On tau functions associated with linear systems

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Abstract This paper considers the Fredholm determinant $\det(I - \Gamma_x)$ of a Hankel integral operator on $L^2(0, \infty)$ with kernel $\phi(s + t + 2x)$, where ϕ is a matrix scattering function. The original contribution of the paper is a related operator R_x such that $\det(I - R_x) = \det(I - \Gamma_x)$ and $-dR_x/dx = AR_x + R_xA$ and an associated differential ring. The paper introduces two main classes of linear systems $(-A, B, C)$ for Schrödinger's equation $-\psi'' + u\psi = \lambda\psi$, namely

(i) $(2, 2)$-admissible linear linear systems which give scattering class potentials, with scattering function $\phi(x) = Ce^{-xA}B$;

(ii) periodic linear systems, which give periodic potentials as in Hill's equation.

The paper introduces the state ring $\mathbf{S}$ for linear systems as in (i) and (ii), and the tau function is $\tau(x) = \det(I + R_x)$.

(i) A Gelfand–Levitan equation relates ϕ and $u(x) = -2\frac{d^2}{dx^2} \log \det(I - R_x)$, which is solved with linear systems as in inverse scattering. Any system of rational matrix differential equations gives rise to an integrable operator K as in Tracy and Widom's theory of matrix models. The Fredholm determinant $\det(I + \lambda K)$ equals $\det(I + \lambda\Gamma_\Phi\Gamma_\Psi)$, where Γ_Φ and Γ_Ψ are Hankel operators with matrix symbols. The paper derives differential equations for τ in terms of the singular points of the differential equation. This paper also introduces an admissible linear system with tau function which gives a solution of Painlevé's equation P_{II}.

(ii) Consider Hill's equation with elliptic potential u. Then u is expressed as a quotient of tau functions from periodic linear systems. If the general solution is a quotient of tau functions from periodic linear systems for all but finitely many complex eigenvalues, then u is finite gap and has a hyperelliptic spectral curve.

The isospectral flows of Schrödinger's equation are given by potentials $u(t, x)$ that evolve according to the Korteweg de Vries equation $u_t + u_{xxx} - 6uu_x = 0$. Every hyperelliptic curve $\mathcal{E}$ gives a solution for KdV which corresponds to rectilinear motion in the Jacobi variety of $\mathcal{E}$. Extending Pöppe's results, the paper develops a functional calculus for linear systems thus producing solutions of the KdV equations. If Γ_x has finite rank, or if A is invertible and e^{-xA} is a uniformly continuous periodic group, then the solutions are explicitly given in terms of matrices.

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1. Introduction

The motivation for this paper is from the theory of random matrices, and the scattering theory of differential equations with rational matrix coefficients. In Tracy and Widom's theory of matrix models [46], the basic data are a 2×2 rational differential equation and a curve. One starts with a system of differential equations

$$J\frac{d}{dx}\begin{bmatrix} f\\ g\end{bmatrix}=\begin{bmatrix}\gamma & \alpha\\ \alpha & \beta\end{bmatrix}\begin{bmatrix} f\\ g\end{bmatrix},\qquad J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\tag{1.1}$$

with α,β and γ rational functions, then one introduces a kernel

$$K(x,y)=\frac{f(x)g(y)-f(y)g(x)}{x-y},\tag{1.2}$$

which due to its special shape is known as an integrable operator. The other essential ingredient of the theory is a prescribed curve $\gamma=\cup_{j=0}^m[a_{2j-1},a_{2j}]$, so that K defines a trace class operator on $L^2(\gamma)$; hence the Fredholm determinant $\det(I-K)$ is defined, and one considers this as a function of the parameters a_j. In particular, one can consider $K: L^2(0,\infty)\to L^2(0,\infty)$ that is trace class and such that $0\leq K\leq I$, so there exists a determinantal random point field on $(0,\infty)$, and $\det(I-K\mathbf{I}_{(s,\infty)})$ is the probability that all random points are in $(0,s)$. In applications to random matrix theory, the random points are eigenvalues of Hermitian matrices with random entries.

Given an $L^2(0,\infty)$ function ϕ, the Hankel integral operator Γ_ϕ with symbol ϕ can be defined on a suitable domain in $L^2(0,\infty)$ by

$$\Gamma_\phi f(x)=\int_0^\infty \phi(x+y)f(y)\,dy.\tag{1.3}$$

When Γ_ϕ belongs to the ideal c^1 of trace class operators on $L^2(0,\infty)$, one can form the determinants $\det(I+\mu\Gamma_\phi)$ and the eigenvalues of $\Gamma_\phi\in c^1$ satisfy multiplicity conditions which are stated in [35, 38]. More generally, one can introduce $\phi_{(x)}(y)=\phi(x+2y)$ and consider

$$\tau(x;\mu)=\det(I+\mu\Gamma_{\phi_{(x)}})\tag{1.4}$$

as a function of $x>0$ and $\mu\in\mathbf{C}$. In this paper, we analyse $\tau(x,\mu)$ by the methods of linear systems. In significant cases of (1.2), such as the Airy kernel or Bessel kernel [46, 47], there exists a Hankel integral operator Γ_ϕ such that $\Gamma_\phi^2=K$; hence one can describe $\det(I-K)$ in terms of $\tau(x,\mu)$. In [8] we showed how one can realise Γ_ϕ by means of linear systems. In the present paper, we take linear systems as the starting point and show how general properties of the linear system are reflected in the τ functions and systems of differential equations so produced.

Definition (Linear system) Let H be a complex Hilbert space, known as the state space, and $\mathrm{B}(H)$ the space of bounded linear operators on H. Let $(e^{-tA})_{t\geq 0}$ be a C_0 semigroup of operators on H such that $\|e^{-tA}\|\leq M$ for all $t\geq 0$ and some $M<\infty$. Let $\mathcal{D}(A)$ be the domain of the generator $-A$ so that $\mathcal{D}(A)$ is itself a Hilbert space for the graph norm

$\|\xi\|^2_{\mathcal{D}(A)} = \|\xi\|^2_H + \|A\xi\|^2_H$, and let $A^\dagger$ be the adjoint of A. Let H_0 be a complex separable Hilbert space which serves as the input and output spaces; let $B : H_0 \to H$ and $C : H \to H_0$ be bounded linear operators. The linear system $(-A, B, C)$ is

$$\begin{aligned} \frac{dX}{dt} &= -AX + BU \\ Y &= CX, \qquad X(0) = 0; \end{aligned} \tag{1.5}$$

so $\phi(x) = Ce^{-xA}B$ is a bounded operator function on H_0, and the corresponding Hankel operator is Γ_ϕ on $L^2((0,\infty); H_0)$, where $\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y)\,dy$.

Definition (Admissible linear system). Let $(-A, B, C)$ be a linear system as above; suppose that the observability operator $\Theta_0 : L^2((0,\infty); H_0) \to H$ is bounded, where

$$\Theta_0 f = \int_0^\infty e^{-sA^\dagger} C^\dagger f(s)\,ds; \tag{1.6}$$

suppose that the controllability operator $\Xi_0 : L^2((0,\infty); H_0) \to H$ is also bounded, where

$$\Xi_0 f = \int_0^\infty e^{-sA} B f(s)\,ds. \tag{1.7}$$

(i) Then $(-A, B, C)$ is an admissible linear system and $\phi(x) = Ce^{-xA}B$ is an admissible scattering function.

(ii) Suppose furthermore that Θ_0 and Ξ_0 belong to the ideal c^2 of Hilbert–Schmidt operators. Then we say that $(-A, B, C)$ is $(2,2)$-admissible.

In [8, Proposition 2.4] we showed that for any $(2,2)$ admissible linear system, the operator

$$R_x = \int_x^\infty e^{-tA} BCe^{-tA}\,dt \tag{1.8}$$

is trace class, and the Fredholm determinant satisfies

$$\det(I + \lambda R_x) = \det(I + \lambda \Gamma_{\phi(x)}) \qquad (x > 0, \lambda \in \mathbf{C}). \tag{1.9}$$

Whereas R_x does not have a direct interpretation in control theory, the notation suggests that R_x has many of the properties of a resolvent operator, as we justify in Lemma 2.1 below. In examples of interest in scattering theory, one can calculate $\det(I + \lambda R_x)$ more easily than the Hankel determinant directly [26, 27]. The operator R_x has additional properties which make it easier to deal with than $\Gamma_{\phi(x)}$.

Definition (Lyapunov equation). Let $-A$ be the generator of a C_0 semigroup on H and let $R : (0,\infty) \to \mathbf{B}(H)$ be a function. The Lyapunov equation is

$$-\frac{dR_z}{dz} = AR_z + R_z A \tag{1.10}$$

with initial condition on the derivative

$$AR_0 + R_0A = BC. \tag{1.11}$$

The definition slightly differs from the equations from [35, 38]. In this paper we take (1.10) as the starting point and in section 2 we solve (1.10) for some $(2,2)$ admissible linear system. Then we use R_x to construct solutions to the associated Gelfand–Levitan equation which involves ϕ. The following definition of u is motivated by scattering theory for Schrödinger's equation $-\psi'' + u\psi = \lambda\psi$ in $L^2(\mathbf{R})$. See [19]

Definition (Potential). For each $(2,2)$ admissible system with $H_0 = \mathbf{C}$, the potential is

$$u(x) = -2\frac{d^2}{dx^2}\log\det(I + \Gamma_{\phi_{(x)}}). \tag{1.12}$$

Theorem 1.1 *(i) Suppose that $(-A, B, C)$ is a $(2,2)$ admissible linear system with A bounded. Then there exists a solution R_x to (1.10) and (1.11) such that $\tau(x) = \det(I + R_x)$ is entire.*

(ii) Alternatively, suppose that $(-A, B, C)$ is a linear system with input and output space H, and (e^{ixA}) is a uniformly continuous and π-periodic group on H. Suppose that there exists a trace class operator E on H such that $AE+EA = BC$. Then there exists a solution to (1.10) and (1.11) such that $\tau(x) = \det(I + R_x)$ is entire and π-periodic.

(iii) In either case u is meromorphic on $\mathbf{C}$.

Part (i) is proved in section 2, while (ii) is proved in section 8. In [9] we introduced examples of periodic linear systems as in (ii), and here develop a systematic theory which shares some common elements of scattering theory from case (i).

The fundamental idea of [35] is to realise Hankel operators with balanced linear systems; we refine this idea by working with admissible linear systems, so that we can define determinants and hence the tau function. In section 2, we solve the Gelfand–Levitan equation by means of the operator R_x and recover u from ϕ. The Lyapunov equation (1.10) is equivalent to the identity

$$\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{d}{dx} - \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}, \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}\right] = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{d}{dx}, \tag{1.13}$$

which turns out to be important when one considers $\det(I - R^2)$.

In section 3 we show how to realise kernels of the form (1.2) from linear systems by means of products of Hankel operators with matricial symbols. The system of differential equations (1.1) depends upon the poles of α, β and γ, hence these are natural parameters for the solution space. Ee recall how Schlesinger's equations [41, 22] arises in this context, and compare various notions of tau functions by the partial differential equations that they satisfy.

Krichever and Novikov considered

$$\left[\frac{\partial}{\partial t_j} - U_j, L\right] = B_j L \tag{1.14}$$

where U_j are matrix functions and B_j are differential operators, a relation which is similar to (1.13). They formulated the notion of an algebo-geometric system. In particular, this applies to finite gap Schrödinger equations, where the spectral parameter may be chosen to be a meromorphic function on a hyperelliptic Riemann surface.

In section 4, we introduce the family of linear systems $\Sigma_\lambda = (-A, (\lambda I+A)(\lambda I-A)^{-1}B, C)$ for λ in the resolvent set of A, and the corresponding tau function $\tau_\lambda(x)$; then we introduce the Baker–Akhiezer function $\psi_{BA}(x,\lambda) = e^{\lambda x}\tau_\lambda(x)/\tau(x)$; here x is the state variable and λ a spectral parameter. We say that $(\Sigma_\lambda)_\lambda$ is a *Picard family* of linear systems if $x \mapsto \psi_{BA}(x,\lambda)$ is meromorphic for all but finitely many λ. This term is introduced by analogy with the terminology of Gesztesy and Weikard [25, Theorem 1.1], who define a meromorphic potential u to be Picard if $-f'' + uf = \lambda f$ has a meromorphic general solution for all but finitely many $\lambda \in \mathbf{C}$. We obtain significant examples of scattering functions which we use in subsequent sections,and mention the linear partial differential equations for scattering functions that correspond to the nonlinear KP equations for the potentials. In subsequent examples, we introduce a compact Riemann surface $\mathcal{E}$ and a meromorphic function $\lambda : \mathcal{E} \to \mathbf{P}^1$ such that $\lambda \mapsto \psi_{BA}(x,\lambda)$ is meromorphic, except possibly at finitely many points. We recall that a compact Riemann surface $\mathbf{X}$ is hyperelliptic if and only if there exists a meromorphic function u on $\mathbf{X}$ that has precisely two poles. In this case, there is a two-sheeted cover $\mathbf{X} \to \mathbf{P}^1$ with $2g+2$ branch points, where g is the genus of $\mathbf{X}$. The elliptic case has $g = 1$.

To realise integrable operators as in (1.2), we need to work with products of Hankel operators. Pöppe [32, 39, 40] proved some remarkable product formulas involving products and traces of Hankel integral operators and applied them to scattering theory, and his work motivated some of the results of this paper. In section 5, we introduce a functional calculus which encompasses Pöppe's ideas, but uses R_x and operators on the state space of a linear system. We suppose that (e^{-tA}) defines a holomorphic semigroup and we can introduce a domain Ω on which $\det(I+R_z)$ is holomorphic and nowhere zero, so $I+R_z$ has a bounded inverse F_z. We introduce a differential ring $\mathbf{S}$ of holomorphic functions from Ω to the space of bounded linear operators on H, which contains A, BC, R_z and F_z, so that we can solve (1.10) and (1.11) inside $\mathbf{S}$. If we can choose $\mathbf{S}$ to be a right Noetherian ring, then we say that $(-A,B,C)$ is finitely generated. Given $\mathbf{S}$, we introduce a space of functions $\mathbf{B}$ and the linear map $\lfloor\,.\,\rfloor : \mathbf{S} \to \mathbf{B}$ such that

$$\lfloor P \rfloor = \frac{d}{dx}\mathrm{trace}\Big(P(F_x - I)\Big). \tag{1.15}$$

We identify a subring $\mathbf{A}$ of $\mathbf{S}$ such that the range of $\lfloor\,.\,\rfloor$ restricted to $\mathbf{A}$ is a differential ring $\lfloor \mathbf{A} \rfloor$ of functions which contains $u(x)$. In these terms, the scattering transform is

$$\phi(x) = Ce^{-xA}B \longleftrightarrow u(x) = -4\lfloor A \rfloor. \tag{1.16}$$

Thus $\lfloor\,.\,\rfloor$ linearizes the determinant.

Gelfand and Dikii [23] considered the ring $\mathbf{A}_0 = \mathbf{C}[u, u', u'', \dots]$ of complex polynomials in u and its derivatives. They showed that if u satisfies the stationary higher order KdV equations (8.1), then $-f'' + uf = \lambda f$ is integrable by quadratures on a spectral curve, which is a hyperelliptic Riemann surface $\mathcal{E}$ of finite genus. Such u are known as finite gap or algebro geometric potentials since $-\frac{d^2}{dx^2}+u$ has a spectrum in $L^2(\mathbf{R})$ that consists of intervals known as bands, separated by finitely many gaps. Then $\mathbf{A}_0$ is a Noetherian ring; see [14, 43]. The ring $\lfloor \mathbf{A} \rfloor$ is analogous to $\mathbf{A}_0$ in the particular examples that we analyse in subsequent sections.

In section 6 we show that if A is a finite matrix with eigenvalues λ_j such that $\Re\lambda_j>0$, then $(-A,B,C)$ is finitely generated. We also recover some determinant formulas from the theory of solitons.

Our next major application is in section 7, concerning the Airy kernel. With $\phi(x)=\mathrm{Ai}(x)$, the integral operator $\Gamma^2_{\phi_x}$ on $L^2(0,\infty)$ has a kernel known as the Airy kernel, which is a universal example in random matrix theory [43]. There $F_2(x)=\det(I-\Gamma^2_{\phi_x}/4)$ is the cumulative distribution function of the Tracy–Widom distribution associated with the soft spectral edge of the Gaussian unitary ensemble. We recover Ablowitz and Segur's result of [1] that $-2(\log F_2)''$ satisfies the Painlevé's second transcendental differential equation P_{II}.

A significant advantage of the R_x operator is that it enables us to analyse periodic linear systems, which seem to lie outside the scope of [32, 39]. In section 8, we introduce linear systems $(-A,B,C)$ such that A is an invertible operator that commutes with BC, and e^{xA} is a uniformly continuous periodic group and the A,B,C are block diagonal matrices. Thus we introduce periodic linear systems with potentials that are either rational trigonometric functions on the complex cylinder $\mathbf{C}/\pi\mathbf{Z}$ or elliptic functions on the complex torus $\mathbf{C}/\pi\mathbf{Z}+i\pi\mathbf{Z}$ as in section 10, and show that these have analogous properties.

The table below summarizes the functions that we produce from explicit linear systems in sections 6,7 and 10. Here g is the genus of the spectral curve, $\wp$ is Weierstrass's elliptic function, θ_1 is Jacobi's theta function [33], u in the fifth column satisfies P_{II} from [20].

equation	$u\in\lfloor \mathbf{A}\rfloor$	$\tau\in\mathbf{L}$	$\mathcal{E}$
Schrödinger	scattering		$\mathbf{R}\to[0,\infty)$
Painlevé	P_{II}	Tracy–Widom F_2	
Hill	finite gap	θ	hyperelliptic
Lamé	$-g(g+1)\wp$	$\theta_1(x)^{g(g+1)/2}$	$\mathcal{Y}_\ell\to\mathcal{T}$
soliton	$-g(g+1)\mathrm{cosech}^2 x$	$(\sinh x)^{g(g+1)/2}$	$\{-g,\dots,-1\}\cup[0,\infty)$

Our most complete results are for elliptic potentials, as in section 10. We obtain a characterization of the elliptic potentials that are finite gap in terms of the general solution of Hill's equation. All elliptic potentials can be realised as quotients of tau functions from periodic linear systems, however, the general solution of Hill's equation can be expressed as a quotient of tau functions from periodic or Gaussian linear systems only if the potential is finite gap. This complements results of Gesztesy and Weikard from [25].

In discussing Hill's equation, Ercolani and McKean [19] observe that the notions of Jacobi variety and theta functions can be extended to the case of infinitely many spectral gaps, whereas the notion of a multiplier curve is somewhat tenuous. Likewise we can introduce tau functions via determinants of linear systems in cases where there is no related algebraic curve. The spectral class of a potential is invariant under flows associated with the Korteweg de Vries equation $u_t+u_{xxx}-6uu_x=0$, which belongs to a hierarchy of partial differential equations which are themselves associated with flows $u(0,x)\mapsto u(t,x)$ on the space of potentials. Indeed,

u is finite gap if it satisfies the stationary KdV equations as in [23, 24, 34]. We therefore consider a family of linear systems $\Sigma_\lambda(t)$, with common $A : H \to H$, and constant input and output spaces, where $t = (t_1, t_2, \dots)$ is a sequence of real parameters and λ is a spectral parameter. Then $\Sigma_\lambda(y)$ has a potential $u_\lambda(x;t)$ with poles depending upon (λ, t); thus the dynamics of the system is reflected in the pole divisor of the potentials, as we describe in section 9.

If u is a finite gap potential for Hill's equation, then the spectral curve is hyperelliptic and has a finite-dimensional complex torus $\mathbf{X}$ as its Jacobi variety, thus the corresponding tau function can be expressed as the restriction of a theta function to a straight line in the tangent space of $\mathbf{X}$ by results of Its and Matveev. In section 9 we formulate a sufficient condition for the tau function of a peridic linear system to be algebraic, in this sense, in terms of the Kadomstev–Petviashvili equations. Soliton solutions of KP occur for spectral curves that are rational curves in the plane that have only regular double points. The term elliptic solitons refers to functions of rational character on the torus, namely elliptic functions.

Some of the linear systems are associated with classical or quantum Hamiltonian systems. Let $H(q,p;x)$ be a Hamiltonian system in canonical coordinates $q = (q_1, \dots, q_n)$ and $p = (p_1, \dots, p_n)$ with time x, and let $S(q, \alpha, x)$ a complete solution of the Hamiltonian–Jacobi equation

$$\frac{\partial S}{\partial x} + H\Big(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, x\Big) = 0 \tag{1.17}$$

depending upon parameters $\alpha = (\alpha_1, \dots, \alpha_n)$, and that $\det[\frac{\partial^2 S}{\partial \alpha_j \partial q_k}] \neq 0$ and $(q_j) \mapsto (\frac{\partial S}{\partial \alpha_k})$ is the Jacobian map. Suppose further that the system is separable and integrable, so that $S(q, \alpha; x) = \sum_{j=1}^n S_j(q_j, \alpha; x)$ where $S_j(q_j, \alpha; x)$ arises by successive processes of Liouville integration, and let $\tau_\alpha(x) = \exp S(q(x), \alpha; x)$. A family of admissible linear systems $\Sigma_\alpha = (-A^\alpha, B^\alpha, C^\alpha)$ is integrable if $\tau(x, \alpha) = e^{S(q(x), \alpha, x)}$ for an integrable Hamiltonian system. In this context, we are concerned with generic values of α, and not with exceptional values. Gelfand and Dikii [23] showed that a finite gap Schrödinger equation is associated with an integrable Hamiltonian system.

When U is a family of unitary operators on H, the tau function of $(-A, UB, CU)$ is generally different to that of $(-A, B, C)$; thus we can make tau functions and potentials evolve. In section 11, we allow B and C to evolve under a unitary group $U(t)$, so that ϕ, u and $\lfloor \,.\, \rfloor$ itself evolve with respect to time as in the KdV flow. Thus we are able to linearize the the KdV flow on functions of rational character, and produce solutions of the higher order KdV equations.

2 Solving Lyapunov's equation and the Gelfand–Levitan equation

We begin with simple existence result, showing how linear systems in continuous time give rise to Hankel matrices. Subsequent results will introduce stronger hypotheses to ensure the existence of Fredholm determinants.

Proposition 2.1 *Suppose that H is a separable Hilbert space, and that*
(i) $C : H \to \mathbf{C}$ and $B : \mathbf{C} \to H$ are bounded linear operators;
(ii) A is a densely defined linear operator in H;
(iii) A is accretive, so $\Re\langle Af, f\rangle \geq 0$ for all $f \in \mathcal{D}(A)$;
(iv) $\lambda I + A$ is invertible for some $\lambda > 0$.

Then $(e^{-tA})_{t>0}$ is a C_0 contraction semigroup on H, so $\phi(x) = Ce^{-xA}B$ is bounded and continuous on $(0,\infty)$; the cogenerator $V = (A-I)(A+I)^{-1}$ satisfies $\|V\| \leq 1$ as an operator on H, and there is a unitary equivalence between Γ_ϕ on $L^2(0,\infty)$ and the Hankel matrix on $\ell^2(\mathbf{N}\cup\{0\})$ that is given by

$$\Gamma_\phi \leftrightarrow \Big[\sqrt{2}CV^{n+m}(I+A)^{-1}B\Big]_{n,m=0}^{\infty}. \tag{2.1}$$

Proof. By the Lumer–Phillips theorem [18], $-A$ generates a C_0 contraction semigroup. Directly from the definition (iii) of an accretive operator and hypothesis (iv), one proves that $\|V\| \leq 1$.

We introduce the Laguerre polynomials of order zero $L_n^{(0)}(x) = (n!)^{-1}e^x(d/dx)^n x^n e^{-x}$ and then the functions $h_n(x) = \sqrt{2}e^{-x}L_n^{(0)}(2x)$, so that $(h_n)_{n=0}^\infty$ gives a complete orthonormal basis of $L^2(0,\infty)$. By integrating by parts, one can verify that

$$\begin{aligned}\int_0^\infty \phi(x)h_n(x)\,dx &= \frac{1}{\sqrt{2}n!}\int_0^\infty Ce^{-(A-I)x/2}B\frac{d^n}{dx^n}\Big(x^ne^{-x}\Big)\,dx \\ &= \sqrt{2}C(A-I)^n(A+I)^{-n-1}B.\end{aligned} \tag{2.2}$$

Peller [38, p.233] shows that Γ_ϕ is unitarily equivalent to the Hankel matrix under the unitary correspondence $(h_n)_{n=0}^\infty \leftrightarrow (e_j)_{j=0}^\infty$, where (e_j) is the standard orthonormal basis of ℓ^2.

□

We introduce Lyapunov's equation, and the existence of solutions for suitable $(-A,B,C)$. The solution R_x is defined by a formula suggested by Heinz's theorem [7, Theorem 9.2] and has properties analogous to the resolvent operator of a semigroup.

Lemma 2.2 *Let $(-A,B,C)$ be a linear system such that $\|e^{-t_0A}\| < 1$ for some $t_0 > 0$, and that B and C are Hilbert–Schmidt operators on H_0 such that $\|B\|_{HS}\|C\|_{HS} \leq 1$. Then $(-A,B,C)$ is $(2,2)$-admissible, so the following hold.*

(i) The trace class operators

$$R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt \qquad (x>0) \tag{2.3}$$

give the solution to (1.8) for $x > 0$ that satisfies (1.9), and the solution to (1.9) is unique.

(ii) The Laplace transform $\hat{R}(s)$ of R_x is holomorphic on $\{s : \Re s > 0\}$ and satisfies

$$s\hat{R}(s) + A\hat{R}(s) + \hat{R}(s)A = R_0. \qquad (\Re s > 0) \tag{2.4}$$

Proof. (i) Since $BC \in c^1$, the integrand of (2.3) takes values in c^1 and is weakly continuous, hence strongly measurable, by Pettis's theorem. By considering the spectral radius, the authors of [15] show that there exist $\delta > 0$ and $M_\delta > 0$ such that $\|e^{-tA}\| \leq M_\delta e^{-\delta t}$ for all $t \geq 0$; hence (2.3) converges as a Bochner–Lebesgue integral with

$$\begin{aligned}\|R_x\|_{c^1} &\leq \int_x^\infty M_\delta^2\|BC\|_{c^1}e^{-2\delta t}\,dt \\ &\leq \frac{M_\delta^2}{2\delta}\|B\|_{HS}\|C\|_{HS}e^{-2\delta x}.\end{aligned} \tag{2.5}$$

Furthermore, A is a closed operator and satisfies

$$A\int_x^T e^{-tA}BCe^{-tA}\,dt+\int_x^T e^{-tA}BCe^{-tA}\,dtA=\int_x^T -\frac{d}{dt}e^{-tA}BCe^{-tA}\,dt$$
$$=e^{-xA}BCe^{-xA}-e^{-TA}BCe^{-TA}$$
$$\to e^{-xA}BCe^{-xA} \tag{2.6}$$

as $T\to\infty$; so $AR_x+R_xA=e^{-xA}BCe^{-xA}$ for all $x\geq 0$. We deduce that $x\mapsto R_x$ is a differentiable function from $(0,\infty)$ to c^1 and that the modified Lyapunov equation (1.8) holds.

