$-invariants of degree $r + 2i$, which we will call v_i.*

If the invariants extend, then by Corollary 3·6 they are unique. First we will show how to construct the invariants on reduced Jordan algebras, and then use [**4**, Proposition 7·2] to extend them to all Jordan algebras.

For a reduced Jordan algebra $J = (M_n(C), ad_q \otimes -)_+$, we call C the *composition algebra associated to J*. It is determined up to isomorphism by the isomorphism class of J (see [**9**] or [**13**, 16]). We will usually denote its norm form ϕ, which is a Pfister form.

LEMMA 4·2. *The quadratic form $\phi \otimes q$ is determined up to similarity by the isomorphism class of the reduced Jordan algebra $J = (M_n(C), ad_q \otimes -)_+$.*

Proof. First consider the case when C is associative, which is to say, $r \neq 3$. By [**9**, p. 210] the Jordan algebra J determines the isomorphism class of the algebra with involution $(M_n(C), \sigma)$. Then Proposition 3·2 lets us associate up to a scalar, the n-dimensional hermitian form h on C. Finally, Lemma 3·1 allows us to determine its trace form $\phi \otimes q$ up to similarity.

In the case $r = 3$, we use the following argument. For any reduced Jordan algebra of degree n, the quadratic (reduced) trace form, $T_J(x) = Trd(x^2) = trace(x^2)$ is determined up to isometry by the isomorphism class of J, and is of the form

$$T_J \cong n\langle 1\rangle \perp \langle 2\rangle\phi \otimes \wedge^2 q.$$

But since q is similar to $\wedge^2(q)$ for 3-dimensional forms, we see that $\phi \otimes q$ is determined up to similarity for $n = 3$, and in particular when $r = 3$.

Remark 4·3. In the $r = 2$ case, this observation was noted in [**7**, lemma 4·2].

LEMMA 4·4. *Let ϕ be an r-fold Pfister form, and q, q' quadratic forms over k. Then $\phi \otimes q \cong \phi \otimes q'$ implies $e_r(\phi)w(q) = e_r(\phi)w(q') \in H^*(k)$.*

Proof. This is an extension of [**3**] or [**14**] where it is shown for $r = 0$. We need the fact from [**21**] that says if $\phi \otimes q \cong \phi \otimes q'$ then q and q' are ϕ-chain equivalent.

Two quadratic forms are *simply ϕ-equivalent* if they can both be diagonalized in such a way that $q \cong \langle a_1, \ldots, a_n\rangle$ and $q'' \cong \langle \lambda a_1, a_2, \ldots, a_n\rangle$, where λ is represented by ϕ. Then two forms are *ϕ-chain equivalent* if there is a finite chain of simple ϕ-equivalences from one to the other.

This immediately reduces the problem to showing that equality holds at each stage of the chain equivalence. This is the same as showing $e_r(\phi)w(\langle a\rangle) = e_r(\phi)w(\lambda\langle a\rangle)$ for λ represented by ϕ. For such a λ, we have $\phi \otimes \langle 1, -\lambda\rangle$ is isotropic, and hence $e_r(\phi)\cdot(\lambda) = 0 \in H^{r+1}(k)$. Expanding $w(\lambda\langle a\rangle) = 1 + (\lambda) + (a)$, the result clearly follows.

So the following Lemma shows that the quadratic form $\phi \otimes q$, where $\det(q) = 1$, is in fact determined up to isometry by the isomorphism class of a reduced Jordan algebra. Since q is odd dimensional, there is always a determinant 1 quadratic form similar to it. We will write $d(q) = \det(q)$ for the element of $k^*/(k^*)^2$ corresponding to $w_1(q) \in H^1(k)$.

LEMMA 4·5. *Let ϕ be an r-Pfister form, $\lambda \in k^*$, and q, q' quadratic forms. If $\phi \otimes q \cong \phi \otimes \lambda q'$ then $\phi \otimes d(q)q \cong \phi \otimes d(q')q'$.*

Proof.

$$\begin{aligned} e_r(\phi)\cdot w_1(q) = e_r(\phi)\cdot w_1(\lambda q') \in H^{r+1}(k) &\iff e_r(\phi)\cdot(\lambda d(q)d(q')) = 0 \\ &\iff \phi \otimes \langle\langle \lambda d(q)d(q')\rangle\rangle \text{ is hyperbolic} \\ &\iff d(q)d(q') = \lambda \bmod D(\phi)^* \\ &\implies \phi \otimes d(q)q \cong \phi \otimes d(q')q'. \end{aligned}$$

Proof of theorem 4·1. First we will show that the invariants $e_r \otimes w_{2i}$ extend to invariants on k-isomorphism classes of reduced Jordan algebras.