Now suppose that $AR_0+R_0A=BC$ and $AW_0+W_0A=BC$, and consider $V_0=R_0-W_0$. Then for $\xi,\eta\in H$, we have

$$\frac{d}{dt}\langle V_0e^{-tA}\xi,e^{-tA^\dagger}\eta\rangle_H=\langle (V_0A+AV_0)e^{-tA}\xi,e^{-tA^\dagger}\eta\rangle_H=0; \tag{2.7}$$

hence $\langle V_0e^{-tA}\xi,e^{-tA^\dagger}\eta\rangle_H$ is constant, and by the hypothesis on A, we have $\langle V_0e^{-tA}\xi,e^{-tA^\dagger}\eta\rangle_H\to 0$ as $t\to\infty$. Hence $\langle V_0\xi,\eta\rangle_H=0$, and so $V_0=0$, and R_0 is unique. See [31, p. 261] for a similar argument.

(ii) Since e^{-tA} is of exponential decay, $R'_x=-e^{-xA}BCe^{-xA}$ has a convergent Laplace transform $\widehat{(R')}(s)$ for all s such that $\Re s>-2\delta$. By integrating by parts, one obtains

$$\int_0^\infty e^{-sx}R_x\,dx=\frac{1}{s}R_0+\frac{1}{s}\int_0^\infty e^{-sx}R'_x\,dx\qquad(\Re s>-2\delta, s\neq 0) \tag{2.8}$$

so R_x also has a Laplace transform, and from Lyapunov's equation, we obtain ().

□

Definition (Gelfand–Levitan equation) The Gelfand–Levitan integral equation is

$$T(x,y)+\Phi(x+y)+\int_x^\infty T(x,z)\Phi(z+y)\,dz=0\qquad(0<x<y) \tag{2.9}$$

where $T(x,y)$ and $\Phi(x+y)$ are 2×2 matrices with operator entries.

Proposition 2.3 *(i) In the notation of Lemma 2.2, there exists $x_0>0$ such that $T_\mu(x,y)=-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B$ satisfies the integral equation () for $x_0<x<y$ and $|\mu|<1$).*

(ii) The determinant satisfies $\det(I+\mu R_x)=\det(I+\mu\Gamma_{\phi_{(x)}})$ *and*

$$\mu\,\mathrm{trace}\,T_\mu(x,x)=\frac{d}{dx}\log\det(I+\mu R_x). \tag{2.10}$$

Proof. (i) We choose x_0 so large that $e^{\delta x_0}\geq M_\delta/2\delta$, then by (2.4), we have $|\mu|\|R_x\|<1$ for $x>x_0$, so $I+\mu R_x$ is invertible. Substituting into the integral equation, we obtain

$$\begin{aligned}&Ce^{-(x+y)A}B-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B\\ &\quad-\mu Ce^{-xA}(I+\mu R_x)^{-1}\int_x^\infty e^{-zA}BCe^{-zA}\,dze^{-yA}B\\ &\quad=Ce^{-(x+y)A}B-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B-\mu Ce^{-xA}(I+\mu R_x)^{-1}R_xe^{-yA}B\\ &\quad=0.\end{aligned} \tag{2.11}$$

(ii) As in (2.?), the operator $\Theta_x : L^2(0,\infty) \to H$ is Hilbert–Schmidt; likewise $\Xi_x : L^2(0,\infty) \to H$ is Hilbert–Schmidt; so $(-A, B, C)$ is $(2,2)$-admissible. Hence $\Gamma_{\phi_{(x)}} = \Theta_x^\dagger \Xi_x$ and $R_x = \Xi_x \Theta_x^\dagger$ are trace class and

$$\det(I + \mu R_x) = \det(I + \mu \Xi_x \Theta_x^\dagger) = \det(I + \mu \Theta_x^\dagger \Xi_x) = \det(I + \mu \Gamma_{\phi_{(x)}}). \tag{2.12}$$

Correcting a typographic error in [8, p. 324], we rearrange terms and calculate the derivative

$$\begin{aligned}\mu T_\mu(x,x) &= -\mu \mathrm{trace}\Big(Ce^{-xA}(I+\mu R_x)^{-1}e^{-xA}B\Big)\\ &= -\mu \mathrm{trace}(I+\mu R_x)^{-1}e^{-xA}BCe^{-xA}\\ &= \mu \mathrm{trace}\Big((I+\mu R_x)^{-1}\frac{dR_x}{dx}\Big)\\ &= \frac{d}{dx}\mathrm{trace}\log(I+\mu R_x).\end{aligned} \tag{2.13}$$

This identity is proved for $|\mu| < 1$ and extends by analytic continuation to the maximal domain of $T_\mu(x,x)$.

Proposition 2.4 *(i) Let $\mathcal{T}$ be the set of τ functions that arise from linear systems as in Lemma 2.2. Then $\mathcal{T}$ is closed under multiplication.*

(ii) Let $u_\pm(x)$ be the potentials that correspond thereby to $(-A, B, \pm C)$ with scattering functions $\pm\phi(x)$. Then $u(x) = u_+(x) + u_-(x)$ satisfies

$$u(x) = -2\frac{d^2}{dx^2}\log\det(I - \Gamma^2_{\phi_{(x)}}), \tag{2.14}$$

where the Hankel square $\Gamma^2_{\phi_{(x)}}$ is the integral operator on $L^2(0,\infty)$ that has kernel

$$\Psi_{(x)}(y,z) = \int_0^\infty \phi(2x+y+s)\phi(2x+z+s)\,ds. \tag{2.15}$$

Proof. (i) Let $(-A_j, B_j, C_j)$ be a linear system with state space H_j and input and out put spaces H_0 for $j = 1, 2$, let ϕ_j be the corresponding scattering function and let τ be the corresponding tau function. Then the linear system

$$\Big(-\begin{bmatrix} A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix} B_1 & 0\\ 0 & B_2\end{bmatrix}, \begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix}\Big) \tag{2.16}$$

has state space $H_1 \oplus H_2$ and input and output space $H_0 \oplus H_0$, it has scattering function $\begin{bmatrix} \phi_1 & 0\\ 0 & \phi_2\end{bmatrix}$ and hence has tau function

$$\tau(x) = \det\Big(\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} - \begin{bmatrix} \Gamma_{\phi_{1,(x)}} & 0\\ 0 & \Gamma_{\phi_{2,(x)}}\end{bmatrix}\Big) = \det(I - \Gamma_{\phi_{1,(x)}})\det(I - \Gamma_{\phi_{2,(x)}}). \tag{2.17}$$

(ii) The Hankel square appears give u since $\det(I - \Gamma^2_{\phi_{(x)}}) = \det(I - \Gamma_{\phi_{(x)}})\det(I + \Gamma_{\phi_{(x)}})$. We observe that

$$\Psi_{(x)}(y,z) = Ce^{-2xA}e^{-yA}R_0e^{-2xA}e^{-yA}B. \tag{2.18}$$

□

3 Tracy–Widom kernels and Schlesinger's differential equations

In random matrix theory, one often encounters kernels that are the products of Hankel integral operators on $L^2(0,\infty)$; see [46, 47] and (3.1) below for examples. In contrast to the previous section, purposefully introduce Hankel operators that have matrix symbols corresponding to vectorial input and output spaces, so that we can introduce admissible linear systems associated with Hankel products.

Definition (Integrable operators) [17] An integrable kernel has the form

$$K(x,y)=\frac{\sum_{j=1}^{n} f_j(x)g_j(y)}{x-y}, \tag{3.1}$$

where f_j, g_j are continuous and bounded functions on $(0,\infty)$, and we suppose further that $\sum_{j=1}^{n} f_j(x)g_j(x)=0$, so K is nonsingular on $x=y$.

In particular, consider the system of differential equations

$$J\frac{d}{dx}\begin{bmatrix} f\\ g\end{bmatrix}=\Omega(x)\begin{bmatrix} f\\ g\end{bmatrix}, \qquad \Omega(x)=\begin{bmatrix}\gamma & \alpha\\ \alpha & \beta\end{bmatrix}, \qquad J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{3.2}$$

with α,β and γ rational functions. Then, as in Tracy and Widom's theory of matrix models [46,47], we introduce the kernel

$$K_{(z)}(x,y)=\frac{f(x+2z)g(y+2z)-f(y+2z)g(x+2z)}{x-y}, \tag{3.3}$$

and $L_{(z)}$ by $(I-L_{(z)})(I+K_{(z)})=I$.

Theorem 3.1 *Suppose that α,β and γ are proper rational functions with n poles of order less than or equal to p, and all poles are in $\mathbf{C}\setminus[0,\infty)$; suppose that $f,g\in L^2(0,\infty)$ are solutions of (3.3) and that $f(x),g(x)\to 0$ as $x\to\infty$.*

(i) Then there exist Hilbert–Schmidt Hankel operators Γ_Φ and Γ_Ψ with $2np^2\times 2np^2$ matrix symbols Φ and Ψ such that

$$\det(I+\lambda K_{(z)})=\det(I+\lambda\Gamma_{\Phi_{(z)}}\Gamma_{\Psi_{(z)}}). \tag{3.4}$$

(ii) There exists x_0 such that $L_{(z)}$ is a bounded integrable operator for all $z\geq x_0$.

(iii) Suppose further that $e^{2\varepsilon x}f(x)\to 0$ and $e^{2\varepsilon x}g(x)\to 0$ as $x\to\infty$ for some $\varepsilon>0$. Then Φ and Ψ are realised by $(2,2)$ admissible linear systems.

Proof. (i) We can write

$$\Omega(x)=E_0+\sum_{k=1}^{n}\sum_{\ell=1}^{p_k}\frac{E_{k,\ell}}{(x-a_k)^\ell}, \tag{3.5}$$

where the E_0 and $E_{k,\ell}$ for $\ell = 1, \dots, p_k$ and $k = 1, \dots, n$ are symmetric 2×2 matrices and the poles a_j lie in $\mathbf{C} \setminus [0,\infty)$. From the differential equation, we have

$$\Bigl(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Bigr)\frac{f(x)g(y) - f(y)g(x)}{x-y} = \Bigl\langle \frac{\Omega(x)-\Omega(y)}{x-y}\begin{bmatrix} f(x)\\ g(x)\end{bmatrix}, \begin{bmatrix} f(y)\\ g(y)\end{bmatrix}\Bigr\rangle \tag{3.6}$$
$$= -\sum_{k=1}^{n}\sum_{\ell=1}^{p_k}\sum_{\nu=0}^{\ell}\Bigl\langle \frac{E_{k,\ell}}{(x-a_k)^{\ell-\nu}}\begin{bmatrix} f(x)\\ g(x)\end{bmatrix}, \frac{1}{(y-a_k)^{\nu+1}}\begin{bmatrix} f(y)\\ g(y)\end{bmatrix}\Bigr\rangle,$$

where we have used the real inner product. Noting that $E_{k,\ell}$ has rank less than or equal to two, let $N = 2np^2$ and introduce scalar-valued functions $\phi_j(x)$ and $\psi_j(y)$ such that the previous sum equals $-\sum_{j=1}^N \phi_j(x)\psi_j(y)$, and since the poles are off $(0,\infty)$, we can ensure that $\int_0^\infty x(|\phi_j(x)|^2 + |\psi_j(x)|^2)dx$ is finite, so ϕ_j and ψ_j give the symbols of Hilbert–Schmidt Hankel operators on $L^2(0,\infty)$. Then one verifies the identity

$$\frac{f(x)g(y) - f(y)g(x)}{x-y} = \int_0^\infty \sum_{j=1}^N \phi_j(x+s)\psi_j(s+y)\,ds; \tag{3.7}$$

indeed by the preceding calculation, the difference between the two sides of (3.7) is a function of $x+y$, which goes to zero as $x\to\infty$ or $y\to\infty$. Finally, we build the $N\times N$ matrices

$$\Phi(x) = \begin{bmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_N(x)\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{bmatrix}, \Psi(y) = \begin{bmatrix} \psi_1(y) & 0 & \dots & 0\\ \psi_2(y) & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \psi_N(y) & 0 & \dots & 0\end{bmatrix} \tag{3.8}$$

so that Γ_Φ and Γ_Ψ are Hilbert–Schmidt matrix operators, and with $\phi_{j,(z)}(x) = \phi_j(x+2z)$ *etc* we have

$$\det(I + \lambda K_{(z)}) = \det\Bigl(I + \lambda\sum_{j=1}^N \Gamma_{\phi_{j,(z)}}\Gamma_{\psi_{j,(z)}}\Bigr) = \det(I + \lambda\Gamma_{\Phi_{(z)}}\Gamma_{\Psi_{(z)}}). \tag{3.9}$$

(ii) We can define $L_{(z)} = K_{(z)}(I+K_{(z)})^{-1}$ for all z such that $\|K_{(z)}\| < 1$. Now let δ be any derivation on the bounded linear operators on $L^2(0,\infty)$, and observe that

$$\delta L = (I+K)^{-1}(\delta K)(I+K)^{-1}. \tag{3.10}$$

In particular, with $Mh(x) = xh(x)$ for $h \in L^2(0,\infty)$, the derivation $\delta K = MK - KM$ is represented by the finite rank kernel $f(x)g(y) - f(y)g(x)$ which vanishes on the diagonal $x = y$; hence $ML - LM$ is also a finite rank kernel which vanishes on the diagonal. In short, we obtain L from the kernel

$$\frac{F(x)G(y) - F(y)G(x)}{x-y}, \qquad \begin{bmatrix} F\\ G\end{bmatrix} = \begin{bmatrix} (I+K)^{-1}f\\ (I+K)^{-1}g\end{bmatrix}. \tag{3.11}$$

Moreover, $\delta K = [d/dx, K]$ is the finite rank integral operator that is represented by the kernel (3.5), so δL is also finite rank.

(iii) Given that f and g are of exponential decay, the integral $\int_0^\infty xe^{2\varepsilon x}|\phi_j(x)|^2dx$ converges, and hence the Hankel operator Γ_j with symbol $e^{\varepsilon x}\phi_j(x)$ is bounded. We decompose $\phi_j = \Re\phi_j + i\Im\phi_j$ so that we can work with the self-adjoint Hankel operators $\Gamma_{\Re\phi_j}$ and $\Gamma_{\Im\phi_j}$; so by theorem 2.1 of [35, p.257], there exist linear systems $(-A_j', B_j', C_j')$ and $(-A_j'', B_j'', C_j'')$ with input and output spaces $\mathbf{C}$, and state space H, and all operators bounded, such that $e^{\varepsilon x}\Re\phi_j(x) = C_j'e^{-xA_j'}B_j'$ and $e^{\varepsilon x}\Im\phi_j(x) = C_j''e^{-xA_j''}B_j''$; then we let

$$(-A_j, B_j, C_j) = \Bigl(-\begin{bmatrix} A_j' & 0 \\ 0 & A_j'' \end{bmatrix}, \begin{bmatrix} B_j' \\ B_j'' \end{bmatrix}, [\, C_j' \quad iC_j''\,]\Bigr), \tag{3.12}$$

so that $e^{\varepsilon x}\phi_j(x) = C_je^{-xA_j}B_j$. Hence we can introduce

$$(-A, B, C) = \Bigl(-\begin{bmatrix} \varepsilon I + A_1 & \dots & 0 \\ 0 & \ddots & \vdots \\ 0 & \dots & \varepsilon I + A_N \end{bmatrix}, \begin{bmatrix} B_1 & \dots & 0 \\ 0 & \ddots & \vdots \\ 0 & \dots & B_N \end{bmatrix}, \begin{bmatrix} C_1 & \dots & C_N \\ 0 & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}\Bigr) \tag{3.13}$$

where $A : H^{2N} \to H^{2N}$, $B : \mathbf{C}^N \to H^{2N}$ and $C : H^{2N} \to \mathbf{C}^N$ are bounded linear operators. Since $\Re\langle A\xi, \xi\rangle_{H^N} \geq \varepsilon\langle \xi, \xi\rangle_{H^N}$ for all $\xi \in H^N$, Lemma 2.2 shows that $(-A, B, C)$ is a $(2, 2)$ admissible linear system. Evidently $(-A, B, C)$ realises Φ, and we can likewise realise Ψ by a $(2, 2)$ admissible linear system.

□

By taking $\alpha = 0$, γ to be a negative proper rational function and $1/\beta$ to be a positive polynomial on $(0, \infty)$, one can produce solutions of (3.2) that satisfy the hypotheses of Theorem 3.1(ii).

Now we show how to calculate the determinant in terms of the Gelfand–Levitan equation. Changing to a more symmetrical notation, we suppose that $(-A_1, B_1, C_1)$ and $(-A_2, B_2, C_2)$ are $(2, 2)$ admissible systems with state spaces H_1 and H_2 and output space $\mathbf{C}^N$ that realise ϕ_1 and ϕ_2. First, let $R_{jk} : H_k \to H_j$ for $j, k = 1, 2$ be the operators

$$R_{jk}(x) = \int_x^\infty e^{-tA_j}B_jC_ke^{-tA_k}\,dt, \tag{3.14}$$

For the first result, we introduce that state space $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ and the output space $H_0 = \mathbf{C}^{2\times N}$ and $A : H \to H$, $B : H_0 \to H$ and $C : H \to H_0$ by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_2 \\ C_1 & 0 \end{bmatrix}; \tag{3.15}$$

so that

$$\Phi(x) = \begin{bmatrix} 0 & \phi_2(x) \\ \phi_1(x) & 0 \end{bmatrix}. \tag{3.16}$$

Proposition 3.2 *(i) For all $\mu \in \mathbf{C}$ such that $|\mu|$ is sufficiently small $I - \mu^2 R_{21}(x)R_{12}(x)$ has an inverse G_x and*

$$T(x,y) = \begin{bmatrix} \mu C_2 e^{-xA_1} G_x R_{21}(x) e^{-yA_1} B_1 & -C_2 e^{-xA_2} G_x e^{-yA_2} B_2 \\ -C_1 e^{-xA_1}(I + \mu^2 R_{12}(x) G_x R_{21}(x)) e^{-yA_1} B_1 & \mu C_1 e^{-xA_1} R_{12}(x) G_x e^{-yA_2} B_2 \end{bmatrix} \tag{3.17}$$

satisfies (2.9) for all $x > x_0$ from some $x_0 > 0$.

(ii) The determinants satisfy

$$\det(I - \mu^2 R_{12}(x)R_{21}(x)) = \det(I - \mu^2 \Gamma_{\phi_{2,(x)}} \Gamma_{\phi_{1,(x)}}). \tag{3.18}$$

and

$$\frac{d}{dx} \log\det(I - \mu^2 \Gamma_{\phi_{2,(x)}} \Gamma_{\phi_{1,(x)}}) = \mu \mathrm{trace}\, T(x,x). \tag{3.19}$$

(iii) In particular, with $A_2 = A_1^\dagger, B_2 = \varepsilon C_1^\dagger$ and $C_2 = B_1^\dagger$ and $\varepsilon = \pm 1$, the identities hold with $\phi_2(x) = \varepsilon \phi_1(x)^\dagger$ so Γ_Φ is self-adjoint with $\varepsilon = 1$ and skew with $\varepsilon = -1$.

Proof. (i) It is easy to check that $\Phi(x) = Ce^{-xA}B$. Likewise, we can compute

$$R_x = \int_x^\infty e^{-tA} BC e^{-tA}\, dt = \begin{bmatrix} 0 & R_{12}(x) \\ R_{21}(x) & 0 \end{bmatrix}, \tag{3.20}$$

which is a trace class operator on H since both $(-A_1, B_1, C_1)$ and $(-A_2, B_2, C_2)$ are $(2,2)$-admissible. For x such that $|\mu|^2 \|R_{12}(x)\| \|R_{21}(x)\| < 1$, we can form the operator $G_x = (I - \mu^2 R_{21} R_{12})^{-1}$ and hence compute

$$F_x = \begin{bmatrix} I & \mu R_{12}(x) \\ \mu R_{21}(x) & I \end{bmatrix}^{-1} = \begin{bmatrix} I + \mu^2 R_{12}(x) G_x R_{21}(x) & -\mu R_{12}(x) G_x \\ -\mu G_x R_{21}(x) & G_x \end{bmatrix}. \tag{3.21}$$

Then we compute $T(x,y) = -Ce^{-xA} F_x e^{-yA} B$ and obtain the matrix from (). One then checks, as in Lemma 2.2, that T satisfies the integral equation (2.9).

(ii) We introduce the observability operators $\Theta_x : L^2((0,\infty); \mathbf{C}^{2\times N}) \to H$ by

$$\Theta_x \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & \Theta_2 \\ \Theta_1 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \int_x^\infty e^{-tA_2^\dagger} C_2 g(t)\, dt \\ \int_x^\infty e^{-tA_1^\dagger} C_1^\dagger f(t)\, dt \end{bmatrix} \tag{3.22}$$

and the controllability operators $\Xi_x : L^2((0,\infty); \mathbf{C}^{2\times N}) \to H$ by

$$\Xi_x \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \Xi_2 & 0 \\ 0 & \Xi_1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \int_x^\infty e^{-tA_2} B_2 f(t)\, dt \\ \int_x^\infty e^{-tA_1} B_1 g(t)\, dt \end{bmatrix} \tag{3.23}$$

such that

$$\Xi_x \Theta_x^\dagger = \begin{bmatrix} 0 & \Xi_2 \Theta_1^\dagger \\ \Xi_1 \Theta_2^\dagger & 0 \end{bmatrix} = \begin{bmatrix} 0 & R_{21} \\ R_{12} & 0 \end{bmatrix} \tag{3.24}$$

as operators on H, and

$$\Theta_x^\dagger \Xi_x = \begin{bmatrix} 0 & \Theta_1^\dagger \Xi_1 \\ \Theta_2^\dagger \Xi_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Gamma_{\phi_{1,(x)}} \\ \Gamma_{\phi_{2,(x)}} & 0 \end{bmatrix} \tag{3.25}$$

as operators on $L^2((0,\infty);\mathbf{C}^{2\times N})$. Now from the determinant identity

$$\det(I+\mu\Theta_x^\dagger\Xi_x)=\det(I+\mu\Xi_x\Theta_x^\dagger) \tag{3.26}$$

we deduce

$$\det\bigl(I-\mu^2\Gamma_{\phi_{2,(x)}}\Gamma_{\phi_{2,(x)}}\bigr)=\det\bigl(I-\mu^2R_{12}(x)R_{21}(x)\bigr). \tag{3.27}$$

The function R_x is differentiable with respect to x, so by Lemma 2.1, we can compute

$$\begin{aligned}\frac{d}{dx}\log\det(I-\mu^2\Gamma_{\phi_{2,(x)}}\Gamma_{\phi_{1,(x)}})&=\frac{d}{dx}\log\det(I+\mu R_x)\\&=\mu\mathrm{trace}\,T(x,x).\end{aligned} \tag{3.28}$$

(iii) We have $\phi_1(x)=C_1e^{-xA_1}B_1$ and $\phi_2(x)=C_2e^{-xA_2}B_2=\varepsilon B_1^\dagger e^{-xA_1^\dagger}C_1^\dagger$.

$\square$

Remarks 3.3 (i) Whereas Theorem 3.1 does not give an explicit form for the admissible linear system $(-A,B,C)$, we can produce one explicitly in several important cases; see () and [9,10].

(ii) In section 5, we introduce a differential ring $\mathbf{S}$, which is directly related to the specific choice of admissible linear system $(-A,B,C)$, so that we can multiply and differentiate potentials. In subsequent sections, we will introduce determinants from linear systems via R_x, thus bypassing the Hankel operators. This enables us to deal with linear systems that are not admissible, such as periodic systems. The first step is to widen the discussion from rational functions on $\mathbf{C}$ to meromorphic functions on algebraic curves, as we consider in section 3. Krichever and Novikov introduced the notion of a spectral curve for a family of commuting differential operators [30].

Definition Let $\mathbf{P}$ be a Riemann surface and let $u_j(t,\mathbf{p})$ be differentiable functions of $t=(t_1,t_2,\dots,t_n)$ with values in M_N, which are meromorphic functions of $\mathbf{p}$, and let $L_j=\frac{\partial}{\partial t_j}-u_j(t,\mathbf{p})$.

(i) Say that L_j form a commutative ensemble if $[L_j,L_k]=0$ for all j,k.

(ii) Given a cummutative ensemble, suppose that there exists a function $W(t,\mathbf{p})$ with values in M_N which is differentiable with respect to t and algebraic in $\mathbf{p}$ on $\mathbf{P}$. Then the ensemble is said to be algebraic if $[L_j,W]=0$ for all j. In this case the spectral curve is

$$\mathcal{E}=\Bigl\{(\mu,\lambda):\det(\mu I_N-W(t,\mathbf{p}))=0;\lambda=\lambda(\mathbf{p})\Bigr\} \tag{3.29}$$

which is actually independent of t.

Suppose that the poles of (3.2) are simple and that the residue matrices are differentiable functions of deformation parameters $t=(t_1,\dots,t_n)$, so that

$$-J\Omega(\lambda,t)=\sum_{j=1}^n\frac{U_j(t)}{\lambda-a_j} \tag{3.30}$$

where $\mathrm{trace}(U_j) = 0$, and consider a family of meromorphic solutions $Y = Y(\lambda; t_1, \dots, t_n)$ of the differential equation $JdY/d\lambda = \Omega(\lambda)Y$ for λ complex that also satisfy the conditions of Theorem 3.1, and as in (3.2) introduce the kernels

$$K^{(t)}_{(z)}(x,y) = \frac{\langle JY(x+2z,t), Y(y+2z,t)\rangle}{x-y}, \qquad Y = \begin{bmatrix} f \\ g \end{bmatrix} \tag{3.31}$$

Proposition 3.4 *Let $\tau(z,t) = \det(I + K^{(t)}_{(z)})$, suppose that $\|K_{(z)}\| < 1$ for all $\Re z > x_0$, and suppose that the differential equations*

$$\frac{\partial Y}{\partial t_j} = \frac{-U_j}{\lambda - a_j} Y \qquad (j = 1, \dots, n) \tag{3.32}$$

are mutually compatible.