Consider the reduced Jordan algebra $J = (M_n(C), ad_q \otimes -)_+$ with $n = 2m + 1$. Then we can assume $\det(q) = 1$. Lemma 4·2 together with Lemma 4·5 show that $\phi \otimes q$ is determined up to isometry by the isomorphism class of J. Then by Lemma 4·4 we can define $v_i(J) = e_r(\phi)w_{2i}(q) \in H^{r+2i}(k)$ on k-isomorphism classes of reduced Jordan algebras, for $1 \leqslant i \leqslant m$. This clearly extends $e_r \otimes w_{2i}$.

Finally, by Lemma 3·4, any odd degree Jordan algebra becomes reduced after an odd degree extension. So by [**4**, proppsition 7·2] these invariants may be extended to non-reduced Jordan algebras as well, and in other words, to all $\mathbf{Aut}(J_n^r)$-torsors. By Corollary 3·6, v_i is the unique invariant extending $e_r \otimes w_{2i}$.

Remark 4·6. For $r = 1$, there is an invariant closely related to v_1 defined on conjugacy classes of algebras with unitary involution of odd degree in [**11**, p. 438, 31·44]. They related it to the Rost invariant.

Now we can state and prove our main theorem.

THEOREM 4·7. $\mathrm{Inv}_{k_0}(\mathbf{Aut}(J_n^r))$ *is a free $H^*(k_0)$-module with a basis consisting of the invariants $\{1, v_0, v_1, v_2, \ldots, v_m\}$.*

Proof. For $r = 0$ this is shown in [**5**, chapter VI], noting that in this case $v_0 = 1$, causing a redundancy in the set of basis elements. So take $r > 0$. From Corollary 3·6 we know that every $\mathbf{Aut}(J_n^r)$-invariant restricts to some

$$1 \otimes a + e_r \otimes b \in \mathrm{Inv}(\mathsf{Pf}_r) \otimes \mathrm{Inv}(\mathsf{Quad}_{n,1}),$$

for some uniquely defined $a, b \in \mathrm{Inv}(\mathsf{Quad}_{n,1})$. We know from [**5**, Chapter VI] that any $b \in \mathrm{Inv}(\mathsf{Quad}_{n,1})$ is in the $H^*(k)$-span of the even Stiefel–Whitney classes, so by Theorem 4·1, $e_r \otimes b$ is the restriction of some $\mathbf{Aut}(J_n^r)$-invariant in the $H^*(k)$-span of $\{v_0, v_1, \ldots, v_m\}$. So all that remains to show is that if $1 \otimes a$ is the restriction of an $\mathbf{Aut}(J_n^r)$-invariant, then a is constant.

Let a' be an $\mathbf{Aut}(J_n^r)$-invariant that restricts to $1 \otimes a$ for some $\mathsf{Quad}_{n,1}$-invariant a. If we let C_s be the split composition algebra of dimension 2^r, then

$$J = (M_n(C_s), ad_q \otimes -)_+$$

is isomorphic to the split algebra J_n^r (by Proposition 3·2 for $r \neq 3$ and by [**20**, corollary 5·8·2] for $r = 3$). So for any such J, we must have that $a'(J) = a(q)$ is a constant, independent of q. Since we can take q to be an arbitrary n-dimensional form of determinant 1, this implies a is constant. This completes the proof.

Remark 4·8. One may ask to what extent do these v_i determine the $\mathbf{Aut}(J_n^r)$-torsors? There are examples of non-isometric quadratic forms with determinant 1 in each dimension $\geqslant 4$ that have equal total Stiefel–Whitney classes [**17**, beispiel 3·4·1]. So one can use these examples to write down two different reduced $\mathbf{Aut}(J_n^r)$-torsors for $n \geqslant 4$ odd, whose invariants agree.