(i) Then $\frac{\partial}{\partial z} \log \tau(z,t)$ is given by Proposition 3.2(ii) and

$$\frac{1}{2}\frac{\partial}{\partial z} \log \tau(z,t) = \sum_{j=1}^{n} \frac{\partial}{\partial t_j} \log \tau(z,t) \qquad (\Re z > x_0). \tag{3.33}$$

(ii) Let j be an index such that $\Re a_j$ is largest, suppose that $\Re a_j > 2x_0$ and that $\langle JU_jY(a_j,t), Y(a_j,t)\rangle \neq 0$. Then $\frac{\partial}{\partial z} \log \tau(z,t)$ has a pole at $z = a_j/2$.

(iii) There exists a hyperelliptic curve $\mathcal{E}$ and a commutative Lie algebra $\mathbf{T}$ such that $\tau(\lambda, t)$ extends to $\mathcal{E} \times \mathbf{T}$.

Proof. (i) By a calculation as in [9, Theorem 3.3], we have

$$\frac{\partial}{\partial t_j} \frac{\langle JY(x+2z,t), Y(y+2z,t)\rangle}{x-y} = -\Big\langle JU_j \frac{Y(x+2z,t)}{x+2z-a_j}, \frac{Y(y+2z,t)}{y+2z-a_j} \Big\rangle \tag{3.34}$$

which decomposes the kernel into a finite sum of rank one integral operators, and likewise

$$\frac{1}{2}\frac{\partial}{\partial z} \frac{\langle JY(x+2z,t), Y(y+2z,t)\rangle}{x-y} = -\sum_{j=1}^{n} \Big\langle JU_j \frac{Y(x+2z,t)}{x+2z-a_j}, \frac{Y(y+2z,t)}{y+2z-a_j} \Big\rangle, \tag{3.35}$$

which gives the identity of finite rank operators

$$\frac{1}{2}\frac{\partial}{\partial z} K^{(t)}_{(z)}(x,y) = \sum_{j=1}^{n} \frac{\partial}{\partial t_j} K^{(t)}_{(z)}(x,y). \tag{3.36}$$

The operator $(d/dz)K_{(z)}$ is of finite rank, and hence is trace class if and only if the constituent functions belong to $(L^2(0,\infty)dx)$. Now as in Theorem 3.1(ii), we choose x_0 so large that $I + K^{(t)}_{(z)}$ is an invertible operator for all $z > x_0$ and then compute $\frac{\partial}{\partial z} \log \tau(z,t) = \mathrm{trace}((I + K^{(t)}_{(z)})^{-1} \frac{\partial}{\partial z} K^{(t)}_{(z)})$; so we deduce the stated result. The identity () asserts that infinitesimally translating z is equivalent to the added effect of infinitesimally moving all the t_j.

In Theorem 3.1, we showed that $\tau(z,t)$ is given by the Fredholm determinant of a product of Hankel operators, and in Proposition 3.2, we expressed $\frac{\partial}{\partial}\log\det(I+\Gamma_{\phi_{1,(z)}}\Gamma_{\phi_{2,(z)}})$ in terms of the solution of a Gelfand–Levitan equation; thus $\frac{\partial}{\partial z}\log\tau(z,t)$ is given in terms of the solution of a Gelfand–Levitan equation.

Note that when a_j-2z lies on $(0,\infty)$ and $Y(a_j)\neq 0$, the function $Y(x+2z)/(x+2z-a_j)$ does not belong to $L^2((0,\infty);dx)$, so there is a possible pole for $\tau'(z,t)/\tau(z,t)$.

(ii) We take $2z-a_j\in\mathbf{C}\setminus(-\infty,0]$ and compute

$$\begin{aligned}\frac{1}{2}\mathrm{trace}\frac{d}{dz}K_{(z)} &= -\sum_{k=1}^{n}\int_0^\infty\frac{\langle JU_jY(x+2z,t),Y(x+2z,t)\rangle}{(x+2z-a_j)^2}dx\\ &= -\frac{\langle JU_jY(a_j,t),Y(a_j,t)\rangle}{2z-a_j}+O(1)\qquad (z\to a_j/2),\end{aligned}\tag{3.37}$$

so $(d/dz)K_{(z)}$ has a simple pole at $a_j/2$. By (), $(d/dz)\log\det(I+K_z)$ has a pole at $a_j/2$.

(iii) Schlesinger observed that the system (3.29) is consistent if and only if the family of solutions satisfies an isomonodromy condition with respect to infinitesimal deformation, or equivalently that a certain family of differential operators commutes.

Let $\mathcal{D}^1$ be the space of first order differential operators in time parameters $t=(t_1,\dots,t_n)$ with coefficients in $M_2(\mathbf{C}(\lambda,t))$, and let

$$L_0=\frac{\partial}{\partial\lambda},\quad L_j=\frac{\partial}{\partial t_j}+\frac{U_j(t)}{\lambda-a_j}\qquad (j=1,\dots,n),\tag{3.38}$$

Garnier observed that

$$\Big[L_j,\sum_{k=1}^{n}\frac{U_k}{\lambda-a_k}\Big]=0\qquad (j=1,\dots,n)\tag{3.39}$$

hence $\{L_j;j=1,\dots,n\}$ gives an algebraic ensemble for the 2×2 matrix

$$W(\lambda,t)=J\Omega(\lambda,t)\prod_{j=1}^{n}(\lambda-a_j)\tag{3.40}$$

which is a polynomial in λ. Consequently,

$$\mathbf{T}=\Big\{\sum_{j=1}^{n}s_jL_j:s_j\in\mathbf{C}\Big\}\tag{3.41}$$

defines a commutative complex Lie subalgebra of $\mathcal{D}^1$. Any solution Y of () and () belongs to

$$H=\{Y=Y(\lambda,t)\in\mathbf{C}^2:[L_j,L_0]Y=0;L_jY=0;j=1,\dots,n\},\tag{3.42}$$

and $\mathbf{T}$ acts on H. The operation of translation on H is described by a flow on a curve. Since $\mathrm{trace}(W)=0$, we observe that

$$\det(\eta I_2+W(\lambda,t))=\eta^2+\det W(\lambda,t)\tag{3.43}$$

which is independent of t by (3.34). Hence $\mathcal{E} = \{(\lambda, \eta) : \eta^2 + \det W(\lambda, t) = 0\}$ defines a hyperelliptic curve independent of t. Thus we can extend the tau function to

$$\tau(\lambda, t) = \det(I + K^{(t)}_{(\lambda)}) \qquad (t \in \mathbf{T}, \mathbf{p} = (\lambda, \eta) \in \mathcal{E}). \tag{3.44}$$

$\square$

Remark. To recover the usual form of Schlesinger's equations [20, 22, 26, 41] one substitutes $t_j = a_j$ after differentiating, and considers the residues at each of the poles.By Schlesinger's results, as interpreted in [26], there exists a multi-valued and locally analytic complex function $\tau_S(a_1, \ldots, a_n)$ on

$$\{(a_1, \ldots, a_n) : a_j \neq a_k; j, k = 1, \ldots, n\} \tag{3.45}$$

such that

$$d \log \tau_S = \sum_{j,k:j<k} \mathrm{trace}(U_j U_k) d \log(a_j - a_k) \tag{3.46}$$

as an identity of differential one forms, so that

$$\sum_{j=1}^{n} \frac{\partial}{\partial a_j} \log \tau_S(a_1, \ldots, a_n) = \sum_{j,k:j\neq k} \frac{\mathrm{trace}(U_j U_k)}{a_j - a_k} = 0. \tag{3.47}$$

This contrasts with (), and indicates that translation has a different role for the two versions of the tau function.

Remark 3.5 There is another case in which Schlesinger's equations give a hyperelliptic spectral curve. Suppose that $W(\lambda; t)$ is a $m \times m$ matrix, a differentiable function in t_1, t_2, t_3 and a quadratic polynomial in λ such that

$$\frac{\partial W}{\partial t_j} = \Big[\frac{U_j(t)}{\lambda - \alpha_j}, W\Big] \qquad (j = 1, 2, 3), \tag{3.48}$$

and that

$$\det(\eta I_m + W(\lambda, t)) = p_{m-2}(\eta)\lambda^2 + p_{m-1}(\eta)\lambda + p_m(\eta) \tag{3.49}$$

where $p_m(\eta), p_{m-1}(\eta)$ and $p_{m-2}(\eta)$ have degrees $m, m-1$ and $m-2$ respectively. Garnier [22] reduced the system () to

$$\begin{aligned} \xi_j'' &= \xi_j\Big(\alpha_j + \sum_{k=2}^{m} \xi_k \eta_k\Big) \qquad (j = 2, \ldots, m) \\ \eta_j'' &= \eta_j\Big(\alpha_j + \sum_{k=2}^{m} \xi_k \eta_k\Big). \end{aligned} \tag{3.50}$$

with $' = d/dt_1$, which he integrated directly in terms of hyperelliptic functions of $m-1$ arguments, $m-2$ of which have received constant values. On the invariant hyperplanes $\eta_j = b_j \xi_j$ with b_k constant, this has the form of coupled anharmonic oscillators constrained to lie on the sphere $\sum_{j=2}^m \xi_j^2 = 1$ under the influence of a quadratic potential. Neumann integrated this system by changing to elliptic spheroidal coordinates.

4. Scattering functions

Thus tau functions have a multiplication rule which is analogous to the addition rule for positive divisors divisors on an algebraic curve. The multiplication $B \mapsto (\lambda I - A)(\lambda I + A)^{-1}B$ is associated with adding a the divisor associated with a pole on the spectral curve. There is a consequent formula for addition of divisors, which the authors of [19] credit to Darboux, as in Proposition 2.5.

Definition (Baker–Akhiezer function) Given an admissible linear system $\Sigma_\infty = (-A, B, C)$ with tau function $\tau_\infty(x) = \det(I + \Gamma_{\phi(x)})$ as in Proposition 2.2, we introduce

$$\Sigma_\lambda = \Bigl(-A, (\lambda I + A)(\lambda I - A)^{-1}B, C\Bigr) \qquad (\Re\lambda > 0) \tag{4.1}$$

with tau function $\tau_\lambda(x)$, and the Baker–Akhiezer function

$$\psi_{BA}(x;\lambda) = \exp\bigl(\lambda x\bigr)\frac{\tau_\lambda(x)}{\tau_\infty(x)}. \tag{4.2}$$

Let $C_0^\infty(\mathbf{R};\mathbf{R})$ denote the space of infinitely differentiable functions $f : \mathbf{R} \to \mathbf{R}$ such that $|x|^j f^{(k)}(x) \to 0$ as $x \to \pm\infty$, and suppose that $u \in C_0^\infty(\mathbf{R};\mathbf{R})$. Then with $\lambda = k^2$, let $s(k)$ be the scattering matrix, which depends analytically upon k, and let $s_{21}(k)$ be the bottom left entry, which satisfies $s_{21} \in C_0^\infty(\mathbf{R};\mathbf{R})$ and $\overline{s_{21}(k)} = s_{21}(-k)$, so that

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} s_{21}(k)\,dk \tag{4.3}$$

gives a real function. Dyson inverted the scattering map $q \mapsto s_{21}$ by the formula (1.10).

Subsequently [27], Kamvissis recovered the determinant formula (1.10) as a limiting case of the Its–Matveev formula for periodic finite-gap potentials as the period tends to infinity. In this paper, we show that finite gap and localized potentials can be treated similarly via linear systems.

Example 4.1 As in [8, Theorem 4.2] and [19, p. 486] we can introduce a linear system and Hankel determinant to realise scattering functions. The following formulas are similar, but slightly different from those in [19]. Let $H = L^2(\mathbf{R};\mathbf{C})$ and let $b_1, b_2 : \mathbf{R} \to \mathbf{C}$ be smooth functions of compact support such that $b_1(-k) = \overline{b_1(k)}$, $b_2(-k) = \overline{b_2(k)}$ and $|b_1(k)| = |b_2(k)|$ for all $k \in \mathbf{R}$, and let

$$\begin{aligned} B &: \mathbf{C} \to H : \alpha \mapsto b_1(k)\alpha; \\ e^{-xA} &: H \to H : f(k) \mapsto e^{ixk} f(k); \\ C &: H \to \mathbf{C} : f(k) \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k) b_2(k)\,dk. \end{aligned} \tag{4.4}$$

The potential u is in $C_0^\infty(\mathbf{R};\mathbf{R})$, and we assume that there are no bound states, so we are in the scattering case of Schrödinger's equation. Then $(-A,B,C)$ has scattering function $\phi(x)=\int_{-\infty}^\infty e^{ixk}b(k)dk/2\pi$, while $\Sigma_{i\kappa}=(-A,(i\kappa I-A)(i\kappa I+A)^{-1}B,C)$ has scattering function $\phi_{i\kappa}(x)=\int_{-\infty}^\infty e^{ixk}b(k)(\kappa+k)(\kappa-k)^{-1}dk/2\pi$, which is unambiguously defined for real κ since the Hilbert transform is bounded on H; the corresponding potential is $u_{i\kappa}(x)=-2\frac{d^2}{dx^2}\log\tau_{i\kappa}(x)$.

The Bloch spectrum is a double cover of $[0,\infty)$ given by $\pm k\mapsto k^2$, where $\pm k$ is associated with the unique $f_{i\kappa}(x,\pm k)$ such that $-f''_{i\kappa}(x,\pm k)+u_{i\kappa}(x)f_{i\kappa}(x,\pm k)=k^2f_{i\kappa}(x,\pm k)$ and $f_{i\kappa}(x,\pm k)-e^{\pm ikx}\to 0$ as $x\to\pm\infty$. The point κ is associated with the function $(k+\kappa)/(k-\kappa)$ which has a simple pole at κ.

Proposition 4.2 *(i) Suppose that the operator $G:L^2(0,\infty)\to L^2(0,\infty)$ defined by $Gf(x)=f(x)+\int_x^\infty T(x,y)f(y)dy$ is invertible. Then there is a gauge transformation*

$$G^{-1}(-d^2/dx^2+u)G=-d^2/dx^2. \tag{4.5}$$

(ii) The multiplication rule

$$s_{21}(k)\mapsto\frac{\kappa+k}{\kappa-k}s_{21}(k) \tag{4.6}$$

is equivalent to the addition rule $u(x)\mapsto u_{i\kappa}(x)$ for potentials as in

$$-2\frac{d^2}{dx^2}\log\psi_{BA}(x,i\kappa)=u_\infty(x)-u_{i\kappa}(x). \tag{4.7}$$

(iii) The Baker–Akhiezer function is given as a series of Fredholm determinants and satisfies $\psi_{BA}(x,ik)-e^{ikx}\to 0$ as $x\to\infty$ and

$$-\psi''_{BA}(x,ik)+u(x)\psi_{BA}(x,ik)=k^2\psi_{BA}(x,ik)\qquad(x\in\mathbf{R}). \tag{4.8}$$

Proof. (i) The operators $-d^2/dx^2$ and $-d^2/dx^2+u$ are essentially self-adjoint on $C_c^\infty(0,\infty)$, so the identity $f_\infty(x,k)=G(e^{ixk})$ for the eigenfunctions shows that G gives a similarity between operators on $L^2(0,\infty)$.

(ii) We can express the difference of the potentials for the systems as

$$u_\infty(x)-u_{i\kappa}(x)=-2\frac{d^2}{dx^2}\log\frac{\tau_\infty(x)}{\tau_{i\kappa}(x)}, \tag{4.9}$$

and then simplify the expressions.

(iii) With $T_{i\kappa}$ and the corresponding potential $u_{i\kappa}(x)=-2\frac{d^2}{dx^2}\log\tau_{i\kappa}(x)$ defined for the linear system $\Sigma_{i\kappa}$, we introduce

$$f_{i\kappa}(x,k)=e^{ikx}+\int_x^\infty T_{i\kappa}(x,y)e^{iky}\,dy. \tag{4.10}$$

By repeated integration by parts, one verifies that $-f''_{i\kappa}(x,\pm k)+u_{i\kappa}(x)f_{i\kappa}(x,\pm k)=k^2 f_{i\kappa}(x,\pm k)$ and $f_{i\kappa}(x,\pm k)-e^{\pm ikx}\to 0$ as $x\to\pm\infty$. In particular, with $i\kappa=\infty$ we can express

$$\begin{aligned} f_\infty(x,k) &= e^{ikx}-Ce^{-xA}(I+R_x)^{-1}\int_x^\infty e^{-yA}Be^{iky}dy \\ &= e^{ikx}\big((1+Ce^{-xA}(I+R_x)^{-1}(ikI-A)^{-1}e^{-xA}B\big) \\ &= e^{ikx}\det\big(I+(ikI-A)^{-1}e^{-xA}BCe^{-xA}(I+R_x)^{-1}\big) \end{aligned} \tag{4.11}$$

where we have used a simple identity for rank-one operators, hence

$$f_\infty(x,k)=e^{ikx}\frac{\det(I+R_x+(ikI-A)^{-1}e^{-xA}BCe^{-xA})}{\det(I+R_x)}, \tag{4.12}$$

and we can finish by using Lyapunov's equation

$$f_\infty(x,k)=e^{ikx}\frac{\det(I+R_x-(ikI-A)^{-1}R'_x)}{\det(I+R_x)}, \tag{4.13}$$

where the determinant on the numerator is

$$\det\big(I+R_x+(ikI-A)^{-1}(AR_x+R_xA)\big)=\det\big(I+R_x(ikI+A)(ikI-A)^{-1}\big). \tag{4.14}$$

As in Fredholm theory, we let

$$D_x=R_x(I+R_x)^{-1}\tau_\infty(x), \tag{4.15}$$

and temporarily write $\tilde{D}_x=D_x(ikI+A)(ikI-A)^{-1}$. We can proceed to compute the kernel of D_x as an integral operator on $L^2(0,\infty)$. The operator R_x on $L^2(0,\infty)$ is represented by the kernel

$$R_x(s,t)=\frac{e^{-ixs}b_1(s)b_2(t)e^{-ixt}}{i(s+t)}, \tag{4.16}$$

so we have a Cauchy determinant

$$\begin{aligned} R_x\begin{pmatrix} s_1 & \dots & s_n\\ t_1 & \dots & t_n\end{pmatrix} &= \det\Big[\frac{e^{-ixs_j}b_1(s_j)b_2(t_\ell)e^{-ixt_\ell}}{i(s_j+t_\ell)}\Big]_{j,\ell=1}^n \\ &= e^{-\sum_{j=1}^n ixs_j}e^{-\sum_{\ell=1}^n ixt_\ell}\prod_{j=1}^n b_1(s_j)\prod_{\ell=1}^n b_2(t_\ell)\frac{\prod_{1\le j<\ell\le n}(s_j-s_\ell)\prod_{1\le j<\ell\le n}(t_j-t_\ell)}{i^n\prod_{j,\ell=1}^n(s_j+t_\ell)} \end{aligned} \tag{4.17}$$

In the usual notation of Fredholm theory, we express the kernel of $D_x(\lambda)$ as the series

$$D_x(s,t;\lambda)=\sum_{n=0}^\infty D_{n,x}(s,t)\lambda^n \tag{4.18}$$

where $D_{0,x}(s,t)=R_x(s,t)$ and

$$D_{n,x}(s,t)=\frac{(-1)^n}{n!}\int_0^\infty\dots\int_0^\infty R_x\begin{pmatrix} s & s_1 & \dots & s_n\\ t & s_1 & \dots & s_n\end{pmatrix}ds_1\dots ds_n. \tag{4.19}$$

To obtain the kernel for $\tilde{D}_x$ we simply multiply by $(k+t)/(k-t)$. Then

$$f_\infty(x,k) = e^{ikx}\Big(1 + \frac{\phi(x)+\phi_{ik}(2x)}{2ik\tau_\infty(x)} + \frac{Ce^{-xA}(D_x+\tilde{D}_x)e^{-xA}B}{2ik\tau_\infty(x)}\Big). \tag{4.20}$$

where D_x is given by the determinant series ().

□

Lemma 4.4 *Any Gaussian function on $\mathbf{R}^N$ can be realised as the scattering function of a linear system.*

Proof. Given any $N<\infty$ and a positive definite real symmetric matrix Q with inverse Q^{-1}, we introduce a linear system with state space $L^2(\mathbf{R}^N)$, with state variables $(x,t) = (x,t_1,\dots,t_{N-1})$ and $\xi = (\xi_0,\dots,\xi_{N-1})$, by

$$B : \mathbf{C}\to H: \quad \alpha \mapsto \alpha(2^N\pi^N\det Q)^{-1/4}\exp\big(-Q^{-1}(\xi,\xi)/4\big)$$

$$U(t)e^{-xA}U(t) : H\to H: \quad f(\xi)\mapsto \exp\Big(-ix\xi_0 - i\sum_{j=1}^{N-1}\xi_j t_j\Big)f(\xi) \tag{4.21}$$

$$C : H\to\mathbf{C}: \quad f\mapsto \int_{\mathbf{R}^N} f(\xi)\exp\big(-Q^{-1}(\xi,\xi)/4\big)\frac{d\xi_0\dots d\xi_{N-1}}{(2^N\pi^N\det Q)^{1/4}}. \tag{4.22}$$

For consistency with the theory of this section, we define this the tau function of the Gaussian linear system to be

$$\tau_0(x,t) = CU(t)e^{-xA}U(t)B = \exp(-Q((x,t),(x,t))/2), \tag{4.23}$$

and $u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\phi(x,t) = q_0$, where q_0 is the coefficient of x^2 in $Q((x,t),(x,t))$.

□

We recall the definition of the tau function in terms of Riemann's theta function for an Abelian variety.

Definition (Theta functions) Let Λ be a lattice in $\mathbf{C}^g$ such that $\mathbf{C}^g/\Lambda$ is a complex torus, which is compact for the quotient topology. A quotient Θ of nonzero entire functions on $\mathbf{C}^g$ is said to be a theta function if there exists a family of linear maps $\mathbf{C}^g\to\mathbf{C} : z\mapsto L_\gamma(z)$ for $\gamma\in\Lambda$ and a map $J:\Lambda\to\mathbf{C}$ such that $\Theta(z+\gamma) = e^{2\pi i(L_\gamma(z)+J(\gamma))}\Theta(z)$ for all $\gamma\in\Lambda$ and $z\in\mathbf{C}^g$. If Q is a quadratic form on $\mathbf{C}^g$, $\psi:\mathbf{C}^g\to\mathbf{C}$ is a linear functional and $c\in\mathbf{C}^\sharp$, then $e^{2\pi i(Q(z,z)+\psi(z)+c)}$ gives a trivial theta function. Evidently the product of theta functions is again a theta function.

Definition (Riemann's theta function) Suppose that Ω_0 and Ω_1 are real symmetric $g\times g$ matrices with Ω_1 positive definite, and let $\Omega = \Omega_0 + i\Omega_1$; then let $\Lambda = \mathbf{Z}^g + \Omega\mathbf{Z}^g$ be a lattice in $\mathbf{C}^g$. Then

$$\theta(x\mid\Omega) = \sum_{m\in\mathbf{Z}^g} e^{2\pi i m^t + \pi i m^t\Omega m} \tag{4.24}$$

is Riemann's theta function for the Abelian variety $\mathbf{X} = \mathbf{C}^g/\Lambda$. Let $\omega \in \mathbf{C}$ have $\Im\omega > 0$; then Jacobi's elliptic theta function for the torus $\mathbf{C}/(\mathbf{Z} + \omega\mathbf{Z})$ is

$$\theta_1(x \mid \omega) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{(2n-1)\pi i x + (n+1/2)^2 \pi i \omega}. \tag{4.25}$$

By Lemma 4.4, one can realise these functions as scattering functions of linear sytems. More importantly, in section 10 we realise θ_1 as the tau function of a linear system.

Zhakharov and Shabat [50] considered the Kadomtsev–Petviashvili equation

$$\frac{\partial}{\partial x}\Big(\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial t}\Big) + 3\frac{\partial^2 u}{\partial y^2} = 0; \tag{4.26}$$

and the associated scattering function Ψ, which satisfies

$$\alpha\frac{\partial \Psi}{\partial t} + \frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial^3 \Psi}{\partial z^3} + \lambda\Big(\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial z}\Big) = 0 \tag{4.27}$$

and

$$\beta\frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} = 0. \tag{4.28}$$

We will use these differential equations to guide us towards significant examples of linear systems with computable tau functions.

Proposition 4.5 *(i) Let $(-A, B, C)$ be a linear system as in Lemma 2.2 with A bounded and $H_0 = \mathbf{C}$. Then*

$$\Psi(x, z; t) = Ce^{t(A^3+\lambda A)/\alpha} e^{-xA} R_0 e^{-zA} e^{t(A^3+\lambda A)/\alpha} B \tag{4.29}$$

is the kernel of a Hankel square and gives a solution to (4.26).

(ii) Let $U(x, z; t)$ be the solution of the integral equation

$$U(x, z; t) - \Psi(2x, z + x; t) + \int_x^\infty U(x, s; t)\Psi(s + x, z + x; t)ds = 0, \tag{4.30}$$

and let $\Psi_{(x)}$ be the integral operator with kernel $\Psi(x + y, y + x; t)$. Then

$$U(x, x; t) = \frac{-1}{2}\frac{d}{dx}\log\det(I + \Psi_{(x)}). \tag{4.31}$$

Proof. (i) This follows by a direct computation.
(ii) As in Proposition 3.2, the kernel $\Psi(x, z; t)$ corresponds to the square of the Hankel operator with symbol $\phi(x; t) = Ce^{t(A^3+\lambda A)/\alpha} e^{-xA} B$ which corresponds to the admissible linear system $(-A, B_0, C_0 e^{t(A^3+\lambda A)/\alpha})$. We then consider the matrix linear system

$$\left(\begin{bmatrix} -A & 0 \\ 0 & -A \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} 0 & Ce^{t(A^3+\lambda A)/\alpha} \\ -Ce^{t(A^3+\lambda A)/\alpha} & 0 \end{bmatrix}\right) \tag{4.32}$$

and obtain a solution to the integral equation () as in Proposition 3.2, which gives an explicit formula for $U(x, y; t)$. The determinant identity follows from Proposition 3.2.

$\square$

Definition Given a solution $U(x, z; t)$ of (4.11) then define $u(x; t) = -2\frac{d}{dx}U(x, x; t)$, so that $u \leftrightarrow \Psi$ is the scattering transform.

In Proposition 8.2, we give an important example in which u also satisfies (4.7). However, we do not have general conditions which ensure that u satisfies (4.7).

In section 5 we show to how produce differential rings of functions from the linear systems, so we can deal with the derivatives and the nonlinear term in KdV. In sections 9 and 10 we produce explicit examples of linear systems such that u satisfies KdV and thereby produce tau functions which are associated with hyperelliptic curves of arbitrary genus; the tau functions in such cases can be expressed in term of determinants, and in terms of Riemann theta functions. We also produce, by similar methods, tau functions which are not associated algebraic curves of finite genus; such examples are already familiar from the theory of Hill's equation. A significant advantage of our approach is that we can deal with periodic potentials, as in Hill's equation, by methods which are formally similar to those used for solitons or scattering potentials. Our results are most complete when u is either trigonometric or elliptic.

5 The state ring associated with an admissible linear system

A linear system with one dimensional input and output that is composed of taps, summing junctions, amplifiers, differentiators and integrators has a transfer function that is real and rational. In [21], the authors consider factorization of transfer functions in rings such as $M_n(\mathbf{R}(\lambda))$. In this paper, we prefer to work with differential rings of operators on the state space so as to integrate various differential equations related to Schrödinger's equation. We introduce these state rings in this section.

Definition (Differential rings) Let H and K be separable complex Hilbert spaces, let $\mathrm{B}(H)$ be the ring of bounded linear operators on H. For $x_0, x_1 \in \mathbf{R}$ let $\mathbf{S}$ be a subring of $C^\infty((x_0, x_1); \mathrm{B}(H))$; that is we suppose that each $T \in \mathbf{S}$ is a differentiable function of $x \in (x_0, x_1)$ and indicate this by writing T_x; we suppose further that $dT_x/dx \in \mathbf{S}$, and that $(d/dx)(ST) = (dS/dx)T + S(dT/dx)$. Then $\mathbf{S}$ is a differential ring with the subring $\{S \in \mathbf{S} : dS/dx = 0\}$ of constants. When $I \in \mathbf{S}$, we identify θI with θ to simplify notation.

Definition (State ring of a linear system) Let $(-A, B, C)$ be a linear system such that A is a bounded linear operator on the state space H. Suppose that:

(i) $\mathbf{S}$ is a differential subring of $C^\infty((x_0, x_1); \mathrm{B}(H))$;

(ii) I, A and BC are constant elements of $\mathbf{S}$;

(iii) e^{-xA}, R_x and $F_x = (I + R_x)^{-1}$ belong to $\mathbf{S}$.

Then $\mathbf{S}$ is a state ring for $(-A, B, C)$ on (x_0, x_1).

(iv) Moreover, if $\mathbf{S}$ is left Noetherian as a ring, then we say that $(-A, B, C)$ is finitely generated.

Remarks. (i) By working with BC in (ii), we suppress the input and output spaces of $(-A, B, C)$ and deal with operators on H.

(ii) When A is algebraic, we can use simple functional calculus to help construct the differential ring. We use this technique in sections 6 and 7.

(iii) We do not assume that $\det F_x$ belongs to $\mathbf{S}$; indeed, the aim is to express this in terms of simpler functions.

Lemma 5.1 *Suppose that $(-A, B, C)$ is a linear system with bounded A and that R_x gives a solution of Lyapunov's equation (1.8) such that $I + R_x$ is invertible for $x > 0$ with inverse F_x. Then the free algebra $\mathbf{S}$ generated by $I, R_0, A, F_0, e^{-xA}, R_x$ and F_x is a state ring for $(-A, B, C)$ on $(0, \infty)$.*

Proof. First we note that $BC = AR_0 + R_0A$ belongs to$\mathbf{S}$, as required. We also note that $(d/dx)e^{-xA} = -Ae^{-xA}$ and that Lyapunov's equation (1.8) gives

$$\frac{d}{dx}(I + R_x)^{-1} = (I + R_x)^{-1}(AR_x + R_xA)(I + R_x)^{-1}, \tag{5.1}$$

which implies

$$\frac{dF_x}{dx} = AF_x + F_xA - 2F_xAF_x. \tag{5.2}$$

with the initial condition

$$AF_0 + F_0A - 2F_0AF_0 = F_0BCF_0. \tag{5.3}$$

Hence $\mathbf{S}$ is a differential ring.

□

Definition (Complex differential rings and state rings) Let Ω be a domain in $\mathbf{C}$ and $\mathbf{M}_\Omega(X)$ the meromorphic functions from Ω to some complex Banach algebra X. If $\mathbf{S}$ as above is also a subring of $\mathbf{M}_\Omega(X)$, then we use the standard complex derivative d/dx and say that $\mathbf{S}$ is a complex state ring for $(-A, B, C)$ on Ω. (In section 8, we work with periodic meromorphic functions and replace Ω by the complex cylinder $\mathbf{C}/\pi\mathbf{Z}$. In section 9, we work with double periodic and meromorphic functions, so we replace Ω by $\mathcal{T} = \mathbf{C}/\Lambda$, where Λ is a lattice.)

Definition (Brackets) Given a state ring for $(-A, B, C)$, let $[X, Y] = XY - YX$ and

$$\lfloor Y \rfloor = Ce^{-xA}F_xYF_xe^{-xA}B. \tag{5.4}$$

The following result is our counterpart of Pöppe's identities [34, 39] from Remark 3.3(ii).

Let $\mathbf{S}$ be a state ring for $(-A, B, C)$ on (x_0, x_1), and let $\mathbf{B}$ be any differential ring of functions on (x_0, x_1) to the bounded linear operators on K. Let

$$\mathbf{A} = \mathrm{span}_{\mathbf{C}}\{A^{n_1}, A^{n_1}F_xA^{n_2} \dots F_xA^{n_r} : n_j \in \mathbf{N}\}. \tag{5.5}$$

Now we introduce a special functional. Let $\lfloor\,.\,\rfloor : \mathbf{S} \to \mathbf{B}$ be a complex linear map such that

$$\lfloor P \rfloor \lfloor Q \rfloor = \lfloor P(AF_x + F_xA - 2F_xAF_x)Q \rfloor \tag{5.6}$$

$$\frac{d}{dx}\lfloor P \rfloor = \Big\lfloor A(I - 2F_x)P + \frac{dP}{dx} + P(I - 2F_x)A \Big\rfloor. \tag{5.7}$$

Lemma 5.2 *(i) Then* $\mathbf{A}$ *defines a differential subring of* $\mathbf{S}$.

(ii) The range $\lfloor \mathbf{S} \rfloor$ *is a differential ring with derivative* d/dx, *and has* $\lfloor \mathbf{A} \rfloor$ *as a differential subring.*

(iii) Suppose that the input and output spaces are $\mathbf{C}$. *Then* $\lfloor X \rfloor = \mathrm{trace}(X(dF_x/dx))$.

Proof. (i) We can multiply elements in $\mathbf{S}$ by concatenating words and taking linear combinations. Since all words in $\mathbf{A}$ begin and end with A, we obtain words of the required form, hence $\mathbf{A}$ is a subring. To differentiate a word in $\mathbf{A}$ we add words in which we successively replace each F_x by $AF_x + F_xA - 2F_xAF_x$, giving a linear combination of words of the required form.

(ii) As in (i), the operations are well defined in the sense that $\lfloor P \rfloor \lfloor Q \rfloor$ and $(d/dx)\lfloor P \rfloor$ are images of elements of $\mathbf{A}$ for all $P, Q \in \mathbf{A}$. Evidently the proposed multiplication is associative and distributive over addition. Using (), one checks that Leibniz's rule holds in the form

$$\frac{d}{dx}\Bigl(\lfloor P \rfloor \lfloor Q \rfloor\Bigr) = \Bigl(\frac{d}{dx}\lfloor P \rfloor\Bigr)\lfloor Q \rfloor + \lfloor P \rfloor \Bigl(\frac{d}{dx}\lfloor Q \rfloor\Bigr). \tag{5.8}$$

(iii) To see that these definition are consistent, observe that when C has range in the scalars, we can remove the trace and write

$$\begin{aligned}\mathrm{trace}\Bigl(Y\frac{d}{dx}F_x\Bigr) &= \mathrm{trace} Y F_x e^{-xA} BCe^{-xA} F_x \\ &= Ce^{-xA} F_x Y F_x e^{-xA} B. \end{aligned} \tag{5.9}$$

□

Let $\mathbf{K}$ be a field of complex functions with differential ∂, and adjoin an element h to $\mathbf{K}$ where either:

(L1) $h = \int g$ for some $g \in \mathbf{K}$, so that $\partial h = g$;

(L2) $h = \exp \int g$ for some $g \in \mathbf{K}$;

(L3) h is algebraic over $\mathbf{K}$.

Then $\mathbf{K}(h)$ is a Liouvillian extension of $\mathbf{K}$ as in [12, 48]. More generally, a field $\mathbf{L}$ is a Liouvillian extension of $\mathbf{K}$ if there exist differential fields $\mathbf{F}_j$ such that $\mathbf{K} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \ldots \subset \mathbf{F}_n = \mathbf{L}$, and each $\mathbf{F}_j$ arises from $\mathbf{F}_{j-1}$ by applying (L1), (L2), or (L3).

Theorem 5.3 *Let* $(-A, B, C)$ *be a linear system as in Lemma 2.2, and suppose furthermore that* A *is bounded and* $H_0 = \mathbf{C}$.

(i) Then $(-A, B, C)$ *has a complex state ring* $\mathbf{S}$ *on* $\mathbf{C}$ *on which* R_z *is unique.*

(ii) The map $\lfloor\,.\,\rfloor : \mathbf{S} \to \mathbf{M}_{\mathbf{C}}(\mathbf{C})$ *satisfies* $\phi(2x) = \bigl\lfloor F_x^{-2} \bigr\rfloor$ *and* $u(x) = -4\bigl\lfloor A \bigr\rfloor$.

(iii) The ranges $\lfloor \mathbf{S} \rfloor$ *and* $\lfloor \mathbf{A} \rfloor$ *are differential rings. The field of fractions* $\mathbf{K}$ *of* $\lfloor \mathbf{A} \rfloor$ *is a differential field, and* $\tau(x) = 1/\det F_x$ *is entire and belongs to a Liouvillian extension* $\mathbf{L}$ *of* $\mathbf{K}$.

(iv) $\mathbf{C}(u, u', \ldots, u^{(k-1)})$ *is a differential subfield of* $\mathbf{K}$, *if and only if* $u^{(k)} = r(u, \ldots, u^{(k-1)})$ *for some rational function* r.

Proof. (i) Mainly this follows from Lemma 2.1 and Proposition 2.4. By Riesz's theory of compact operators, the $F_x = (I + R_x)^{-1}$ defines a meromorphic operator valued function on $\mathbf{C}$. Hence we can select $\mathbf{S}$ to be the subring of meromorphic functions from Ω to $\mathrm{B}(H)$ generated by I, A, BC, R_x, e^{-xA} and F_x. On $\{x : R_x + R_x^\dagger > -2I\}$, the function F_x is holomorphic and satisfies $F_x' = FA + AF - 2FAF$.

(ii) Evidently
$$\lfloor F^{-2} \rfloor = Ce^{-2xA}B = \phi(2x),$$
while we can write (2.3) as $(d/dx)\log\det(I + R_x) = \lfloor F^{-1} \rfloor$ and differentiate using (2.8).

(iii) From the definition of R_x, we have $AR_x + R_xA = e^{-xA}BCe^{-xA}$, and hence

$$F_x e^{-xA}BCe^{-xA}F_x = AF_x + F_xA - 2F_xAF_x, \tag{5.10}$$

which implies

$$\begin{aligned}\lfloor P \rfloor \lfloor Q \rfloor &= Ce^{-xA}F_xPF_xe^{-xA}BCe^{-xA}F_xQF_xe^{-xA}B\\ &= Ce^{-xA}F_xP(AF_x + F_xA - 2F_xAF_x)QF_xe^{-xA}B\\ &= \lfloor P(AF_x + F_xA - 2F_xAF_x)Q \rfloor. \end{aligned}\tag{5.11}$$

Moreover, the first and last terms in $\lfloor P \rfloor$ have derivatives

$$\frac{d}{dx}Ce^{-xA}F_x = Ce^{-xA}F_xA(I - 2F_x), \qquad \frac{d}{dx}F_xe^{-xA}B = (I - 2F_x)AF_xe^{-xA}B, \tag{5.12}$$

which implies (5.8). Hence by Lemma 5.3(ii), the image of $\lfloor\,.\,\rfloor$ is a differential ring.

Now $\lfloor \mathbf{A} \rfloor$ is a subring of $\mathbf{M_C(C)}$ and hence is an integral domain with a field of fractions $\mathbf{K}$. We have $2(d/dx)^2\log\det F_x = u(x) \in \mathbf{K}$, so we can recover $\det F_x$ by integration and exponential integration. By (2.3) and Morera's theorem, R_x is an entire c^1-valued function, hence $\det(I + R_x)$ is entire.

(iv) By (ii), u and all its derivatives belong to $\mathbf{K}$. Evidently $\mathbf{C}(u, \dots u^{(k-1)})$ is a differential field if and only if such a differential equation holds.

□

Remarks 5.4 (i) Airault, McKean and Moser [2] consider the cases of Theorem 5.3(iv) given by $u''' = 12uu'$ for u rational, trigonometric and elliptic.

(ii) Pöppe [39, 40] introduced a linear functional $\lceil\,.\,\rceil$ on Fredholm kernels $K(x, y)$ on $L^2(0,\infty)$ by $\lceil K \rceil = K(0, 0)$. In particular, let K, G, H, L be integral operators on $L^2(0,\infty)$ that have smooth kernels of compact support, let $\Gamma = \Gamma_{\phi_{(x)}}$ have kernel $\phi(s + t + 2x)$, let $\Gamma' = \frac{d}{dx}\Gamma$ and $G = \Gamma_{\psi_{(x)}}$ be another Hankel operator; then the trace satisfies

$$\lceil \Gamma \rceil = -\frac{d}{dx}\operatorname{trace}\Gamma \tag{5.13}$$
$$\lceil \Gamma KG \rceil = -\frac{1}{2}\frac{d}{dx}\operatorname{trace}\Gamma KG \tag{5.14}$$
$$\lceil (I + \Gamma)^{-1}\Gamma \rceil = -\operatorname{trace}\big((I + \Gamma)^{-1}\Gamma'\big), \tag{5.15}$$
$$\lceil K\Gamma \rceil \lceil GL \rceil = -\frac{1}{2}\lceil K(\Gamma'G + \Gamma G')L \rceil, \tag{5.16}$$

where (4) is known as the product formula. The easiest way to prove these is to observe that $\Gamma' G+\Gamma G'$ is the integral operator with kernel $-2\phi_{(x)}(s)\psi_{(x)}(t)$, which has rank one, as in (3.8) below. These ideas were subsequently revived by McKean [32].

(iii) Mulase [36] considers differential rings over $\mathbf{C}$ that are also closed under (L1) and (L2); an important example is the Noetherian ring $\mathbf{C}[[x]]$ of formal complex power series. However, $\mathbf{C}[[x]]$ does not contain functions with poles. Krichever [29] considered an algebraic curve with a preferred point P_0, and functions that are holomorphic except for poles at P_0. Note that $\{f(z)=\sum_{k=-n}^{\infty} a_k z^k; n\in \mathbf{N}; a_k\in \mathbf{C}\}$ is a Noetherian differential ring, but it is not closed under (L1) or (L2). So we prefer to start in a smaller ring and then control the extensions that are formed by making quadratures.

6. Finite dimensional state spaces

In this section, we are concerned with complex differential rings for linear systems $(-A,B,C)$ that have finite dimensional state spaces. While we seek to realise $\mathbf{S}$ by the approach of Remark 5.3, we do not assume commutativity of A and BC, and we do not assume that e^{-xA} is stable.

Hypotheses. Throughout this section, we let A be a $n\times n$ complex matrix with eigenvalues λ_j with geometric multiplicity n_j such that $\lambda_j+\lambda_k\neq 0$ for all j and k; if all the eigenvalues are geometrically simple, then let $\mathbf{K}=\mathbf{C}(e^{-\lambda_1 t},\dots,e^{-\lambda_n t})$; otherwise, let $\mathbf{K}=\mathbf{C}(e^{-\lambda_1 t},\dots,e^{-\lambda_n t},t)$. Also, let $B=(b_j)\in\mathbf{C}^{n\times 1}$ and $C=(c_j)\in\mathbf{C}^{1\times n}$.

The following result extends a special case of the Sylvester–Rosenblum theorem [7].

Lemma 6.1 *Let $\mathbf{S}=\mathbf{C}[I,A,BC]$. Then there exists $R_0\in\mathbf{S}_0$ such that $R_0A+AR_0=-BC$, and the equations (1.9) and (1.8) have a unique solution.*

Proof. Let Σ be a chain of circles that go once round each λ_j in the positive sense and have all the points $-\lambda_k$ in their exterior. Then by [7], the matrix

$$R_0=\frac{1}{2\pi i}\int_\Sigma (A+\lambda I)^{-1}BC(A-\lambda I)^{-1}d\lambda \tag{6.1}$$

gives the unique solution to the equation (1.8). Furthermore, by the Cayley–Hamilton theorem, $(A\mp\lambda I)^{-1}$ is a polynomial in λ, A,I and $\det(A\mp\lambda I)^{-1}$ for all λ on γ; hence R_0 belongs to the algebra $\mathbf{S}_0$.

The function $R_x=e^{-xA}R_0e^{-xA}$ is entire and of exponential growth, and gives a solution of (1.9) and (1.8). Since R_x is of exponential growth, it has a Laplace transform which satisfies $s\hat R(s)+A\hat R(s)+R(s)A=R_0$, and for all $s>2\|A\|$ the solution is unique and may be expressed as

$$\hat R(s)=\int_{-i\infty}^{i\infty}\big((\lambda+s/2)I+A\big)^{-1}R_0\big((-\lambda+s/2)I+A\big)^{-1}\frac{d\lambda}{2\pi i}. \tag{6.2}$$

Hence R_x is the unique solution of (1.8) and (1.9).

$\square$

Theorem 6.2 *Let $R_x=e^{-xA}R_0e^{-xA}$; then let $\mathbf{S}=\mathbf{K}[I,A,BC]$.*

(i) Then $(-A,B,C)$ is finitely generated since $\mathbf{S}$ is a left Noetherian ring with respect to the standard multiplications.

(ii) The linear map $\lfloor\,.\,\rfloor : \mathbf{S} \to \mathbf{H_C} : \lfloor P \rfloor = Ce^{-xA}F_xPF_xe^{-xA}B$ *satisfies* $\phi(2x) = \lfloor F_x^{-2} \rfloor$ *and* $u(x) = -4\lfloor A \rfloor$. *Also,* $\tau, \tau/\tau_\lambda \in \mathbf{K}$.

Proof (i) The complex algebra generated by I, BC and A is finite-dimensional and hence left Noetherian; so by Hilbert's basis theorem, $\mathbf{S}$ as a subalgebra of $M_n(\mathbf{K})$ is also Noetherian; see [14, p. 106]. Observe that $(\lambda I - A)(\lambda I + A)^{-1} \in \mathbf{S}$ for all $-\lambda$ in the resolvent set of A.

By the Riesz functional calculus, we can introduce a sum of cycles going round each λ_j once in the positive sense, so that

$$e^{-tA} = \frac{1}{2\pi i}\int_\Sigma (\lambda I - A)^{-1} e^{-t\lambda} d\lambda; \tag{6.3}$$

hence there exist complex polynomials p_j and q_j, and integers $m_j \geq 0$ such that

$$e^{-tA} = \sum_{j=1}^n q_j(t) e^{-t\lambda_j} p_j(A), \tag{6.4}$$

where $q_j(t)$ is constant if the corresponding eigenvalue is simple. Hence $R_x \in \mathbf{S}$, and likewise all the entries of R_x belong to $\mathbf{S}$. Moreover, for any $B \in \mathbf{C}^{n\times 1}$ and $C \in \mathbf{C}^{1\times n}$, there exist constants α_j and polynomials q_j such that

$$\phi(x) = Ce^{-xA}B = \sum_{j=1}^n \alpha_j q_j(x) e^{-\lambda_j x}. \tag{6.5}$$

Now introduce the minors $\sigma_j \in \mathbf{K}$ of $I + R_x$ such that

$$\det(\mu I - (I + R_x)) = \mu^n + \sigma_{n-1}(x)\mu^{n-1} + \ldots + \sigma_1(x)\mu + (-1)^n\theta(x), \tag{6.6}$$

and recall that by the Cayley–Hamilton theorem

$$(I + R_x)\Big((I + R_x)^{n-1} + \sigma_{n-1}(x)(I + R_x)^{n-2} + \ldots + \sigma_1(x)I\Big) + (-1)^n\theta(x)I = 0 \tag{6.7}$$

so F_x belongs to $\mathbf{S}$. Hence $\mathbf{S}$ is a complex differential ring for $(-A, B, C)$. By the usual expansion of the determinant, $\tau \in \mathbf{K}$.

(ii) This follows as in Theorem 2.5. Observe also that ϕ and u belong to $\mathbf{K}$, and all elements of $\mathbf{K}$ are meromorphic on $\mathbf{C}$.

□

Lemma 6.3 (The Cauchy determinant formula) *Let* x_r *and* y_s *be complex numbers such that* $x_r y_s \neq 1$. *Then*

$$\det\Big[\frac{1}{1 - x_j y_k}\Big]_{j,k=1}^n = \frac{\prod_{1\leq j<k\leq n}(x_j - x_k)\prod_{1\leq m<p\leq n}(y_m - y_p)}{\prod_{1\leq r,s\leq n}(1 - x_r y_s)}. \tag{6.8}$$

Proposition 6.4 *Suppose that* $B = (b_j)_{j=1}^n \in \mathbf{C}^{n\times 1}$, $C = (c_j)_{j=1}^n \in \mathbf{C}^{1\times n}$ *and* A *is the* $n \times n$ *diagonal matrix with simple eigenvalues* λ_j *such that* $\lambda_j + \lambda_k \neq 0$ *for all* $j = 1, \ldots, n$.

(i) *Then R_x gives rise to the determinant*

$$\det(I+\mu R_x) = 1+\mu\sum_{j=1}^n \frac{b_jc_je^{-2\lambda_jx}}{2\lambda_j} + \mu^2 \sum_{(j,k),(m,p):j\neq m;k\neq p} (-1)^{j+k+m+p}\frac{b_jb_mc_kc_pe^{-(\lambda_j+\lambda_k+\lambda_m+\lambda_p)x}}{(\lambda_j+\lambda_m)(\lambda_k+\lambda_p)}+\dots + \mu^n\prod_{j=1}^n b_jc_j \prod_{1\leq j<k\leq n}\frac{(\lambda_j-\lambda_k)^2}{(\lambda_j+\lambda_k)^2}e^{-2\sum_{j=1}^n\lambda_jx}. \tag{6.9}$$

Proof. (i) The proof is by induction on n. There is an expansion

$$\det\Bigl[\delta_{jk}+\frac{\mu b_jc_ke^{-(\lambda_j+\lambda_k)x}}{\lambda_j+\lambda_k}\Bigr]_{j,k=1}^n = \sum_{\sigma\subseteq\{1,\dots,n\}}\mu^{\sharp\sigma}\det\Bigl[\frac{b_jc_ke^{-\lambda_jx-\lambda_kx}}{\lambda_j+\lambda_k}\Bigr]_{j,k\in\sigma} \tag{6.10}$$

in which each subset σ of $\{1,\dots,n\}$ of order $\sharp\sigma$, contributes a minor indexed by $j,k\in\sigma$. Letting $x_r=\lambda_r$ and $y_r=-1/\lambda_r$ in the Cauchy determinant formula, we obtain the identity

$$\det\Bigl[\frac{b_jc_ke^{-\lambda_jx-\lambda_kx}}{\lambda_j+\lambda_k}\Bigr]_{j,k\in\sigma} = \prod_{j\in\sigma}\frac{b_jc_je^{-2\lambda_jx}}{2\lambda_j}\prod_{j,k\in\sigma:j\neq k}\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}. \tag{6.11}$$

□

Remarks 6.5 (1) The results of this section apply in particular when A is a finite matrix such that all the eigenvalues have $\Re\lambda_j>0$.

(2) Kronecker's theorem asserts that a bounded Hankel integral operator has finite rank if and only if the transfer function $\hat\phi$ is a rational function with all its poles in $\{z\in\mathbf{C}:\Re z<0\}$. Such rational functions are known as stable. In [19], the authors consider factorization of the transfer function in $M_{n\times n}(\mathbf{C}(\lambda))$ and the subring of stable matrix rational functions. Their results describe the properties of $\hat{\mathbf{S}}$ rather than $\mathbf{S}$ itself.

7. The differential ring associated with the Painlevé II equation

In this section we consider a linear system which is important in random matrix theory. Whereas the state ring $\mathbf{S}$ is finitely generated, the linear system is not integrable in the sense that τ does not emerge from $\mathbf{C}(x)$ by successive Liouville integrations. Let $H(p,q;x)$ be a Hamiltonian which is rational in the canonical variables (p,q) and a meromorphic function of time x, and let $(p(s),q(s))$ be solutions of the canonical equations of motion, and suppose momentarily that these are meromorphic functions of s. Then the corresponding tau function is

$$\tau(x)=\exp\int_0^x H(p(s),q(s);s)\,ds, \tag{7.1}$$

where the integral is taken along an orbit in phase space; so the value of τ is locally independent of the path of integration, provided the path avoids poles.

The Hamiltonians which arise on random matrix theory have additional properties which are described in the following result, which is a variant of Theorem 1 in Okamoto's paper [37].

Proposition 7.1 *Suppose that the Hamiltonian $H(p,q;x)$ is rational in x, a polynomial in q, and a quadratic polynomial in p, let u be the potential that corresponds to τ, and let $\mathbf{K}=\mathbf{C}(x,q)$. Then there exist $E,F,G\in\mathbf{K}$ such that*

$$\mathbf{K}(u)\big[\sqrt{F^2-4E(u-G)}\big] \tag{7.2}$$

gives a differential field with respect to d/dx under the canonical equations of motion.