In the case of $n = 3$, on the other hand, for $r = 0$ or 2, the torsors are determined completely by v_0 and v_1. This is because they are determined by their quadratic trace form [**20**, section 5]. But for $r = 1$ and $r = 3$ this is not the case, because (for $n = 3$) the trace form of any non-reduced algebra is isometric to the trace form of some reduced algebra. Nevertheless, for $r = 1$ one may define a degree 2, mod 3 invariant, which together with v_0 and v_1, classify $\mathbf{Aut}(J_3^1)$-torsors [**11**, section 19·B, section 30·C]. For $r = 3$, one may define a degree 3, mod 3 invariant, but it is an open problem whether this invariant together with v_0 and v_1, classify $\mathbf{Aut}(J_3^3)$-torsors [**19**, 9·4].

5. *Essential dimension*

Given an algebraic group G over k_0, and a G-torsor E over k, the essential dimension of E is defined to be the minimum transcendence degree over k_0 of all fields of definition of E. The essential dimension of an algebraic group is defined to be the supremum of the essential dimensions of all of its torsors ([**1**, **2**, **15**]). The essential dimension of many simple algebraic groups is unknown. We will determine the value of the essential dimension at 2 for some groups G, which we will denote by $\mathrm{ed}(G; 2)$ (see [**2**] for a definition of the essential dimension at a prime). In all cases that we consider, $\mathrm{ed}(G; 2)$ is equal to the lower bound given in [**2**, theorem 1] (or [**15**, theorem 7·8] for characteristic 0).

PROPOSITION 5·1. *For $n \geqslant 3$ odd, we have* $\mathrm{ed}(\mathbf{Aut}(J_n^r); 2) = r + n - 1$.

Proof. By the surjectivity in Lemma 3·5 we have that for any $\mathbf{Aut}(J_n^r)$-torsor J over k, there is an odd degree extension L/k such that J_L is reduced. So by using [**1**, lemma 1·11] we have that $\mathrm{ed}(J;2) \leqslant \mathrm{ed}_L(J_L) \leqslant \mathrm{ed}(\mathsf{Pf_r}) + \mathrm{ed}(\mathsf{Quad}_{n,1}) = r+n-1$. This gives us the upper bound $\mathrm{ed}(\mathbf{Aut}(J_n^r);2) \leqslant r+n-1$.

The lower bound follows from [**2**, theorem 1]. Alternatively, we could deduce the lower bound by using the non-triviality of the degree $r+n-1$ cohomological invariant v_m. This follows from a slight modification of [**1**, corollary 3·6], that if there is a degree d invariant mod 2, then the essential dimension at 2 is at least d.

Let us consider what Proposition 5·1 says for different r and $n \geqslant 3$ odd.

For $r=0$ we get the well-known fact that $\mathrm{ed}(\mathbf{SO}_n) = \mathrm{ed}(\mathbf{SO}_n;2) = n-1$.

For $r=1$ we get $\mathrm{ed}(\mathbb{Z}/2\mathbb{Z} \ltimes \mathbf{PGL}_n) \geqslant \mathrm{ed}(\mathbb{Z}/2\mathbb{Z} \ltimes \mathbf{PGL}_n;2) = n$. The exact value of $\mathrm{ed}(\mathbb{Z}/2\mathbb{Z} \ltimes \mathbf{PGL}_n)$ is unknown to the author for any $n \geqslant 3$.

For $r=2$ we get $\mathrm{ed}(\mathbf{PSp}_{2n}) = \mathrm{ed}(\mathbf{PSp}_{2n};2) = n+1$, since all $\mathbf{PSp}_{2n}$-torsors are reduced. Previously, the best known upper bound for $\mathrm{ed}(\mathbf{PSp}_{2n})$ was $2n^2-3n-1$, which holds for n even as well [**12**].

For $r=3$ we get $\mathrm{ed}(F_4) \geqslant \mathrm{ed}(F_4;2) = 5$, which is the best known lower bound for $\mathrm{ed}(F_4)$. The best published upper bound for the essential dimension is $\mathrm{ed}(F_4) \leqslant 19$ in [**12**]. [**10**] claimed to show that $\mathrm{ed}(F_4) = 5$, but there was a mistake in the proof.

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