Proof. We write $H=A(q,x)p^2+B(q,x)p+C(q,x)$. Then the canonical equations are $\frac{dq}{dx}=\frac{\partial H}{\partial p}$ and $\frac{dp}{dx}=-\frac{\partial H}{\partial q}$. Hence $\mathbf{C}(x)[p,q]$ is a commutative and Noetherian differential ring for the derivative $\frac{d}{dt}=\frac{\partial}{\partial x}+\frac{\partial H}{\partial p}\frac{\partial}{\partial q}-\frac{\partial H}{\partial q}\frac{\partial}{\partial p}$. Using the special form of the Hamiltonian, we have

$$q''=-2A\Big\{\frac{\partial A}{\partial q}\Big(\frac{q'-B}{2A}\Big)^2+\frac{\partial B}{\partial q}\Big(\frac{q'-B}{2A}\Big)+\frac{\partial C}{\partial q}\Big\}+\frac{q'-B}{A}\Big(\frac{\partial A}{\partial q}q'+\frac{\partial A}{\partial x}\Big)+\Big(\frac{\partial B}{\partial q}q'+\frac{\partial B}{\partial x}\Big). \tag{7.3}$$

so $q''=f(x,q,q')$ where f is rational in x and q and quadratic in q', so $\mathbf{K}[q']$ is a differential ring for d/dx. Likewise, the potential that corresponds to τ is

$$u(x)=-2\frac{\partial H}{\partial x}=-2\frac{\partial A}{\partial x}\Big(\frac{q'-B}{2A}\Big)^2-2\frac{\partial B}{\partial x}\Big(\frac{q'-B}{2A}\Big)-2\frac{\partial C}{\partial x}, \tag{7.4}$$

hence there exist nonzero $E,F,G\in\mathbf{K}$ such that $Eq'^2+Fq'+G=u$, so $\mathbf{K}(u)[q']$ is a differential field, and which we can identify with a quadratic extension of $\mathbf{K}(u)$.

$\square$

Okamoto [37] has shown that each of the Painlevé transcendental differential equations $P_I,\dots,P_{VI}$ arises from a Hamiltonian as in Lemma 7.1, and τ is meromorphic on a suitable covering surface. Conversely, let $v''=F(v,v';x)$ be a differential equation such that $F(v,v';x)$ is meromorphic in x and rational in v and v' and such that the general solution has no movable singularities other than poles. Then the equation may be reduced by change of variables to a Painlevé equation.

For $x\in\mathbf{C}$ and a complex constant α, let

$$H_{II}(p,q;x)=\frac{1}{2}\Big(p-\frac{x}{2}\Big)^2+\Big(q^2+\frac{x}{2}\Big)\Big(p-\frac{x}{2}\Big)-\alpha q+\frac{x^2}{8}. \tag{7.5}$$

Proposition 7.2 *Under the canonical equations of motion with Hamiltonian H_{II},*

(i) q satisfies $P_{II}:q''=xq+2q^3+\alpha$ and the corresponding τ function is

$$\tau(x)=\exp\Big(-\frac{1}{2}\int_x^\infty(s-x)q(s)^2\,ds\Big); \tag{7.6}$$

(ii) p satisfies $p'''+6pp'-(2p+xp')=0$ and $U(x,t)=(3t)^{-2/3}p(3^{-1}t^{-1/3}x)$ satisfies

$$\frac{\partial^3U}{\partial x^3}+\frac{2U}{3^{1/3}}\frac{\partial U}{\partial x}-\frac{1}{9}\frac{\partial U}{\partial t}=0. \tag{7.7}$$

Proof. (i) The canonical equations of motion are satisfied in the polynomial ring $\mathbf{S}_\alpha = \mathbf{C}[x,q,p]$ with the derivatives

$$\frac{dq}{dx} = -p - q^2 \quad \text{and} \quad \frac{dp}{dx} = (2p-x)q - \alpha. \tag{7.8}$$

Hence $\mathbf{K} = \mathbf{C}(x,q,p)$ is a differential field, and by Lemma 7.1 the potential is $u = q^2$, which belongs to $\mathbf{K}$. We deduce that q satisfies P_{II}.

(ii) Now p satisfies

$$K_2: \qquad p'' + 2p^2 - xp + \frac{\alpha(\alpha+1) + p' - (p')^2}{2p - x} = 0. \tag{7.9}$$

One can then verify that U satisfies KdV; see [1] for further discussion.

$\square$

Now we show how to solve P_{II} by means of determinants associated with integrable kernels. We introduce Airy's function $\mathrm{Ai}(x) = \int_{-\infty}^{\infty} e^{i\xi x + i\xi^3/3} d\xi/(2\pi)$, which satisfies $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$. Let $\phi(x) = \mathrm{Ai}(x)$ and let $\zeta = \phi'/\phi$; then $\mathbf{S} = \mathbf{C}[x, \phi(x), \zeta(x)]$ is a differential ring with respect to d/dx. In the context of Theorem 7.3(iii) below the integrable kernel

$$R_0^2(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \tag{7.10}$$

is known as Airy's kernel, which is associated with soft edges of eigenvalue distributions. The Fredholm determinants of R_0^2 lead to a solution of the Painlevé II nonlinear differential equation. Ablowitz and Segur solved P_{II} by a slightly different method, Borodin and Deift [11] obtained a solution by considering a matrix Riemann–Hilbert problem involving (7.5) and we include the proof of (iii) to illustrate the general theory of linear systems.

In previous sections we started from an admissible linear system and produced a Hankel integral operator Γ_ϕ. In this section we begin with a technical result which realises a typical Hilbert–Schmidt Hankel operator Γ_ϕ from an explicit linear system $(-A, B, C)$ chosen for ϕ. Here A is defined on $\mathcal{D}(A) = \{f \in L^2(0,\infty); f' \in L^2(0,\infty)\}$ and C is bounded on $\mathcal{D}(A)$. Suppose that ϕ and ψ are continuous functions on $\mathbf{R}$ such that $\int_0^\infty (1+t)(|\phi(t)|^2 + |\psi(t)|^2)\,dt < \infty$. Then we let $H = L^2(0,\infty)$ and introduce the operators

$$\begin{aligned} A &: f(x) \mapsto -f'(x) \qquad f \in \mathcal{D}(A);\\ B &: \beta \mapsto \phi(x)\beta;\\ E &: \beta \mapsto \psi(x)\beta;\\ C &: g(x) \mapsto g(0) \qquad (g \in \mathcal{D}(A)), \end{aligned} \tag{7.11}$$

so that $\phi(x) = Ce^{-xA}B$ and $\psi(x) = Ce^{-xA}E$. We introduce the operators on H given by $R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt$ and $S_x = \int_x^\infty e^{-tA}ECe^{-tA}\,dt$. In terms of Proposition 2.1, the cogenerator V is unitarily equivalent via the Fourier transform to the coisometry on the Hardy space H^2 on the upper half plane

$$V: f(z) \mapsto \frac{(1-iz)f(z) - 2f(i)}{1+iz} \qquad (f \in H^2), \tag{7.12}$$

so $V^\dagger$ is the shift. This is consistent with Beurling's canonical model of a linear system in [7]. We also introduce the observability Gramian $Q_x = \int_x^\infty e^{-tA^\dagger} C^\dagger C e^{-tA}\, dt$ and we observe that Q_x is the orthogonal projection $Q_x : L^2(0,\infty) \to L^2(0,x)$. We consider the Gelfand–Levitan integral equation (2.7) where $T(x,y)$ and $\Phi(x+y)$ are 2×2 matrices, and

$$\Phi(x) = \begin{bmatrix} 0 & \psi(x) \\ \phi(x) & 0 \end{bmatrix}. \tag{7.13}$$

Lemma 7.3 *(i) For $|\mu|$ sufficiently small, the operator $I - \mu^2 R_x S_x$ has inverse $G_x \in \mathrm{B}(H)$ and the matrix function*

$$\hat T(x,y) = \begin{bmatrix} \mu C e^{-xA} G_x S_x e^{-yA} B & -Ce^{-xA} G_x e^{-yA} E \\ -Ce^{-xA-yA}B - \mu^2 Ce^{-xA} R_x G_x S_x e^{-yA} B & \mu C e^{-xA} R_x G_x e^{-yA} E \end{bmatrix} \tag{7.14}$$

satisfies the Gelfand–Levitan equation (2.7).

(ii) The determinants satisfy

$$\det(I - \mu^2 R_x S_x) = \det(I - \mu^2 \Gamma_{\psi_{(x)}} \Gamma_{\phi_{(x)}}). \tag{7.15}$$

and

$$\operatorname{trace} \hat T(x,x) = \frac{d}{dx} \log\det(I - \mu^2 \Gamma_{\phi_{(x)}} \Gamma_{\psi_{(x)}}). \tag{7.16}$$

Proof. (i) We introduce

$$\hat A = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \hat B = \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}, \quad \hat C = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \tag{7.17}$$

and follow the computations of Proposition 2.3(i) to find T.

(ii) We observe that $R_x f(z) = \int_x^\infty \phi(z+u) f(u)\, du$, so R_x is a Hilbert–Schmidt operator, and R_0 is the Hankel operator Γ_ϕ; likewise S_x is Hilbert–Schmidt; hence $R_x S_x$ is trace class. The identity (7.9) follows from Proposition 2.3.

Whereas $R_x S_x$ is not a differentiable function of x, and we cannot adopt the direct approach of Proposition 2.3(iii), we can differentiate the Hankel product

$$\frac{d}{dx} \int_0^\infty \phi(2x+s+u)\psi(2x+t+u)\, du = -2\phi(2x+s)\psi(2x+t), \tag{7.18}$$

so

$$\frac{d}{dx} \Gamma_{\phi_{(x)}} \Gamma_{\psi_{(x)}} = -2 e^{-2xA} B C e^{-2xA} S_0, \tag{7.19}$$

where the right-hand side is a rank one and bounded linear operator. We recall from [38] the following identities regarding the shift and Hankel operators

$$e^{-xA^\dagger}e^{-xA}=Q_x,\qquad e^{-xA}e^{-xA^\dagger}=I,\qquad e^{-xA^\dagger}R_0e^{-xA^\dagger}=R_0 \tag{7.20}$$

and the following special identities which may be checked by looking at the kernels

$$R_x=R_0Q_x,\quad e^{-xA}R_0=R_xe^{-xA^\dagger},\quad \Gamma_{\phi_{(x)}}=e^{-xA}R_0e^{-xA^\dagger}. \tag{7.21}$$

Hence we can differentiate using (7.12), obtaining

$$\begin{aligned}
&\frac{d}{dx}\log\det\big(I-\mu^2\Gamma_{\phi_{(x)}}\Gamma_{\psi_{(x)}}\big)\\
&\quad=2\mu^2\mathrm{trace}\Big(\big(I-\mu^2\Gamma_{\phi_{(x)}}\Gamma_{\psi_{(x)}}\big)^{-1}e^{-2xA}BCe^{-2xA}S_0\Big)\\
&\quad=2\mu^2Ce^{-2xA}S_0\big(I-\mu^2e^{-xA}R_0e^{-xA^\dagger}e^{-xA}S_0e^{-xA^\dagger}\big)^{-1}e^{-2xA}B\\
&\quad=2\mu^2Ce^{-xA}S_xe^{-xA^\dagger}\big(I-\mu^2e^{-xA}R_0e^{-xA^\dagger}e^{-xA}S_0e^{-xA^\dagger}\big)^{-1}e^{-2xA}B.
\end{aligned} \tag{7.22}$$

We now use the identity $K(I+LK)^{-1}=(I+KL)^{-1}K$ to shuffle terms around, and obtain

$$\begin{aligned}
&=2\mu^2Ce^{-xA}S_x(I-\mu^2e^{-xA^\dagger}e^{-xA}R_0e^{-xA^\dagger}e^{-xA}S_0e^{-xA^\dagger})^{-1}e^{-xA^\dagger}e^{-xA}e^{-xA}B\\
&=2\mu^2Ce^{-xA}S_x(I-\mu^2Q_xR_0Q_xS_0e^{-xA^\dagger})^{-1}Q_xe^{-xA}B\\
&=2\mu^2Ce^{-xA}S_x(I-\mu^2R_xS_x)^{-1}e^{-xA}B\\
&=2\mu^2Ce^{-xA}(I-\mu^2S_xR_x)^{-1}S_xe^{-xA}B;
\end{aligned} \tag{7.23}$$

which is a multiple of the top left entry of $T(x,x)$, and likewise

$$2\mu^2Ce^{-xA}R_x(I-\mu^2S_xR_x)^{-1}e^{-xA}E=2\mu^2Ce^{-xA}R_xG_xe^{-yA}E; \tag{7.24}$$

as in the bottom left entry of $T(x,x)$ so we obtain the expected result

$$\frac{d}{dx}\log\det(I-\mu^2R_x^2)=\mu\mathrm{trace}\,T(x,x). \tag{7.25}$$

□

We consider the Gelfand–Levitan integral equation (2.7) where $T(x,y)$ and $\Phi(x+y)$ are 2×2 matrices, and

$$T(x,y)=\begin{bmatrix}U(x,y)&V(x,y)\\-V(x,y)&U(x,y)\end{bmatrix},\qquad \Phi(x)=\begin{bmatrix}0&\phi(x)\\-\phi(x)&0\end{bmatrix}. \tag{7.26}$$

Theorem 7.4 *Let $(-A,B,C)$ be as in Lemma 7.3.*

(i) For $|\mu|$ sufficiently small, $I+\mu^2R_x^2$ is invertible with inverse Z_x and matrix function

$$\hat T(x,y)=\begin{bmatrix}-\mu Ce^{-xA}Z_xR_xe^{-yA}B&-Ce^{-xA}Z_xe^{-yA}B\\Ce^{-xA}Z_xe^{-yA}B&-\mu Ce^{-xA}R_xZ_xe^{-yA}B\end{bmatrix} \tag{7.27}$$

satisfies the Gelfand–Levitan equation (2.7).

(ii) The determinant satisfies

$$\mu \mathrm{trace}\,\hat{T}(x,x) = \frac{d}{dx}\log\det(I+\mu^2\Gamma^2_{\phi_{(x)}}). \tag{7.28}$$

(iii) In particular, let $\phi(x) = \mathrm{Ai}(x/2)$. *Then* $V(x,x)$ *satisfies Painlevé's equation*

$$P_{II} \qquad 1\frac{d^2}{dx^2}V(x,x) = xV(x,x) - 8\mu^2 V(x,x)^3 \tag{7.29}$$

and $V(x,x) \asymp -\mathrm{Ai}(x)$ *as* $x \to \infty$.

Proof. We introduce the 2×2 matrices with entries that are operators given by

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} -B & 0 \\ 0 & B \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}, \tag{7.30}$$

so that $\Phi(x) = \hat{C}e^{-x\hat{A}}\hat{B}$ and

$$\hat{R}_x = \int_x^\infty e^{-t\hat{A}}\hat{B}\hat{C}e^{-t\hat{A}}\,dt = \begin{bmatrix} 0 & -R_x \\ R_x & 0 \end{bmatrix}. \tag{7.31}$$

Here R_x^2 is trace class, and when $|\mu|\int_0^\infty t|\phi(t)|^2 dt < 1$, the operator $I+\mu^2 R_x^2$ is invertible for all $x>0$, so $I+\mu\hat{R}_x$ has an inverse

$$\hat{F}_x = \begin{bmatrix} I & -\mu R_x \\ \mu R_x & I \end{bmatrix}^{-1} = \begin{bmatrix} I-\mu R_x^2 Z_x & \mu R_x Z_x \\ -\mu R_x Z_x & Z_x \end{bmatrix}. \tag{7.32}$$

Hence we can solve the integral equation (2.7) using $\hat{T}(x,y) = -\hat{C}e^{-x\hat{A}}\hat{F}_x e^{-y\hat{A}}\hat{B}$, and we obtain (7.18).

(ii) This follows from Proposition 7.1(ii).

(iii) First, note that $V(x,x) = -Ce^{-xA}(I+\mu^2R_x^2)^{-1}e^{-xA}B$ where $\mathrm{Ai}(x/2) = Ce^{-xA}B$, so $V(x,x)$ is asymptotic to $-\mathrm{Ai}(x)$ as $x\to\infty$.

It follows from the Gelfand–Levitan equation that

$$V(x,y) + \phi(x+y) + \mu^2\int_x^\infty\int_x^\infty V(x,z)\phi(z+s)\phi(s+y)\,dzds = 0. \tag{7.33}$$

Let $L = (\frac{\partial}{\partial x}+\frac{\partial}{\partial y})^2 - \frac{x+y}{2}$ and $\phi(x) = \mathrm{Ai}(x/2)$, so that $L\phi(x+y) = 0$. Also from Airy's equation, we obtain

$$\frac{y-z}{2}\int_x^\infty \phi(z+t)\phi(t+y)\,dt = 4\big(\phi'(z+x)\phi(x+y) - \phi(z+x)\phi'(x+y)\big), \tag{7.34}$$

and by repeatedly integrating by parts, we can reduce (7.27) to the expression

$$\begin{aligned} LV(x,y) - 4\mu^2\Big(\frac{d}{dx}\int_x^\infty V(x,z)\phi(z,x)\,dz\Big)\phi(x+y) \\ + \mu^2\int_x^\infty\int_x^\infty LV(x,z)\phi(z+s)\phi(s+y)\,dzds = 0, \end{aligned} \tag{7.35}$$

which is a multiple of the original equation (7.24) by

$$-4\mu^2 \frac{d}{dx} \int_x^\infty V(x,z)\phi(z+x)\, dz = -8\mu^2 V(x,x)^2. \tag{7.36}$$

To see (7.29), we use the definition of V to compute

$$\begin{aligned} -4\mu^2 \int_x^\infty V(x,z)\phi(z+x)\, dz &= 4\mu^2 \int_x^\infty Ce^{-xA} Z_x e^{-zA} BCe^{-zA} e^{-xA} B dz \\ &= 4\mu^2 Ce^{-xA} Z_x R_x e^{-xA} B, \end{aligned} \tag{7.37}$$

and then we use the basic identity (1.8) to calculate

$$\begin{aligned} \frac{d}{dx}\Big(4\mu^2 Ce^{-xA} Z_x R_x e^{-xA} B\Big) &= 4\mu^2 Ce^{-xA}\Big(-AZ_x R_x + \mu^2 Z_x (AR_x^2 + R_x^2 A + 2R_x A R_x) Z_x R_x \\ &\quad - Z_x (AR_x + R_x A) - Z_x R_x A\Big) e^{-xA} B \\ &= -8\mu^2 Ce^{-xA} Z_x (AR_x + R_x A) Z_x e^{-xA} B, \end{aligned} \tag{7.38}$$

where we have repeatedly used the rule $\mu^2 Z_x R_x^2 = I - Z_x$ to simplify. Meanwhile, the product rule gives

$$V(x,x)^2 = Ce^{-xA} Z_x e^{-xA} BCe^{-xA} Z_x e^{-xA} B = Ce^{-xA} Z_x (AR_x + R_x A) Z_x e^{-xA} B, \tag{7.39}$$

and hence we obtain (7.26). On multiplying (7.26) by $-8\mu^2 V(x,x)^2$ and using uniqueness, we deduce that

$$LV(x,y) = -8\mu^2 V(x,x)^2 V(x,y), \tag{7.40}$$

and on the diagonal we have

$$P_{II} \qquad \frac{d^2}{dx^2} V(x,x) - xV(x,x) = -8\mu^2 V(x,x)^3. \tag{7.41}$$

□

Corollary 7.5 *(i) The entries of $T(x,x)$ all lie in $\mathbf{S}_0$, and the potential is*

$$u(x) = -8\mu^2 V(x,x)^2. \tag{7.42}$$

(ii) The cumulative distribution function of the Tracy–Widom distribution [47] satisfies

$$F_2(x) = \det((I - \Gamma^2_{\phi_{(x)}}/4). \tag{7.43}$$

Proof. (i) All the terms vanish as $x \to \infty$, so $\alpha = 0$. By the identities (8.20) and (8.21), we have

$$\begin{aligned} u(x) &= -2\mu \frac{d}{dx} \operatorname{trace} \hat{T}(x,x) \\ &= 4\mu^2 \frac{d}{dx} Ce^{-xA} R_x Z_x e^{-xA} B \\ &= -8\mu^2 V(x,x)^2. \end{aligned} \tag{7.44}$$

Hence we can write, with $v(x) = V(x,x)$

$$-2\mu \frac{d}{dx}\hat{T}(x,x) = \begin{bmatrix} -4\mu^2 v(x)^2 & -2\mu v'(x) \\ 2\mu v'(x) & -4\mu^2 v(x)^2 \end{bmatrix}, \tag{7.45}$$

so the trace is $-8\mu^2 v(x)^4$. Moreover, the differential equation gives

$$\int_x^\infty v(t)^2\, dt = -xv(x)^2 + v'(x)^2 - v(x)^2, \tag{7.46}$$

which are all elements of $\mathbf{S}_0$, so the entries of $\hat{T}(x,x)$ are all in $\mathbf{S}_0$.

(ii) With $\mu = i/2$, the potential gives rise to the Tracy Widom distribution function

$$F_2(x) = \exp\Big(-2^{-1} \int_x^\infty (s-x)u(s)\, ds\Big) \tag{7.47}$$

that is associated with the soft spectral edge of the Gaussian unitary ensemble; see [46, 47 (1.17)].

□

8. The differential ring of a periodic linear system

In this section we obtain analogues of Theorem 6.2 for periodic groups. For periodic and meromorphic u, the differential equation $-\psi'' + u\psi = \lambda\psi$ is known as the complex Hill's equation. We consider special periodic linear systems such that u is a function of rational character on the cylinder or u is doubly periodic and of rational character on some elliptic curve $\mathcal{T}$.

For periodic linear systems, the defining integral for R_x in Lemma 2.1 does not converge, and the contour integral for R_0 in Lemma 6.1 is inapplicable; nevertheless, we can adapt a result of Bhatia, Dacis and McIntosh discussed in [7] and otherwise construct R_x satisfying (1.8).

Lemma 8.1 *Let B be a trace class operator and C be a bounded operator on H, and let $(e^{-tA})_{t\in\mathbf{R}}$ be a bounded C_0 group of operators on H such that the spectrum of A does not intersect the spectrum of $-A$. Then there exists a solution to the Lyapunov equation $-\frac{d}{dx}R_x = AR_x + R_xA$ such that $AR_0 + R_0A = BC$ and R_x is trace class for all $x \in \mathbf{R}$.*

Proof. The main problem is to find E such that $EA + AE = BC$. By a theorem of Sz.-Nagy, the group (e^{-tA}) is similar to a group of unitaries, so there exists an invertible operator S and a unitary group $(U_t)_{t\in\mathbf{R}}$ such that $e^{-tA} = SU_tS^{-1}$. Hence the spectrum of A lies on $i\mathbf{R}$ and is a closed subset. By hypothesis, there exists $\delta > 0$ such that the spectra of A and $-A$ are separated by δ and $\sigma(A) \cup \sigma(-A)$ does not intersect $(-\delta, \delta)$. By Plancherel's theorem, we can construct an integrable function f such that $\hat{f}(\xi) = 1/\xi$ for all $\xi \in \mathbf{R}$ such that $|\xi| \geq \delta$. Then the integral

$$E = \int_{-\infty}^\infty e^{-xA} BCe^{-xA} f(x)dx \tag{8.1}$$

has a weakly continuous integrand in the trace class operators, and is absolutely convergent with

$$\|E\|_{c^1} \leq \int_{-\infty}^{\infty} \|B\|_{c^1}\|C\|_{B(H)}M^2|f(x)|\,dx \tag{8.2}$$

hence E is trace class. Using the spectral representation of U_t, one can show that $AE+EA = BC$. Next we introduce $R_x = e^{-xA}Ee^{-xA}$ which gives a one parameter family of trace class operators such that $-\frac{dR_x}{dx} = AR_x + R_xA$.

□

Definition (Periodic linear system) Let $(e^{-xA})_{x\in\mathbf{R}}$ be a uniformly continuous group of operators on H such that $e^{2\pi A} = I$ and A is invertible. Suppose further that B and E are trace class operators on H, and that C is a bounded linear operator on H, such that $AE+EA = BC$. Then $\Sigma_\infty = (-A, B, C; E)$ is a periodic linear system with input, output and state spaces all equal to H. Whenever we define a parametrized family Σ_t of periodic linear systems, the input, output and state spaces are taken to be fixed; furthermore, A is taken fixed in the family.

We let $\mathcal{C} = \mathbf{C}/\pi\mathbf{Z}$ be the complex cylinder formed by identifying $w \sim z$ if $z - w \in \pi\mathbf{Z}$; we can choose equivalence class representatives in the strip $\{z : -\pi/2 < \Re z \leq \pi/2\}$; then we identify each π-periodic $f : \mathbf{C} \to X$ with a function $f : \mathcal{C} \to X$. Let $\mathbf{C}_\mathcal{C} = \mathbf{C}[\sin 2z, \cos 2z]$ and let $\mathbf{K}_\mathcal{C} = \mathbf{C}(\sin 2z, \cos 2z)$ be the field of trigonometric functions, which consists of functions of rational character on $\mathcal{C}$ in the sense that the elements are rational functions of $t = \tan z$. The space of entire π periodic functions on $\mathbf{C}$ may be identified with the space of holomorphic functions $\mathbf{H}_\mathcal{C}$ on $\mathcal{C}$, which is differential subring of the meromorphic functions $\mathbf{M}_\mathcal{C}$ on $\mathcal{C}$.

Definition (Operators) Adjusting the definitions of section 5 in a natural way, we let $\Phi(x) = Ce^{-xA}B$ be the operator scattering function so that $\phi(x) = \operatorname{trace}\Phi(x)$ is the scattering function and let $R_x = e^{-xA}Ee^{-xA}$, then we introduce $F_x = (I+e^{-xA}Ee^{-xA})^{-1}$, and $\tau_\infty(x) = -\det F_x$, then let $u(x) = -2\frac{d^2}{dx^2}\log\tau_\infty(x)$ be the potential. Let $\operatorname{Spec}(A)$ be the spectrum of A as an operator, and introduce the periodic linear system

$$\Sigma_\lambda = (-A, (\lambda I+A)(\lambda I-A)^{-1}B, C; (\lambda I+A)(\lambda I-A)^{-1}E) \qquad (\lambda \in (\mathbf{C}\cup\{\infty\})\setminus\operatorname{Spec}(A)) \tag{8.3}$$

and its accompanying tau function τ_λ. We also introduce the (noncommutative) algebra $\mathbf{S} = \mathbf{K}_\mathcal{C}\{I, A, BC, F_x\}$, and then let $\mathbf{A}$ be the subring of $\mathbf{S}$ spanned by A^{n_1} and $A^{n_1}FA^{n_2}\dots FA^{n_r}$ for $n_j \in \mathbf{N}$. We also introduce $\lfloor\,.\,\rfloor : \mathbf{S} \to \mathbf{M}_\mathcal{C}(c^1) : \lfloor P\rfloor = Ce^{-xA}FPFe^{-xA}B$. Let $\mathbf{A}_0 = \{\operatorname{trace}\lfloor P\rfloor : P \in \mathbf{A}\}$, which is analogous to the differential ring generated by the potential u.

The family $\{\Sigma_\lambda : \lambda \in (\mathbf{C}\cup\{\infty\})\setminus\operatorname{Spec}(A))\}$ is an operator model for the spectral curve in the sense that it serves as the domain of τ_λ. In Proposition 8.5, we show how to define τ_λ on the spectral curve of $-f''+uf = \lambda f$.

Theorem 8.2 *Let $(-A, B, C; E)$ be a periodic linear system.*

(i) Then $\phi(2x) \in \mathbf{C}_\mathcal{C}$, and $\mathbf{S}$ is a complex differential ring for $(-A, B, C; E)$ and for Σ_λ;

(ii) $\lfloor\mathbf{A}\rfloor$ is a complex differential ring on $\mathcal{C}$;

(iii) the derivatives $u^{(j)}$ of the potential belong to $\mathbf{M}_\mathcal{C}$ and to $\mathbf{A}_0$.

(iv) If $e^{-A\pi/2}Ee^{-A\pi/2} = -E$ then $T(x,y) = -Ce^{-xA}F_x e^{-yA}B$ satisfies

$$\Phi(x+y) + T(x,y) + \frac{1}{2}\int_x^{x+\pi/2} T(x,z)\Phi(z+y)\,dz = 0. \tag{8.4}$$

Proof. (i) First we show that A is an algebraic operator. By periodicity, the group $(e^{-xA})_{x\in\mathbf{R}}$ is bounded and hence by Sz.-Nagy's theorem, e^{xA} is similar to a unitary group on H, so A is similar to a skew symmetric operator. By uniform continuity, A is bounded, and hence has spectrum contained in $\{-iN,\dots,iN\}$ for some integer N; see [18]. Consequently, there exists a monic polynomial p such that $p(A)=0$.

Hence A is an invertible algebraic operator, so as in (6.5), A^{-1} is a polynomial in A and $(\lambda I + A)(\lambda I - A)^{-1} \in \mathbf{S}$ for all λ in the resolvent set of A. We also introduce polynomials p_j for each point in the spectrum of A such that $p_j(ik) = \delta_{jk}$, and since A is similar to a skew operator, we deduce that

$$e^{-xA} = \sum_{j=-N;j\neq 0}^{N} p_j(A)e^{-ijx}, \tag{8.5}$$

so $\Phi(x) = Ce^{-xA}B$ is a trigonometric polynomial with coefficients in c^1 and of degree less than or equal to N. Hence $\phi(2x)$ is π-periodic.

By (8.5) and (8.1), the operator E belongs to $\mathbf{S}$ and hence $R_x = e^{-xA}Ee^{-xA}$ also belongs to $\mathbf{S}$. Hence we have

$$\frac{d}{dx}R_x = -e^{-xA}AEe^{-xA} - e^{-xA}EAe^{-xA} = -e^{-xA}BCe^{-xA} \tag{8.6}$$

and so $AF + FA - 2FAF = Fe^{-xA}BCe^{-xA}F$, hence

$$\frac{dF}{dx} = AF + FA - 2FAF; \tag{8.7}$$

so $\mathbf{S}$ is a differential ring for $(-A,B,C)$.

(ii) From (8.7), we have the product rule

$$\lfloor P\rfloor\lfloor Q\rfloor = \lfloor P(AF+FA-2FAF)Q\rfloor, \tag{8.8}$$

and just as in Theorem 2.4

$$\frac{d}{dx}\lfloor P\rfloor = \Big\lfloor A(I-2F)P + \frac{dP}{dx} + P(I-2F)A\Big\rfloor. \tag{8.9}$$

As in Lemma 3.2,

$$\lfloor \mathbf{A}\rfloor = \mathrm{span}_{\mathbf{C}}\Big\{Ce^{-xA}FA^{n_1}Fe^{-xA}B, Ce^{-xA}FA^{n_1}FA^{n_2}\dots FA^{n_r}Fe^{-xA}B; n_j\in\mathbf{N}\Big\} \tag{8.10}$$

is a differential ring.

(iii) Since e^{-xA} is an entire operator function, we deduce that θ is entire, and π periodic since $\tau_\infty(x) = \det(I + e^{2xA}E)$ and $e^{2\pi A} = I$. When $\tau_\infty(x) \neq 0$, we have

$$\begin{aligned} \frac{d}{dx} \log\det(I + e^{-xA}Ee^{-xA}) &= -\mathrm{trace}\big((I + e^{-xA}Ee^{-xA})^{-1}e^{-xA}(AE + EA)e^{-xA}\big) \\ &= -\mathrm{trace}\big((I + e^{-xA}Ee^{-xA})^{-1}e^{-xA}BCe^{-xA}\big) \\ &= -\mathrm{trace}\big(Ce^{-xA}(I + e^{-xA}Ee^{-xA})^{-1}e^{-xA}B\big) \\ &= -\mathrm{trace}\big(Ce^{-xA}Fe^{-xA}B\big), \end{aligned} \tag{8.11}$$

and hence

$$\begin{aligned} u &= -2\frac{d^2}{dx^2} \log\det(I + e^{-xA}Ee^{-xA}) \\ &= -4\mathrm{trace}\, Ce^{-xA}FAFe^{-xA}B \\ &= -4\mathrm{trace}\, \lfloor A \rfloor; \end{aligned} \tag{8.12}$$

so u belongs to $\mathbf{A}_0 = \{\mathrm{trace}\lfloor P \rfloor : P \in \mathbf{A}\}$. Likewise, the derivatives $u^{(j)}$ belong to $\mathbf{A}_0$ since $\lfloor \mathbf{A} \rfloor$ is a differential ring.

(iv) One can verify this by direct computation, and the crucial identity is

$$\int_x^{x+\pi/2} e^{-zA}BCe^{-zA}\, dz = \big[-e^{-zA}Ee^{-zA}\big]_x^{x+\pi/2} = 2e^{-xA}Ee^{-xA}. \tag{8.13}$$

□

Remarks (i) If $(\pi/4)\|\Phi\|_\infty < 1$ in Theorem 8.2(iv), then

$$\frac{\partial^2}{\partial x^2}T(x,y) - \frac{\partial^2}{\partial y^2}T(x,y) = -2\Big(\frac{d}{dx}T(x,x)\Big)T(x,y), \tag{8.14}$$

as one can prove by substituting in the integral equation. This motivates the definition of u as the scalar potential, since $u(x) = -2\frac{d}{dx}\mathrm{trace}\, T(x,x)$ by (8.10).

If we assume more commutativity, the proofs simplify and the results become stronger.

Corollary 8.3 *Suppose further that $ABC = BCA$, and let $E = 2^{-1}A^{-1}BC$.*

(i) Then R_x satisfies (1.8) and (1.9);

(ii) $(-A, B, C)$ is finitely generated, since the algebra **S** *is commutative and Noetherian, and a complex state ring for $(-A, B, C)$ on $\mathcal{C}$.*

Proof. (i) Since A^{-1} and C are bounded and B is trace class, E is also trace class. Now $R_x = e^{-xA}Ee^{-xA}$ is an entire and trace class valued function, and using commutativity, one checks that Lyapunov's equation (1.8) holds. Unlike in Lemmas 2.1 and 6.1, we do not assert that the solution is unique.

(ii) Here e^{-xA} is a polynomial in A, e^{ix} and e^{-ix}, hence e^{-xA} and likewise R_x belong to $\mathbf{K}_{\mathcal{C}}[I, E, A]$. Observe that the set $S = \{(I + e^{-xA}Ee^{-xA})^n : n = 0, 1, \ldots\}$ is multiplicatively closed and does not contain 0 since $I + e^{-xA}Ee^{-xA}$ is invertible in the Calkin algebra of $\mathrm{B}(H)$

modulo the compact operators on H. Hence we can identify $\mathbf{S}$ with the ring of fractions of $\mathbf{K}_{\mathcal{C}}[A, BC]$ modulo S. There is a natural surjective ring homomorphism $\mathbf{K}_{\mathcal{C}}[X_1, X_2, X_3] \to \mathbf{S}$ given by $X_1 \mapsto A$, $X_2 \mapsto BC$, $X_3 \to F_x$, so by Hilbert's basis theorem, $\mathbf{S}$ is Noetherian as a commutative ring.

(iii) An ideal $\mathbf{p}$ of $\mathbf{S}$ is maximal, if and only if $\{\mathbf{p}\}$ is closed in the prime spectrum $\mathrm{Spec}(\mathbf{S})$, with the Zariski topology, in which case the field $\mathbf{S}/\mathbf{p}$ is isomorphic to a finite algebraic extension of $\mathbf{K}_{\mathcal{C}}$, by the weak form of Nullstellensatz as in [4].

Now for each $\alpha \in \mathbf{S}/\mathbf{p}$ there exist $a_j \in \mathbf{K}_{\mathcal{C}}$, with $a_n \neq 0$, such that $\sum_{j=0}^{j} a_j\alpha^j = 0$. By changing variables to $t = \tan x/2$, and multiplying by a suitable polynomial in t, we can introduce $q_j(z) \in \mathbf{C}[z]$ such that $\sum_{j=0}^{n} q_j(t)\alpha^j = 0$; thus (α, t) is associated with the curve $\{(w,t) : \sum_{j=0}^{n} q_j(t)w^j = 0\}$, which determines a Riemann surface $\mathcal{Y}$ which covers $\mathbf{P}^1$ finitely.

$\square$

We now consider the tau functions of periodic linear systems $(-A, B, C; D)$. By taking traces or forming determinants, we carry out limiting processes which generally take us from $\mathbf{K}_{\mathcal{C}}$ to $\mathbf{M}_{\mathcal{C}}$. The scattering function conveys information about the spectrum of A, while the zeros of τ_∞ determine the poles of u. This is made precise in the following result.

Proposition 8.4 *Let $(-A, B, C; E)$ be a periodic linear system as in Theorem 8.2, and let τ_λ be the tau function of Σ_λ.*

(i) The function $x \mapsto \tau_\lambda(x)$ is entire, while $\lambda \mapsto \tau_\lambda(x)$ is holomorphic on $\mathbf{C} \setminus \mathrm{Spec}(A)$.

(ii) $\tau_\infty \in \mathbf{H}_{\mathcal{C}}$ satisfies $\log_+\log_+|\tau_\infty(z)| \leq 2N|z| + c_1$ for some $c_1 > 0$ and all z, where N is the spectral radius of A.

(iii) Let $(\tau_\lambda) = \{z \in \mathbf{C} : \tau_\lambda(z) = 0\}$ for all $\lambda \in (-\infty, \infty) \cup \{\pm\infty\}$, which is either empty or countably infinite. Every zero of τ_λ gives rise to a double pole of $u_\lambda = -2(\log\tau_\lambda)''$.

(iv) If E has finite rank, then τ_∞ is of exponential type and in $\mathbf{C}_{\mathcal{C}}$. Conversely, if τ_∞ is of exponential type, then there exist $\alpha_j \in \mathcal{C}$, $\alpha \in \mathbf{Z}$ and $\beta \in \mathbf{C}$ such that

$$\tau_\infty(z) = e^{2i\alpha z+\beta}\prod_{j=1}^{m}\sin 2(z-\alpha_j) \tag{8.15}$$

and

$$u(z) = \sum_{j=1}^{m}\frac{8}{\sin^2 2(z-\alpha_j)}. \tag{8.16}$$

Proof. (i) Observe that $(\lambda I + A)(\lambda I - A)^{-1}$ is a polynomial in A with coefficients that are rational functions of λ, and holomorphic except when λ is in the spectrum of A; in particular it is holomorphic on $\{\lambda : |\lambda| < 1\} \cup \{\lambda : |\lambda| > \|A\|\}$. Hence τ_λ is a holomorphic function of λ, except at the points where λ is in the spectrum of A, which is a finite set.

(ii) The approximation numbers a_j satisfy $a_n(e^{-zA}Ee^{-zA}) \leq \|e^{-zA}\|^2 a_n(E)$ and hence by a standard bound on the determinant

$$\log|\det(I + e^{-zA}Ee^{-zA})| \leq c_0 e^{2N|z|}\sum_{j=1}^{\infty} a_j(E). \tag{8.17}$$

(iii) If $\tau_\lambda(z)=0$, then $\tau_\lambda(z+k\pi)=0$ for all $k\in\mathbf{Z}$.

(iv) There exists a projection P of finite rank ρ such that $PEP=E$ and hence $\tau_\infty(z)=\det(I+PEPe^{-2zA}P)$, where $Pe^{-2zA}P$ is a finite matrix with entries that are in $\mathbf{C}_{\mathcal{C}}$; in particular, the entries are functions of exponential growth. Hence from the expansion of this determinant, we deduce that there exist $c_1,c_2>0$ such that $\vert\tau_\infty(z)\vert\leq c_1e^{2\rho N\vert z\vert+c_2}$ for all z.

Suppose conversely that τ is of exponential type. Then by Jensen's formula, the number of zeros of τ_∞ inside a circle of radius r grows like c_3r+c_4 for some $c_3,c_4>0$, and since τ_∞ is also π-periodic, we deduce that there exists $m<\infty$ such that the only zeros of τ_∞ in $\{z:-\pi/2<\Re z\leq\pi/2\}$ are $\alpha_1,\dots,\alpha_m$; there there exists an entire function g such that

$$\tau_\infty(z)=e^{g(z)}\prod_{j=1}^m\sin 2(z-\alpha_j), \tag{8.18}$$

where g is an entire function such that $g(z+\pi)-g(z)=2\pi i\ell$ for some $\ell\in\mathbf{Z}$. Since $\vert\sin(x+iy)\vert\to\infty$ as $y\to\infty$, we deduce that $\vert g(z)\vert\leq c_5\vert z\vert+c_6$ for some $c_5,c_6>0$, and we finally obtain $g(z)=2i\alpha z+\beta$ where $\alpha\in\mathbf{Z}$.

By computing $u=-2(\log\tau_\infty)''$, we obtain a potential as in (8.10), which is a rational function of e^{ix} and e^{-ix}. In particular, when $m=1$ we have $u(z)=8/\sin^2 2(z-\alpha_1)$, so we can rescale this to the familiar case of $C\mathrm{sech}^2 z$ for some C.

□

Remark 8.4 The potential (8.16) can be interpreted in terms of a simple model in electrodynamics, considered by Sutherland [45]. Consider m fixed unit charges placed at points $e^{i\alpha_j}$ on a circular ring, and a further unit charge which has variable position e^{ix} on the ring. Then the electrostatic energy of the moving charge is u. In section 10, we show how this can otherwise be realised as a limiting case of periodic linear systems with elliptic potentials.

9. Tau functions and the Baker–Akhiezer function

Tau functions are intended to generalize Riemann's theta function on an algebraic curve. For any compact Riemann surface $\mathcal{E}$ of genus g, one can define a homology basis and a g-dimensional space of Abelian differentials of the first kind. Then one defines a corresponding lattice Λ of periods and a Jacobi variety $\mathbf{J}=\mathbf{C}^g/\Lambda$ with a period matrix Ω, and hence Riemann's theta function $\theta(x\mid\Omega)$ by (). Schottky [36, 44] asked how one can characterize the θ functions that arise from compact Riemann surfaces amongst all the possible functions $\theta(x\mid\Omega)$ on Abelian varieties as in (4.1). In this section consider the Kadomtsev–Petviashvili system of differential equations [50] that characterize those τ functions that arise from complete algebraic curves. The KP differential equations reduce to KdV equations in specific cases, and the KdV hierarchy is specifically associated with hyperelliptic curves. By considering the specific form of Hankel operators, we deduce that hyperelliptic curves give the theta functions that are most naturally associated with Hankel determinants as in Proposition 2.3. In this section, we restrict attention to the case in which the input and output space are both $\mathbf{C}$.

Given a tau function from a periodic linear system $(-A,B,C;E)$, we consider the conditions under which τ arises from the theta functions on a compact algebraic curve. First we consider families of linear systems as in Theorem 8.2, with common A, which are parametrized by

$\lambda \in \mathbf{P}^1 \setminus \mathrm{Spec}(A)$ and time parameters $(t_1, t_2, \dots)$, giving tau functions $\tau_\lambda(x,t)$. Initially x and t_j are real, and $\tau_\lambda(x,t)$ is π periodic in each variable, hence $\tau_\lambda(x,t)$ gives a periodic function on the infinite real torus $\mathbf{R}^\infty/\pi\mathbf{Z}^\infty$; then we extend to complex x and t_j, so that $\tau_\lambda(x,t)$ is entire. By forming quotients of such functions, we aim to realise typical tau functions.

To introduce the required linear systems, we let

$$\mathbf{T} = \{(x, t_1, t_2, \dots) \in \mathbf{R}^\infty : \limsup_{j\to\infty} |t_j|^{1/j} = 0\} \tag{9.1}$$

which gives an abelian group under addition, and for $(x,t) \in \mathbf{T}$, let $U(t) = \exp(-\sum_{j=1}^\infty t_j A^{2j+1})$, which gives a multi parameter group of operators such that $U(s+t) = U(s)U(t)$. Then we replace $\Sigma_\infty(0) = (-A, B, C; E)$ of Theorem 8.2 by

$$\Sigma_\lambda(t) = \Big(-A, (\lambda I + A)(\lambda I - A)^{-1}U(t)B, CU(t), (\lambda I + A)(\lambda I - A)^{-1}U(t)EU(t)\Big) \tag{9.2}$$

for $\lambda \in \mathbf{P}^1 \setminus \mathrm{Spec}(A)$. Each $\Sigma_\lambda(t)$ gives a space $\mathbf{A}_0(t,\lambda)$ of potentials as in Theorem 8.2(iii), while λ is a spectral parameter as in Proposition 8.5. Let $(\mathbf{A_0}, d/dx)$ be the differential ring generated by Σ as in Theorem 8.3(ii), and let $(\mathbf{A}_\infty, \partial/\partial x, \partial/\partial t_j)$ be the differential ring generated by all the $\Sigma_\lambda(t)$; then $\mathbf{A}_0 \subseteq \mathbf{A}_\infty$, and the inclusion splits by mapping $t_j \mapsto 0$ for all $j = 1, 2, \dots$.

Definition (Baker–Akhiezer function) We define the quotient

$$\psi_{BA}(x,t;\lambda) = \exp\Big(x\lambda + \sum_{j=1}^\infty t_j\lambda^{2j+1}\Big)\frac{\tau_\infty\Big(x - \frac{1}{\lambda}, t_1 - \frac{1}{3\lambda^3}, t_2 - \frac{1}{5\lambda^5}, \dots\Big)}{\tau_\infty(x, t_1, t_2, \dots)} \tag{9.3}$$

to be the Baker–Akhiezer function of the periodic linear system $(-A, B, C; E)$ under $U(t)$.

This definition is consistent with section 2, but we cannot expect a precise analogue of Proposition 2.5(iii), which expresses eigenfunctions in terms of ψ_{BA}. The term Baker–Akhiezer function is used in various senses in the literature, as we briefly review.

Krichever [29] defines Baker–Akhiezer functions $\psi(x,\lambda)$ for λ in a nonsingular algebraic curve $\mathcal{E}$, except at a distinguished finite set of points $p_j \in \mathcal{E}$ which are independent of x, so that $\lambda \mapsto \psi(x,\lambda)$ is meromorphic, and $\psi(x,\lambda)$ has an exponential asymptotic expansion near p_j in terms of local coordinates; see [25]. One can construct such a function from quotients of Riemann's theta function. To deal with commuting families of differential operators of rank greater than one, he introduces matricial $\psi(x,\lambda)$ in [30].

Given a nonsingular algebraic curve $\mathcal{E}$ with distinguished point p, Shiota [44] introduces Baker–Akhiezer functions as quotients of Riemann's theta functions, so they are meromorphic on $\mathcal{E} \setminus \{p\}$ by construction. In contrast, our ψ_{BA} ius defined for linear systems, irrespective of whether there exists a suitable $\mathcal{E}$.

Lemma 9.2 *(i) The scattering function* $\Phi_\lambda(x,y) = CU(t)e^{-xA}(\lambda I + A)(\lambda I - A)^{-1}U(t)B$ *for* $\Sigma_\lambda(t)$ *satisfies*

$$\frac{\partial^{2j+1}}{\partial x^{2j+1}}\Phi_\lambda(x,t) + \frac{\partial}{\partial t_j}\Phi_\lambda(x,t) = 0. \tag{9.4}$$

(ii) $\tau_\lambda(x,t)$ is holomorphic for $(x,t,\lambda) \in \mathcal{C} \times \mathcal{C}^\infty \times (\mathbf{P}^1 \setminus \mathrm{Spec}(A))$, where $\mathcal{C} = \mathbf{C}/\pi\mathbf{Z}$ is the complex cylinder.

(iii) $\lambda \mapsto \psi_{BA}(x,t,\lambda)$ is holomorphic on $\mathbf{C} \setminus \mathrm{Spec}(A)$, while $(x,t) \mapsto \psi_{BA}(x,t,\lambda)$ is meromorphic and quasiperiodic with respect to the lattice $\pi\mathbf{Z}^\infty$ in $\mathbf{C}^\infty$.

Proof. (i) Since $U(t)$ is actually analytic in each t_j this is a straightforward computation.

(ii) First we observe that

$$\tau_\lambda(x,t) = \tau_\infty\Big(x - \frac{1}{\lambda}, t_1 - \frac{1}{3\lambda^3}, t_2 - \frac{1}{5\lambda^5}, \dots\Big), \tag{9.5}$$

which shows that our definition is consistent with Shiota's [44]. To see this, we start with the numerator and use the elementary identity,

$$(\lambda I + A)(\lambda I - A)^{-1} = \exp\Big(2\sum_{j=1}^\infty \frac{A^{2j+1}}{(2j+1)\lambda^{2j+1}}\Big), \tag{9.6}$$

where the series $\sum_{j=1}^\infty A^{2j+1}/(2j+1)\lambda^{2j+1}$ converges for $|\lambda > \|A\|$, so we can use this as a definition of the right-hand side for all λ outside the spectrum of A. Hence we rearrange the factors in the determinant

$$\tau_\lambda(x,t) = \det\Big(I + (\lambda I + A)(\lambda I - A)^{-1} U(2t) e^{-2xA} E\Big). \tag{9.7}$$

to obtain (9.4). Hence $\lambda \mapsto \tau_\lambda(x,t)$ is holomorphic on $\mathbf{P}^1 \setminus \mathrm{Spec}(A)$, and $(x,t) \mapsto \tau_\lambda(x,t)$ is entire in each variable since A is bounded. The spectrum of A^{2j+1} is contained in $\{-iN^{2j+1}, -i(N-1)^{2j+1}, \dots, iN^{2j+1}\}$, so $e^{2\pi A^{2j+1}} = I$, and by Theorem 8.2 $\tau_\lambda(x+\pi,t) = \tau_\lambda(x,t)$; likewise $\tau_\lambda(x,t)$ is unchanged by adding π to t_j; so $\tau_\lambda(x,t)$ is periodic with respect to $\pi\mathbf{Z}^\infty$ in $\mathbf{C}^\infty$.

(iii) The function $\sum_{j=1}^\infty t_j\lambda^{2j+1}$ is entire by the choice of $(x,t) \in \mathbf{T}$, so $\lambda \mapsto \psi_{BA}(x,t,\lambda)$ is holomorphic on $\mathbf{C} \setminus \mathrm{Spec}(A)$. With $(e_j)_{j=0}^\infty$ the standard unit vector basis in $\mathbf{T}^\infty$, we deduce from (ii) that $\psi_{BA}(x, t+\pi e_j, \lambda) = e^{2\pi\lambda^{2j+1}}\psi_{BA}(x,t,\lambda)$, and $(x,t) \mapsto \psi_{BA}(x,t,\lambda)$ is meromorphic.

$\square$

In particular, suppose that $\tau(t)$ is the tau function that arises from a periodic linear system as in Theorem 8.2. Given a linear map $\alpha : \mathbf{C}^g \to \mathbf{C}^\infty$ of rank g such that $\alpha(e_j) \in \mathbf{Z}^\infty$ has only finitely many non-zero entries with resepct to the standard bases, then $\alpha^t : \mathbf{C}^\infty \to \mathbf{C}^g$ satisfies $\alpha^t(\mathbf{Z}^\infty) \subseteq \mathbf{Z}^g$. Then $\tau \circ \alpha : \mathbf{C}^g \to \mathbf{C}$ is entire and periodic with respect to $\mathbf{Z}^g$.

Proposition 9.3 (Shiota and Mulase) *Suppose that $\tau \circ \alpha(t) = \theta(t \mid \Omega)$, where θ is Riemann's theta function for an Abelian variety $\mathbf{X} = \mathbf{C}^g/\Lambda$ of dimension g; let $Q(x,y,s)$ be a quadratic form, let $\beta, \gamma, \delta, \zeta \in \mathbf{C}^g$ with $\beta \neq 0$, and for*

$$\sigma(x,y,s;\zeta) = e^{Q(x,y,s)}\theta(\beta x + \gamma y + \delta s + \zeta \mid \Omega), \tag{9.8}$$

let $u(x,y,s;\zeta) = -2\frac{\partial^2}{\partial x^2}\log\sigma(x,y,s;\zeta)$. Then the following two conditions are equivalent:

(i) the θ divisor is irreducible, and u satisfies the KP equation

$$\frac{\partial}{\partial x}\Big(\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}-4\frac{\partial u}{\partial s}\Big)+3\frac{\partial^2 u}{\partial y^2}=0, \tag{9.9}$$

for all $\zeta\in \mathbf{C}^g$;

(ii) $\mathbf{X}$ is isomorphic to the Jacobian variety of a complete algebraic curve.

Proof. See [36, 44].

□

The solution u to KP is associated with a scattering function $\Psi(x,z;s)$ as in (4.23). We impose the extra condition $\Psi(x,z;s)=\phi(x{+}z;s)$, so that we can realise τ from the determinant of a linear system. This in turn imposes additional conditions on the algebraic curve, as in the following result. Following Krichever and Novikov [30], we consider the operators

$$L_1=\frac{\partial}{\partial x}-\begin{bmatrix}0&1\\ u-k&0\end{bmatrix}, L_2=\frac{\partial}{\partial y}-\begin{bmatrix}-k&0\\ 0&-k\end{bmatrix},$$

$$L_3=\frac{\partial}{\partial t}-\begin{bmatrix}\frac{1}{4}\frac{\partial u}{\partial x}&-k-\frac{u}{2}\\ k^2-\frac{ku}{2}-\frac{u^2}{2}+\frac{1}{4}\frac{\partial^2 u}{\partial x^2}&-\frac{1}{4}\frac{\partial u}{\partial x}\end{bmatrix} \tag{9.10}$$

note that $k\mapsto L_j$ is a polynomial for $j=1,2,3$, and that $\mathrm{trace}(L_j)=0$.

Lemma 4.5 *Suppose that $\phi(x;t)=Ce^{\lambda At+2A^3t/\alpha}e^{-xA}B$*

(i) Then $\Psi(x,z;t)=\phi(x+z;t)$ satisfies the scattering equations () for the KP equation.

(ii) Suppose that $u(x,t)$ satisfies KdV

$$\frac{\partial u}{\partial t}=\frac{1}{4}\frac{\partial^3 u}{\partial x^2}-\frac{3}{2}u\frac{\partial u}{\partial x}. \tag{9.11}$$

Then u gives a solution to KP, and L_1, L_2 and L_3 commute.

Proof. Then (3.6) implies that $\partial\Phi/\partial y=0$, and hence reduces (4.13) to

$$\frac{\alpha}{2}\frac{\partial\phi}{\partial t}+\frac{\partial^3\phi}{\partial x^3}+\lambda\frac{\partial\phi}{\partial x}=0 \tag{9.12}$$

which is the linear version of KdV. Further the KP equation degenerates to an equation of KdV type, hence u gives a solution to KP. One checks by direct computation that the L_j commute.

□

By Lemma 4.5, solutions of KdV give solutions of KP, and the corresponding scattering functions give Hankel integral operators.

Consider Hill's equation

$$\frac{d}{dx}\begin{bmatrix}f\\ v\end{bmatrix}=\begin{bmatrix}0&1\\ u-\lambda&0\end{bmatrix}\begin{bmatrix}f\\ v\end{bmatrix}\qquad(-\infty<x<\infty) \tag{9.13}$$

where u is continuous, complex-valued and π-periodic on $\mathbf{R}$. Invoking Floquet's theorem, we let $f_\pm(x) = e^{\pm i\alpha x}p_\pm(x)$ be solutions, where $p_\pm(x)$ are π-periodic. Suppose momentarily that $e^{2\pi i\alpha} \neq 1$. Now let

$$F_\lambda(x) = \begin{bmatrix} f_+(x) & f_-(x) \\ f'_+(x) & f'_-(x) \end{bmatrix} \tag{9.14}$$

and note that $\det F_\lambda(0) \neq 0$; then let $U_\lambda(x) = F_\lambda(x)F_\lambda(0)^{-1}$, which gives a fundamental solution matrix with $U_\lambda(0) = I$. Now let $M_\lambda = U_\lambda(0)^{-1}U_\lambda(\pi)$ be the monodromy matrix, which has the same eigenvalues $e^{i\alpha\pi}$ and $e^{-i\alpha\pi}$ as $F_\lambda(0)^{-1}F_\lambda(\pi)$; then let $\Delta(\lambda) = \operatorname{trace} M_\lambda$ be the discriminant of Hill's equation. Observe that when α is real, or equivalently $e^{i\alpha\pi} + e^{-i\alpha\pi} \in [-2, 2]$, the matrix F_λ gives bounded solutions $f_\pm(x)$ to Hill's equation on the real line. Hence the Bloch spectrum $\{\lambda \in \mathbf{R} : \Delta(\lambda)^2 \leq 4\}$ consists of those points such that Hill's equation has a pair of independent bounded solutions. Each oval O_n is associated with a gap in the Bloch spectrum [19].

Definition The multiplier curve is $\{(\lambda, z) : z^2 - \Delta(\lambda)z + 1 = 0\}$, and potentials are said to belong to the same spectral equivalence class if their multiplier curves are equal.

We now consider how the results of section 7 relate to the notions of Liouville integrability and finite gap integration. The results of this section are essentially corollaries of some subtle results proved elsewhere, and the most interesting relate to elliptic potentials.

Definition (Stationary KdV hierarchy) (i) Let $g_1 = -(1/4)u$. Then the KdV recursion formula is

$$4\frac{d}{dx}g_{m+1}(x) = 8g_1(x)\frac{d}{dx}g_m(x) + 8\frac{d}{dx}\big(g_1(x)g_m(x)\big) + \frac{d^3}{dx^3}g_m(x). \tag{9.15}$$

The solutions may depend upon constants of integration; if the constants of integration are chosen all to be zero, so that $g_2 = (3/16)u^2 - (1/16)u''$ *etc*, then the g_m give the homogeneous KdV hierarchy. In this case, the differential equations $g_m = 0$ are known as Novikov's equations; see [23, 24].

(ii) If u satisfies $g_m = 0$ for all m greater than or equal to some m_0, then u satisfies the KdV hierarchy and is said to be an algebro-geometric (finite gap) potential.

The solutions of (8.1) turn out to be complicated polynomials in u and its derivatives, as one can prove by induction. Nevertheless, we can express a solution g_m simply in terms of $\lfloor \mathbf{A} \rfloor$. The following proposition is a compilation of known results, and included for completeness.

Proposition 8.8 *Let $(-A, B, C; E)$ be as in Theorem 8.2.*

(i) Then the functions $g_m(x) = \lfloor A^{2m-1} \rfloor$ for $m = 1, 2, \dots$ satisfy the KdV recurrence relation (8.1).

(ii) The complex vector space spanned by the g_m is finite-dimensional.

(iii) If $\lfloor A^{2m-1} \rfloor = 0$ for some m, then u is finite gap and there exists a hyperelliptic curve $\mathcal{E}$ such that $u \in \mathbf{K}_\mathcal{E}$ and $-\psi'' + u\psi = \lambda\psi$ is Liouville integrable over $\mathbf{K}_\mathcal{E}$.

Proof. (i) By repeatedly using (8.1), one can prove that

$$\frac{d^3}{dx^3}\lfloor A^{2m+1} \rfloor = -96\lfloor A^{2m+4}(I - 2F)(F - F^2) \rfloor + 8\lfloor A^{2m+4}(I - 2F) \rfloor, \tag{9.16}$$

and by (8.5) and (8.6)

$$\frac{d}{dx}\big(\lfloor A\rfloor\lfloor A^{2m+1}\rfloor\big) = 8\lfloor A^{2m+4}(I-2F)(F-F^2)\rfloor; \tag{9.17}$$

and the recurrence relation follows from such identities.

(ii) Let m be the minimal polynomial of degree N for the algebraic operator A. Then for each entire function f, either $f(A)=0$ or there exists a polynomial r of degree less than or equal to N such that $f(A)=r(A)$. Hence the span of the A^{2m-1} for $m=1,2,\dots$ is finite-dimensional, and hence its image under $\lfloor\,.\,\rfloor$ is also finite-dimensional.

(iii) By Lemma 8.3, $g_m=0$, and so from the recurrence relation we deduce that $g_n=0$ for all $n\geq m$, so u is finite gap and $\mathbf{C}[\lambda,u,u',u'',\dots]=\mathbf{C}[\lambda,u,u',u'',\dots,u^{(m+1)}]$ is a differential ring. Any solution of the stationary KdV equations is meromorphic on $\mathbf{C}$ [42, 6.10]. Let $\lambda_0<\lambda_1<\dots<\lambda_{2g}$ be the simple zeros of $4-\Delta(\lambda)^2=0$, and introduce the spectral curve

$$\mathcal{E} = \Big\{(z,w): w^2=\prod_{j=0}^{2g}(z-\lambda_j)\Big\}\cup\{(\infty,\infty)\Big\}, \tag{9.18}$$

Now there exists a solution $\rho(x,\lambda)$ to Drach's equation (8.2)

$$\mu^2 = -\frac{1}{2}\rho(x,\lambda)\rho''(x,\lambda)+\frac{1}{4}\rho'(x,\lambda)^2+(u(x)+\lambda)\rho(x,\lambda)^2 \tag{9.19}$$

such that $\mu(\lambda)$ is independent of x and $\lambda\mapsto\rho(x,\lambda)$ is a polynomial, which we factor as $\rho(x,\lambda)=\prod_{j=1}^{g}(\lambda-\gamma_j(x))$. Brezhnev [12] gives the solution

$$\psi_\pm(x) = \exp\Big(\sum_{j=1}^{g}\int^{\gamma_j(x)}\frac{(w\pm\mu)dz}{(z-\lambda)w}\Big), \tag{9.20}$$

where the integral is taken along $\mathcal{E}$. Here u and its derivatives are rational functions on $\mathcal{E}$; see [29, 43]. For such a potential u, the functions $\psi_\pm$ of () give locally meromorphic solutions to Schrödinger's equation.

Definition (Torus) Let $(\tau_\infty)=\{p_n: n=1,2,\dots\}$ and O_n be the real oval in $\cup_{\lambda\in(-\infty,\infty]}(\tau_\lambda)$ that is based upon p_n. Then let $\mathcal{T}^\infty_{\mathbf{R}}=\prod_{n=1}^\infty O_n$ and consider $\mathbf{z}_\lambda=\{z_n: n=1,2,\dots\}=(\tau_\lambda)$ with $z_n\in O_n$. Then $\mathbf{z}_\lambda\in\mathcal{T}^\infty_{\mathbf{R}}$ is the pole divisor of $\psi_{BA}(x,\lambda)$ in the infinite real torus $\mathcal{T}^\infty_{\mathbf{R}}$.

Proposition 9.2 *(i) The Baker–Akhiezer function $\psi_{BA}(x;\lambda)$ belongs to a Liouvillian extension of the field of fractions of $\mathbf{A}_0$ and satisfies, in the notation of Theorem 8.2,*

$$\psi_{BA}(x,\lambda) = e^{\lambda x}\det\Big(I-\int_x^\infty T(x,y)e^{\lambda(y-x)}dy\Big) \qquad (\Re\lambda<0). \tag{9.21}$$

(ii) Suppose that Σ is a block diagonal direct sum $\oplus_{j=1}^\infty\Sigma_j$, where Σ_j is a periodic linear system with T_j as in Theorem 8.2. Then

$$\psi_{BA}(x,\lambda) = e^{\lambda x}\prod_{j=1}^\infty\det\Big(I-\int_x^\infty T_j(x,y)e^{\lambda(y-x)}dy\Big) \qquad (\Re\lambda<0). \tag{9.22}$$

(iii) Suppose that B and C have rank one. Then

$$-\psi''_{BA}(x,\lambda)+u(x)\psi_{BA}(x,\lambda)=-\lambda^2\psi_{BA}(x,\lambda). \tag{9.23}$$

(iv) If τ_λ has only simple zeros, then each zero of $\psi_{BA}(z,\lambda)$ in (τ_λ) processes in a real oval based at a pole of $\psi_{BA}(z,\lambda)$ in (τ_∞) as λ describes $(-\infty,\infty)$. The pole divisor defines a map $\Sigma_\lambda\mapsto \mathbf{z}_\lambda$ from the periodic linear system to the real torus $\mathcal{T}^\infty_{\mathbf{R}}$.

(v) If E has finite rank, then $\lambda\mapsto\psi_\lambda$ is meromorphic on $\mathbf{C}$ with the only possible poles being on the spectrum of A.

(vi) Suppose that u has finite gap, so that its spectral curve $\mathcal{E}$ is hyperelliptic, and let p_0 be a branch point. Then there exists a meromorphic function λ on $\mathcal{E}$, and a pair of distinct points $p_j,q_j\in\mathcal{E}$ for each point $ij\in$ Spec(A), all independent of x, such that $\lambda\mapsto\psi_{BA}(x,\lambda)$ is holomorphic on $\mathcal{E}\setminus\{p_j,q_j: j=0; ij\in\mathrm{Spec}(A)\}$.

Proof. (i) We have $u(x,\lambda)=-2(\log\tau_\lambda)''$ in $\mathbf{A}_0$ by Theorem 8.2, hence $\psi''_{BA}(x,\lambda)$ belongs to $\mathbf{A}_0$; we integrate this to obtain ψ_{BA} in some Liouville extension. By some simple manipulations, we have

$$\det\Bigl(I+R_x(\lambda I+A)(\lambda I-A)^{-1}\Bigr)=\det(I+R_x)\det\Bigl(I+(\lambda I-A)^{-1}(AR_xx+R_xA)(I+R_x)^{-1}\Bigr) \tag{9.24}$$

where $AR_x+R_xA=e^{-xA}BCe^{-xA}$, and hence

$$\frac{\det\Bigl(I+R_x(\lambda I+A)(\lambda I-A)^{-1}\Bigr)}{\det(I+R_x)}=\det\Bigl(I+Ce^{-xA}(I+R_x)^{-1}(\lambda I-A)^{-1}e^{-xA}B\Bigr), \tag{9.25}$$

and $\int_x^\infty e^{\lambda(y-x)}e^{-yA}=-(\lambda I-A)^{-1}e^{-xA}$, which leads to the stated identity. Moreover, the right-hand side is analytic in λ when $|\lambda|>\|A\|$, and $\psi_{BA}(x,\lambda)=e^{\lambda x}(1+O(\lambda^{-1}))$ as $|\lambda|\to\infty$.

By the proof of Theorem 8.2, $\frac{d^2}{dx^2}\log\psi_{BA}(x;\lambda)$ belongs to $\mathbf{A}_0$.

(ii) This follows immediately from (i).

(iii) We reduce to the case of the admissible linear system $(-A-\varepsilon I,B,C)$, which has input and output space $\mathbf{C}$, as in Proposition 2.5. For $\varepsilon>0$, let $R_x^{(\varepsilon)}=e^{-2\varepsilon x}e^{-xA}Ee^{-xA}$, so that $R_x^{(\varepsilon)}\to 0$ exponentially fast as $x\to\infty$, and $R_x^{(\varepsilon)}$ satisfies the Lyapunov equations

$$-\frac{d}{dx}R_x^{(\varepsilon)}=(A+\varepsilon I)R_x^{(\varepsilon)}+R_x^{(\varepsilon)}(A+\varepsilon I), \tag{9.26}$$

with

$$-\frac{d}{dx}R_x^{(\varepsilon)}|_{x=0}=BC+2\varepsilon E. \tag{9.27}$$

Since BC and E are trace class, we can introduce $\tau_\infty^{(\varepsilon)}(x)=\det(I+R_x^{(\varepsilon)})$ and

$$\tau_\lambda^{(\varepsilon)}(x)=\det(I+R_x^{(\varepsilon)}(\lambda I+\varepsilon I+A)(\lambda I-\varepsilon I-A)^{-1}), \tag{9.28}$$

whenever $\lambda - \varepsilon$ is in the resolvent set of A; likewise we can introduce $u^{(\varepsilon)}(x) = -2\frac{d^2}{dx^2}\log \tau_\infty^{(\varepsilon)}(x)$. Now the Baker–Akhiezer function

$$f^{(\varepsilon)}(x,k) = e^{ikx}\,\frac{\tau^{\varepsilon}_{ik}(x)}{\tau^{(\varepsilon)}_\infty(x)} \tag{9.29}$$

satisfies

$$-\frac{d^2}{dx^2}f^{(\varepsilon)}(x) + u^{(\varepsilon)}(x)f^{(\varepsilon)}(x,k) = k^2 f^{(\varepsilon)}(x,k); \tag{9.30}$$

letting $\varepsilon \to 0$, we obtain

$$-\frac{d^2}{dx^2}f(x) + u(x)f(x,k) = k^2 f(x,k); \tag{9.31}$$

as required.

(iv) Clearly the poles of $\psi_{BA}(z,\lambda)$ occur at the zeros of $\tau_\infty(z)$, and hence form the set (τ_∞), for all λ. The zeros of $\psi_{BA}(z,\lambda)$ form the set (τ_λ), which does vary with λ. The subset $\{(\lambda I - A)(\lambda I + A)^{-1} : \lambda \in \mathbf{R}\}$ of $B(H)$ is compact in the norm topology since the spectrum of A is separated from $\mathbf{R}$; hence τ_λ gives a compact family of holomorphic functions for the topology of uniform convergence on compact sets, with $\tau_{-\infty}(z) = \tau_\infty(z)$. For each bounded open subset Ω of $\mathbf{C}$, the set $\{z \in \Omega : \tau_\lambda(z) = 0\}$ has a uniformly bounded number of terms for $-\infty \leq \lambda \leq \infty$, by Jensen's formula and the Lemma 8.4. Each zero depends continuously upon λ by the inverse function theorem, and describes an oval for $-\infty \leq \lambda \leq \infty$.

(v) Suppose that E has finite rank, and note that $(\lambda I + A)(\lambda I - A)^{-1}E$ is a rational function with values in the space of operators on a finite-dimensional Hilbert space. Hence the determinant τ_λ is meromorphic as a function of λ on $\mathbf{P}^1$.

(vi) Suppose that $\mathcal{E}$ has genus $g \geq 2$, and choose p_0 to be one of the $2g+2$ branch points of the holomorphic two sheeted cover $\mathcal{E} \to \mathbf{P}$, and then observe that there exists a meromorphic function λ on $\mathcal{E}$ such has precisely one pole, namely a double pole at p_0, and hence has degree two (When $g = 1$, we can use $\lambda(p) = \wp(p - p_0)$).

The exponential $e^{x\lambda}$ gives an essential singularity in the variable λ for p close to p_0. As in (iv), $\lambda \mapsto (\lambda I + A)(\lambda I - A)^{-1}E$ is a rational function, with trace class values, and the only possible poles are on the spectrum of A; hence $p \mapsto \tau_\lambda$ gives a holomorphic function, except at finitely many points of $\mathcal{E}$, which we list as p_j, q_j for ij in the spectrum of A.

$\square$

Definition Say that a periodic linear system $(-A, B, C; E)$ is a Picard system if $-\psi'' + u\psi(x) = \lambda^2\psi$ has a meromorphic general solution ψ for all but finitely many $\lambda \in \mathbf{C}$. See [25].

Suppose that $(-A, B, C; E)$ is a Picard system. Then by elementary Floquet theory, there exists a nontrivial solution ψ such that $\psi(x + \pi) = \rho\psi(x)$ for all x.

In section 11, we will produce linear flows on $\mathcal{T}^\infty_{\mathbf{R}}$ from group actions on the linear system.

Given $u \in \mathbf{K}_{\mathcal{C}}$, one can ask whether u is finite gap, and seek to find the spectral curve. Gesztesy and Weikard found a conceptually simple characterization of elliptic potentials that are finite gap, namely those that are Picard potentials. In the next section, we realise some elliptic potentials u that are finite gap in terms of linear systems.

10. Linear systems with elliptic potentials

In this section we produce explicit examples of periodic linear systems such that u is finite gap, and the corresponding spectral curve $\mathcal{E}$ is of arbitrary genus.

Definition (Elliptic functions) Suppose that $\Lambda = \mathbf{Z}2\omega_1 + \mathbf{Z}2w_2$ with $\Im(\omega_2/\omega_1) > 0$ is a lattice, and let $\mathcal{T} = \mathbf{C}/\Lambda$ is the torus, and $\mathcal{C} = \mathbf{Z}/2\pi\mathbf{Z}$ the cylinder. A meromorphic function on $\mathbf{C}$ is elliptic (of the first kind) if it is doubly periodic with respect to Λ; let $\mathbf{K}^1_{\mathcal{T}}$ be the differential field of elliptic functions. A meromorphic function is elliptic of the second kind if there exist multipliers $\rho_j \in \mathbf{C}$ such that $f(z+2\omega_j) = \rho_j f(z)$; so that f is quasi-periodic with respect to the lattice; let $\mathbf{K}^2_{\mathcal{T}}$ be the field of elliptic functions of the second kind. Also let $\mathbf{K}^3_{\mathcal{T}}$ be the set of elliptic functions of the third kind, namely the meromorphic functions on $\mathbf{C}$ that satisfy $f(z+2\omega_j) = e^{a_j z + b_j} f(z)$ for $j = 1, 2$ and some $a_j, b_j \in \mathbf{C}$. Let $\mathbf{M}_{\mathcal{C}}$ be the differential field of 2π-periodic meromorphic functions; then $\mathbf{K}_{\mathcal{T}} \subset \mathbf{K}^2_{\mathcal{T}} \subset \mathbf{K}^3_{\mathcal{T}} \subset \mathbf{M}_{\mathcal{C}}$, where all there spaces are closed under multiplication. See [33].

First we shall obtain a representation for the coordinate ring $\mathbf{C}_{\mathcal{T}}$ of regular functions on elliptic curve

$$\mathcal{T} = \{(X, Z) : Z^2 = 4(X - e_1)(X - e_2)(X - e_3)\} \cup \{(\infty, \infty)\}. \tag{10.1}$$

Let θ_1 be Jacobi's elliptic theta function, $\theta_1^*(z)$ be the entire function $\theta_1^*(z) = \overline{\theta_1(\bar{z})}$ and let $\wp$ be Weierstrass's elliptic function with real constants $e_3 < e_2 < e_1$. Then $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ so a typical point on $\mathcal{T}$ is $(X, Z) = (\wp, \wp')$; moreover $\mathbf{K}^1_{\mathcal{T}} = \mathbf{C}(\wp)[\wp']$.

Definition (Realising elliptic theta functions) (i) We refine the basic construction from [10] so as to ensure that the various matrices commute. Let $H = \oplus_{n=0}^{\infty} \mathbf{C}^2$ be expressed as a space of column vectors and let

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{10.2}$$

then for an elliptic nome $0 < \mathrm{q} < 1$, we introduce the block diagonal matrices on H with 2×2 blocks, in which each top left corner is exceptional:

$$A_0 = \begin{bmatrix} (1/2)J & 0 & 0 & 0 & \dots \\ 0 & J & 0 & 0 & \dots \\ 0 & 0 & J & 0 & \dots \\ 0 & 0 & 0 & J & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad B_0 = -\begin{bmatrix} iI & 0 & 0 & 0 & \dots \\ 0 & 2\mathrm{q}^2 I & 0 & 0 & \dots \\ 0 & 0 & 2\mathrm{q}^4 I & 0 & \dots \\ 0 & 0 & 0 & 2\mathrm{q}^8 I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$C_0 = \begin{bmatrix} I & 0 & 0 & 0 & \dots \\ 0 & J & 0 & 0 & \dots \\ 0 & 0 & J & 0 & \dots \\ 0 & 0 & 0 & J & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad E_0 = -\begin{bmatrix} -iJ & 0 & 0 & 0 & \dots \\ 0 & \mathrm{q}^2 I & 0 & 0 & \dots \\ 0 & 0 & \mathrm{q}^4 I & 0 & \dots \\ 0 & 0 & 0 & \mathrm{q}^8 I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{10.3}$$

Then, with $A^\dagger$ standing for the Hermitian conjugate of A, we introduce

$$\begin{aligned} A &= \begin{bmatrix} A_0 & 0 \\ 0 & A_0^\dagger \end{bmatrix}, \quad B = \begin{bmatrix} B_0 & 0 \\ 0 & B_0^\dagger \end{bmatrix} \\ C &= \begin{bmatrix} C_0 & 0 \\ 0 & C_0^\dagger \end{bmatrix}, \quad E = \begin{bmatrix} E_0 & 0 \\ 0 & E_0^\dagger \end{bmatrix} \end{aligned} \tag{10.4}$$

Given $\lambda \in \mathbf{C} \setminus \{\pm i\}$, we introduce α by $(\lambda I - J)(\lambda I + J)^{-1} = I \cos 2\alpha - J \sin 2\alpha$; so the effect of multiplying B by $(\lambda I - A)(\lambda I + A)^{-1}$ is equivalent to $x \mapsto x + \alpha$.

Proposition 10.1. *(i) The hypotheses of Theorem 8.4 are satisfied, so $e^{-xA}Ee^{-xA}$ defines a trace class operator on H, and $\det(I + e^{-xA}Ee^{-xA})$ is an elliptic function of the third kind which satisfies*

$$\theta_1(x)\theta_1^*(x) = \det\big(I + e^{-xA}Ee^{-xA}\big)|\mathrm{q}|^{1/2} \prod_{n=1}^\infty (1 - q^{2n})^2, \tag{10.5}$$

where $\theta_1(x)\theta_1^*(x)$ is entire and nonzero on $\mathbf{C} \setminus \{j\pi + ik \log \mathrm{q} : j, k \in \mathbf{Z}\}$.

(ii) Let $\mathbf{S} = \mathbf{K}_{\mathcal{C}}[I, A, B, C, F]$. Then $\mathbf{S}$ is a commutative and Noetherian ring of block diagonal matrices with entries from $\mathbf{K}_{\mathcal{C}}$; furthermore, $\mathbf{S}$ is a complex differential ring for $(-A, B, C)$ on $\mathbf{C}/4\pi\mathbf{Z}$.

(iii) The potential $u(x) = -4\mathrm{trace}\lfloor A \rfloor$ is the elliptic function

$$u(x) = 4\wp(x) - 4e_1 - 2\big(\log \theta_1\theta_1^*\big)''(1/2). \tag{10.6}$$

(iv) Then $u(x, t) = u(x - ct)$ gives the general travelling wave solution of the Korteweg–de Vries equation

$$\frac{\partial^3 u}{\partial x^3} = 3u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \tag{10.7}$$

that has speed $c = 4(e_1 + e_2 + e_3) - 3e_1 - (3/2)\big(\log \theta_1\theta_1^*\big)''(1/2)$.

(v) Let $\mathbf{A}_0 = \mathrm{span}_{\mathbf{C}}\{1, \wp^{(j)}(x) : j = 0, 1, 2, \dots\}$ and $\mathbf{A}$ be as in Lemma 3.2. Then $\mathbf{A}_0 = \mathbf{C}[\mathcal{T}]$, and every element of $\mathbf{A}_0$ with zero constant term is the trace of some element of $\mathbf{A}$.

(vi) The scattering function satisfies

$$\phi(x) = \frac{-8\mathrm{q}^2}{1 - \mathrm{q}^2} \sin x. \qquad (x \in \mathbf{R}) \tag{10.8}$$

Proof. (i) The matrix J satisfies the identities $e^{-xJ} = I \cos x - J \sin x$ and $\det(I - \mathrm{q}^{2n}e^{-2xJ}) = (1 - 2\mathrm{q}^{2n} \cos 2x + \mathrm{q}^{4n})$. We deduce that e^{-xA} belongs to $\mathbf{S}$ and defines a unitary operator on Hilbert space ℓ^2; evidently E is trace class. One can calculate

$$\begin{aligned} \det(I + e^{-xA_0}E_0e^{-xA_0}) &= 2i \sin x \prod_{n=1}^\infty (1 - 2\mathrm{q}^{2n} \cos 2x + \mathrm{q}^{4n}) \\ &= \frac{i\theta_1(x)}{\mathrm{q}^{1/4} \prod_{n=1}^\infty (1 - \mathrm{q}^{2n})}. \end{aligned} \tag{10.9}$$

which reduces to a multiple of Jacobi's function as in [33].

Let $\mathrm{q} = e^{i\pi\omega}$. Then $\theta_1(x+\pi) = -\theta_1(x)$ and $\theta_1(x+2\pi\omega) = e^{-4ix-4i\pi\omega}\theta_1(x)$, so $\theta_1\theta_1^*$ is periodic with period π, and $\theta_1(x+2\pi\omega)\theta_1^*(x+2\pi\omega) = e^{-8i(x+\pi\omega)}\theta_1(x)\theta_1^*(x)$, hence $\theta_1\theta_1^*$ is elliptic of the third kind. Using (8.13), one can easily show that the zero set of θ_1 is $\{j\pi + ik\log \mathrm{q} : j,k \in \mathbf{Z}\}$, and this coincides with the zero set of θ_1^*.

(ii) First note that $(A^2+I)(A^2+I/4) = 0$, so $E = 2^{-1}A^{-1}BC$ belongs to $\mathbf{S}$. It follows directly from Theorem 8.2 that $\mathbf{S}$ is a differential ring for $(-A,B,C)$. In this case A is similar to $-A$, so there exists an invertible S such that $AS+SA=0$, so the solution to (1.9) is not unique.

Note that the 2×2 matrices satisfy $(I+iJe^{-xJ})(I-iJe^{xJ}) = 2i\sin xI$ and

$$(I-\mathrm{q}^{2n}e^{-2xJ})(I-\mathrm{q}^{2n}e^{2xJ}) = (1-\mathrm{q}^{2n}\cos 2x+\mathrm{q}^{4n})I, \tag{10.10}$$

so F is a block diagonal matrix with entries from $\mathbf{K}_{\mathcal{C}}[I,J]$. In terms of $t = \tan x/2$, the n^{th} block has determinant $1+\mathrm{q}^{4n}-2\mathrm{q}^{4n}(1+t^4-6t^2)/(1+t^2)^2$, which has simple zeros and double poles for all n.

(iii) Using the identity (8.1), one checks that

$$\frac{d^2}{dx^2}\log\theta_1\theta_1^* = 2\mathrm{trace}\lfloor A\rfloor, \tag{10.11}$$

then a standard result from elliptic function theory [33, p. 132] gives

$$\wp(x) = -\big(\log\theta_1(x)\big)'' + e_1 + \big(\log\theta_1\big)''(1/2), \tag{10.12}$$

hence the result follows from (8.7).

(iv) We have the basic differential equation

$$\wp'' = 6\wp^2 - 4(e_1+e_2+e_3)\wp + 2(e_1e_2+e_1e_3+e_2e_3). \tag{10.13}$$

One can show that $u(x-ct)$ is a solution by differentiating (8.35) again and then adjusting the constants. Conversely, the expression $u''' = 3uu' - cu'$ reduces to

$$(u'/4)^2 = 4\big((u/4)^3 - (c/4)(u/4)^2 + \beta(u/4) + \gamma\big), \tag{10.14}$$

where β and γ are constants. By integrating this ordinary differential equation, we obtain Weierstrass's function.

(v) By induction, one can prove that for each $n = 0,1,2,\dots$, there exists a polynomial $q_n(X)$ of degree $n+1$ such that $\wp^{(2n)}(x) = q_n(X)$; likewise by induction one can prove that there exists a polynomial $p_n(X)$ of degree n such that $\wp^{(2n+1)}(x) = p_n(X)Z$. Hence

$$\mathrm{span}\{1,\wp^{(j)}(x) : j = 0,1,\dots,2N\} \subseteq \mathrm{span}\{X^j, X^kZ : j = 0,\dots,N+1; k = 0,\dots,N-1\} \tag{10.15}$$

and both spaces have dimension $2N+2$, so we have equality. We deduce that $\mathbf{A}_0 = \{p(X)Z + q(X) : p(X), q(X) \in \mathbf{C}[X]\}$, which is isomorphic to the ring $\mathbf{C}[X,Z]$ modulo the ideal $(Z^2 -$

$4(X-e_1)(X-e_2)(X-e_3))$, which is an integral domain since $\mathcal{T}$ is irreducible. Hence $\mathbf{A}_0 = \mathbf{C}[\mathcal{T}]$.

By repeatedly differentiating and using (8.18), we obtain

$$\begin{aligned} \wp(x) &= -\mathrm{trace}\lfloor A\rfloor + e_1 + (1/2)\big(\log\theta_1\theta_1^*\big)''(1/2),\\ \wp'(x) &= -2\mathrm{trace}\lfloor A(I-2F)A\rfloor \end{aligned} \tag{10.16}$$

and likewise $\wp^{(j)}$ is the trace of a $\lfloor P_j\rfloor$ for some $P_j \in \mathbf{A}$ for $j = 2, 3, \dots$.

(vi) By definition $\lfloor F^{-2}\rfloor = Ce^{-2xA}B$. We observe also that $C_0 e^{-xA_0}B_0$ equals

$$\begin{bmatrix} -iI\cos(x/2) + iJ\sin(x/2) & 0 & 0 & \dots \\ 0 & -\mathrm{q}^2 J\cos x - \mathrm{q}^2 I\sin x & 0 & \dots \\ 0 & 0 & -\mathrm{q}^4 J\cos x - \mathrm{q}^4 I\sin x & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{10.17}$$

so when we take the trace, we get

$$\mathrm{trace}\, C_0 e^{-xA_0}B_0 = -2i\cos(x/2) - \frac{4\mathrm{q}^2}{1-\mathrm{q}^2}\sin x, \tag{10.18}$$

and we obtain the stated result when we add the complex conjugate to get $\mathrm{trace}\, Ce^{-xA}B$.

□

On a compact Riemann surface $\mathcal{E}$, the divisor group $D(\mathcal{E}) = \{\delta = \sum_j n_j(z_j) : n_j \in \mathbf{Z}, z_j \in \mathcal{E}\}$ is the free abelian group generated by the points of $\mathcal{E}$, and the degree of the divisor δ is $\deg(\delta) = \sum_j n_j$. We let $\mathbf{K}^\sharp_{\mathcal{E}}$ be the multiplicative group of non zero meromorphic functions on $\mathcal{E}$, where we identify $f \sim g$ if $f = \lambda g$ for some $\lambda \in \mathbf{C}\setminus\{0\}$. Then by Liouville's theorem, each $f \in \mathbf{K}^\sharp_{\mathcal{E}}$ corresponds to the principal divisor $\delta(f) = \sum_j n_j\delta(z_j) - \sum_j m_j\delta(p_j)$, where z_j is a zero of f of order n_j and p_j a pole of f of order m_j; moreover, $\deg(f) = 0$. By extension, we can consider the notion of a divisor for $\psi_{BA}(z;\lambda)$ as in Proposition 8.4, with the understanding that there are infinitely many zeros and poles on $\mathbf{C}$ in a periodic array. In particular, elliptic functions of the third kind give rise to divisors on the torus. We now use the notations τ and σ to defer to tau functions of periodic linear systems as in Theorem 8.2. See [43].

First consider the group $G_{\mathcal{C}} = \{\tau/\sigma : \tau, \sigma \in \mathbf{C}_{\mathcal{C}}\}$ generated by linear systems with E of finite rank. Then each $\tau/\sigma \in \mathbf{K}^\sharp_{\mathcal{C}}$ may be transformed by the change of variable $t = \tan z$ to $\tau/\sigma \in \mathbf{K}^\sharp_{\mathbf{P}^1}$ and hence gives a divisor $\delta(\tau/\sigma)$ on the Riemann sphere. One can check that all divisors of degree zero on the Riemann sphere arise in this way.

Next consider $\mathbf{C}/\Lambda$. The torus $\mathcal{T}$ may be identified with the quotient group of divisors of degree zero modulo the group of principal divisors, known as the Jacobi variety. We consider

$$\tau(x) = e^{ax^2+bx+c}\frac{\prod_{j=1}^n \theta_1(x-a_j)}{\prod_{j=1}^m \theta_1(x-b_j)}, \tag{10.19}$$

which is meromorphic with divisor $(\tau) = \sum_j (a_j) - \sum_k (b_k)$ on some cell of the quotient space $\mathbf{C}/\Lambda$ so $\deg(\tau) = n - m$. The following results are consequences of Abel's theorem [29].

(1) If $\deg(\tau)=0$, $\sum_{j=1}^n (a_j-b_j)\in\Lambda$ and $a=b=0$, then τ is elliptic of the first kind.

(2) If $\deg(\tau)=0$ and $a=0$, then τ is elliptic of the second kind.

(3) τ is elliptic of the third kind and $u=-2(\log\tau)''$ is elliptic of the first kind. If $m=0$, then τ is entire, and u has poles at the a_j for $j=1,\dots,n$.

Lemma 10.2 *(i) For each positive divisor (δ) on $\mathbf{C}/\Lambda$, there exists a periodic linear system with tau function τ as in Theorem 8.1 such that (δ) equals the divisor of the zeros of τ.*

(ii) Any trivial theta function arises from the quotient of theta functions for Gaussian linear systems on $\mathbf{R}$. The effect of multiplying by a trivial theta function $\tau\mapsto e^{-Q/2}\tau$ is to take $u\mapsto u+q_0$ for some constant q_0.

Proof See above.

$\square$

Consider elliptic functions for the curve $\mathcal{T}=\{(x,y): y^2=4x^3-g_2x-g_3\}\cup\{(\infty,\infty)\}$ with Klein's invariant $J=g_2^3/(g_2^3-27g_3^2)$. By forming the trace in $u=-4\mathrm{trace}\lfloor A\rfloor$, we are undergoing a limiting process which takes us from $\mathbf{K}_{\mathcal{C}}$ to $\mathbf{M}_{\mathcal{C}}$, which includes the elliptic functions. Thus we can obtain an analogue of Corollary 8.3 for the elements that appear in finite algebraic extensions of the elliptic function field on $\mathcal{T}$. If u is algebraic over the elliptic function field, then u is realised via a periodic linear system.

Remark 10.3. (Integrable quantum systems) Having constructed the potential $\wp$ from a periodic linear system, we can produce a family of Hankel kernels and potentials from standard limiting arguments which are associated with exactly solvable problems in quantum mechanics. Consider an interacting system of N identical particles at positions x_j on the real line which interact only pairwise, and where the strength of the mutual interaction of particles j and k depends only upon their separation x_j-x_k via a potential u; then the Hamiltonian is

$$H=-\frac{1}{2}\sum_{j=1}^N\frac{\partial^2}{\partial x_j^2}+\sum_{1\le j<k\le N}u(x_j-x_k). \tag{10.20}$$

In each of the following, γ and the potential u are meromorphic functions on a Riemann surface $\mathcal{E}$ and ψ satisfies the addition rule

$$\psi(x+y)=\frac{\psi'(x)\psi(y)-\psi(x)\psi'(y)}{\gamma(x)-\gamma(y)}. \tag{10.21}$$

where $u(x)=\psi(x)^2+\gamma(x)+c$ with c constant.

$$\begin{array}{ccccc} \mathcal{E} & u(x) & \psi(x) & \gamma(x) & \tau(x)\\ \mathbf{P}^1 & g(g+1)/x^2 & (g+1)/x & -(g+1)/x^2 & x^{g(g+1)/2}\\ \mathbf{C}/\pi\mathbf{Z} & g(g+1)\mathrm{cosec}^2x & (g+1)\cot x & -(g+1)\mathrm{cosec}^2x & (\sin x)^{g(g+1)/2}\\ \mathbf{C}/\pi i\mathbf{Z} & g(g+1)\mathrm{cosech}^2x & (g+1)\coth x & -(g+1)\mathrm{cosech}^2x & (\sinh x)^{g(g+1)/2}\\ \mathbf{C}/\Lambda & 2\wp(x\mid\Lambda) & \psi_2(x,\alpha) & -\wp(x\mid\Lambda) & \theta_1(x\mid\Lambda) \end{array} \tag{10.22}$$

In the last line of this array we have introduced

$$\psi_2(x,\alpha)=-2q^{1/4}e^{(\zeta(\alpha)-2\alpha\eta_1/\pi)x}\prod_{n=1}^\infty(1-q^{2n})^3\frac{\theta_1(x-\alpha)}{\theta_1(\alpha)\theta_1(x)}, \tag{10.23}$$

which satisfies Lamé's equation $-\frac{d^2}{dx^2}\psi_2(x,\alpha)+2\wp(x)\psi_2(x,\alpha)=-\wp(\alpha)\psi_2(x,\alpha)$, and is such that $\alpha\mapsto\psi_2(x,\alpha)$ is elliptic and $x\mapsto\psi_2(x,\alpha)$ is elliptic of the second kind; moreover $\psi_2(x,\alpha)\psi_2(-x,\alpha)=\wp(\alpha)-\wp(x)$. By Lemma 10.2, $\psi_2(x,\alpha)$ can be expressed as a quotient of tau functions from periodic linear systems as in Lemma 8.1, and Gaussian linear systems as in Lemma 4.4.

The example in the final line is fundamentally important since one can obtain the periodic and rational potentials as limiting cases of the elliptic potential. We write $\Lambda=\mathbf{Z}2\omega_1+\mathbf{Z}2\omega_2$ where $\omega_1,\omega_2/i>0$. Then we have the thermodynamic limit

$$2\wp(x\mid\mathbf{Z}2\omega_1+\mathbf{Z}2\omega_2)\to 2(\pi/2\omega_2)^2\mathrm{cosech}^2(\pi x/2\omega_2)-\pi^2/6\omega_2^2\qquad(\omega_1\to\infty)\tag{10.24}$$

and in contrast the high density limit

$$2\wp(x\mid\mathbf{Z}2\omega_1+\mathbf{Z}2\omega_2)\to 2(\pi/2\omega_2)^2\mathrm{cosec}^2(\pi x/2\omega_1)-\pi^2/6\omega_1^2\qquad(\omega_2/i\to\infty);\tag{10.25}$$

when one limit is applied after the other, we have the limiting potential $u(x)=2/x^2$. Krichever shows that the system with $u(x)=2\wp(x)$ is integrable in the sense that there exists a compact Riemann surface $\mathcal{Y}_N$ which covers the elliptic curve N-fold, and the solution of the Hamiltonian dynamical system can be expressed in action-angle variables with the angles in the Jacobi variety of $\mathcal{Y}_N$.

For any finite gap $u\in\mathbf{M}_{\mathcal{C}}$, the solutions of $-\psi''+u\psi=\lambda\psi$ are parametrized by the hyperelliptic spectral curve $\mathcal{Y}$, punctured at infinity, and there is a meromorphic covering map $\mathcal{Y}\to\mathbf{P}^1$. One can use Lam'e's equation to produce explicit covering maps $\mathcal{Y}\to\mathcal{T}$ of the elliptic curve by hyperelliptic curves of arbitrary genus. To describe such elliptic covers in terms of linear systems, one can use the following result.

Proposition 10.4 *Let u be a nonconstant elliptic function on $\mathcal{T}$.*

(i) $\mathbf{K}=\mathbf{C}(u)[u']$ is a differential field, which is produced from a periodic linear system.

iii) Let $\mathbf{A}$ be a finitely generated algebra over $\mathbf{K}$, let $\mathbf{P}$ be a maximal ideal in $\mathbf{A}$ and $z\in\mathbf{A}/\mathbf{P}$. Then there exists an algebraic curve $\mathcal{Y}$ with a finite cover $\mathcal{Y}\to\mathcal{T}$ such that z may be identified with a rational function on $\mathcal{Y}$, and $\mathbf{K}[z]$ is a differential field.

(iii) For generic values of J and $\ell=1,2,\dots$, there exists a hyperelliptic curve $\mathcal{Y}_\ell$ of genus ℓ and a holomorphic covering map $\mathcal{Y}_\ell\to\mathcal{T}$ of degree $\ell(\ell+1)/2$.

Proof. (i) By a classical theorem, there exists a polynomial $P\in\mathbf{C}[x,y]$ such that $P(u,u')=0$, and hence u' is algebraic over $\mathbf{C}(u)$, so $\mathbf{K}$ is a field, and closed under differentiation.

(ii) By the weak Nullstellensatz [4], $\mathbf{A}/\mathbf{P}$ is a finite algebraic extension of $\mathbf{K}$. Hence we let $u_0,\dots,u_{n-1}\in\mathbf{K}$ be elliptic function, which may be realised as in Proposition 10.1 as quotients of tau functions from periodic linear systems. By classical results on Riemann surfaces, the characteristic equation

$$\det\begin{bmatrix} z & -1 & 0 & \dots \\ 0 & z & -1 & \dots \\ \vdots & \vdots & \ddots & \ddots \\ u_0(x) & u_1(x) & \dots & z+u_{n-1}(x)\end{bmatrix}=0\tag{10.26}$$

determines an algebraic function $z(x)$. Thus we can produce a Riemann surface $\mathcal{Y}$ and an n-sheeted holomorphic covering $\pi : \mathcal{Y} \to \mathcal{T}$. Then $\mathbf{K}[z]$ is a finite algebraic extension of the differential field $\mathbf{K}$, and hence a differential field.

(iii) Whereas it is not known which curves $\mathcal{Y}$ give covers of typical $\mathcal{T}$, one can produce explicit examples by means of Lamé's covers, as in [31]. Lamé's equation $-y''+\ell(\ell+1)\wp y = \nu^2 y$ is the prototypical example of an elliptic finite gap potential, and has solutions are thoroughly described in [27]. By introducing new variables $(X, Z) = (\wp(x), \wp'(x))$ for $\mathcal{T}$ and a fixed $g \in \mathbf{N}$, we can express Lamé's equation as

$$-\Big(Z\frac{d}{dX}\Big)^2\Psi(X) - 2\kappa\Big(Z\frac{d}{dX}\Big) + \ell(\ell+1)X\Psi(X) + B\Psi(X) + \kappa^2\Psi(X) = 0 \tag{10.27}$$

and use $d/dx = Zd/dX$. Clearly, the elliptic function field is $\mathbf{K}_0 = \mathbf{C}(X)[Z]$.

For each integer ℓ, let L_ℓ be Lamé's spectral polynomial of degree $2\ell+1$, and $\mathcal{Y}_\ell = \{(B,\nu) : \nu^2 = L_\ell(B)\}$ the corresponding hyperelliptic curve. For generic values of Klein's invariant $g_2^3/(g_2^3 - 27g_3^2)$, the curve $\mathcal{Y}_\ell$ is nonsingular and has genus $g = \ell$; whereas for the exceptional values $\mathcal{Y}_\ell$ is singular and g may decline to $\ell - 1$. The exceptional values of J are given by the Cohn polynomials as listed in [31]. There is a covering map $\pi_\ell : \mathcal{Y}_\ell \to \mathcal{T}$, and the resulting values (x, y) on $\mathcal{T}$ are explicit rational functions of (B, ν) on $\mathcal{Y}_\ell$ which are given in terms of the Lamé and twisted Lamé polynomials in [31]. Thus one can produce specific examples of hyperelliptic curves of genus $g = 2, 3, \dots$ which give finite covers $\pi_\ell : \mathcal{Y}_\ell \to \mathcal{T}$. Then $f \mapsto f \circ \pi_\ell$ gives a field homomorphism $\mathbf{K}_{\mathcal{T}} \to \mathbf{K}_{\mathcal{Y}_\ell}$, and for each nonconstant $g, f \circ \pi$ in $\mathbf{K}_{\mathcal{Y}_\ell}$ there exists a non-zero polynomial P such that $P(g, f \circ \pi) = 0$.

The differential equation is finite gap, in the sense that the Bloch spectrum is $[E_0, E_1] \cup [E_2, E_3] \cup \ldots \cup [E_{2\ell}, \infty)$, where E_j are the zeros of $L_\ell(B) = 0$. The spectral curve has points of ramification E_j for $j = 0, \dots, 2\ell$, and (10.9) has solutions of the first kind

$$\Psi = (C(X) + D(X)Z)\exp\Big(\kappa\int\frac{dX}{Z}\Big) \tag{10.28}$$

or of the second kind

$$\Psi = \Big(E(X)\sqrt{X - e_j} + \frac{F(X)Z}{\sqrt{X - e_j}}\Big)\exp\Big(\kappa\int\frac{dX}{Z}\Big) \tag{10.29}$$

where $\kappa \in \mathbf{C}$ is a spectral parameter and $C(X), D(X), E(X)$ and $F(X)$ are complex polynomials, depending on ℓ, e_j, κ and B; see [27]. For $B = E_j$, with $j = 0, \dots, 2\ell$, one can take $\kappa = 0$, and obtain Lamé's polynomial solutions of the first or second kind for (10.9). However, for typical spectral points, one requires $\kappa \neq 0$ and the solutions involve Lamé polynomials twisted by the exponential factor. There exists a polynomial P_ℓ of degree $\ell - 1$ such that

$$\int\frac{P_\ell(B)dB}{\sqrt{L_\ell(B)}} = \int\frac{dx}{\sqrt{4x^3 - g_2x - g_3}} \tag{10.30}$$

reduces the hyperelliptic integral on the left-hand side to the elliptic integral on the right, which is the inverse function of $\wp$.

□

Definition (Differential Galois group [46]) Let U be a fundamental solution matrix of Hill's equation

$$\frac{d}{dx}\begin{bmatrix}\psi\\ \xi\end{bmatrix}=\begin{bmatrix}0 & 1\\ u-\lambda & 0\end{bmatrix}\begin{bmatrix}\psi\\ \xi\end{bmatrix} \tag{10.31}$$

with $\det U(0)=1$ and let $\mathbf{PV}$ be the Picard–Vessiot ring over $\mathbf{C}$ that is generated by the entries of U; then let $\mathbf{L}$ be the field of fractions of $\mathbf{PV}$. The differential Galois group $DGal(\mathbf{L};\mathbf{C})$ is the set of $\mathbf{C}$-linear field automorphisms of $\mathbf{L}$ that commute with d/dx.

We now characterize finite gap elliptic potentials in terms of periodic linear systems.

Theorem 10.5 *Consider (10.31), where u is elliptic.*

(i) Then u may or may not be finite gap.

(ii) Suppose that (10.31) has a general solution $\psi_\lambda(x)$ that is a quotient of τ functions from periodic linear systems for all but finitely many $\lambda\in\mathbf{C}$. Then u is finite gap.

(iii) Conversely, suppose that u is finite-gap. Then for all but finitely many $\lambda\in\mathbf{C}$, (10.31) has a solution $\psi_\lambda(x)$ that is the quotient of tau functions arising from periodic linear systems as in Theorem 8.2 and Gaussian linear systems. Also, $\deg[\mathbf{K}^1_{\mathcal{T}}:\mathbf{K}_0]$ *is finite.*

(iv) Let M be a finite dimensional differentiable manifold of elliptic functions on the torus that is invariant and differentiable with respect to the flow associated with KdV and that some $u\in M$, where u is finite gap. Then there exists a family $\Sigma_t=(-A,B(t),C;E(t))$ of periodic linear systems such that $u(x,t)$ is the potential from Σ_t, and Σ_t evolves according to a finite-dimensional Hamiltonian system.

Proof. (i) The Treibich–Verdier potentials of the form

$$u(z)=a_0+\sum_{j=1}^{4}c_j\wp(z-a_j), \tag{10.32}$$

are finite gap if and only if $c_j=d_j(d_j+1)$ for some $d_j\in\mathbf{Z}$ for $j=1,\dots,4$, $a_0\in\mathbf{C}$ and the the poles satisfy $a_3=a_1+a_2$ and $a_4=0$. The corresponding tau function is

$$\tau(z)=\prod_{j=1}^{4}\theta_1(z-a_j)^{d_j(d_j+1)/2}\in\mathbf{K}^3_{\mathcal{T}}, \tag{10.33}$$

where the exponents are triangular numbers. Whereas one can realise such tau functions from periodic linear systems by means of Proposition 10.1, one can likewise produce tau functions corresponding to elliptic potentials that are not of the form (10.33).

(ii) Gesztesy and Weikard [25] considered $-\psi''+u\psi=\lambda\psi$ for $u\in\mathbf{K}^1_{\mathcal{T}}$, and showed that u is finite gap if and only if u is a Picard potential. If ψ is a quotient of τ functions, then ψ is meromorphic and hence u is a Picard potential.

(iii) Suppose that (10.31) has a meromorphic solution. Then by a theorem of Picard [25], there exists a solution ψ that is elliptic of the second kind, hence has the form (10.19) with $a=0$ and degree zero. By inspecting the differential equation, we see that the only possible poles of u are contained in the set $\{a_1,\dots,a_n;b_1,\dots,b_n\}$. By Proposition 10.7, each factor

$\theta_1(x-a_j)$ or $\theta_1(x-b_j)$ arises from the tau function of a periodic linear system, while the factor e^{bx} is a quotient of Gaussian tau functions. By [29, p. 96], u' is algebraic over $\mathbf{C}(u)$ and we have $\mathbf{K}_0 = \mathbf{C}(u)[u']$ and $\deg[\mathbf{K}^1_{\mathcal{T}} : \mathbf{K}_0] < \infty$. Let V_λ be the solution space of (10.15), and observe that $DGal(\mathbf{L};\mathbf{K}_0)$ operates on V_λ component-wise; in particular, the monodromy operators $T_j : \Psi(z) \mapsto \Psi(z+2\omega_j)$ are commuting operators such that $T_j(V_\lambda) \subseteq V_\lambda$ for $j=1,2$ since u is elliptic, so we can take Λ to be the group generated by T_1 and T_2. Let $\Psi_1 \sim \Psi_2$ if $\Psi_1 = c\Psi_2$ for some constant $c \in \mathbf{C} \setminus \{0\}$; then let $V^*_\lambda = (V_\lambda \setminus \{0\})/\sim$. Then with ψ the solution that is elliptic of the second kind, $\Psi = \mathrm{column}[\,\psi \quad \psi'\,] \in V_\lambda$, gives a common eigenvector $T_1\Psi = \rho_1\Psi$ and $T_2\Psi = \rho_2\Psi$, so $\gamma\Psi \sim \Psi$ for all $\gamma \in \Lambda$; hence Ψ gives an element of $(V^*_\lambda : \Lambda)$. Furthermore, if T_1 or T_2 has distinct eigenvalues as an operator on V_λ, then there exists a fundamental system of elliptic functions of the second kind, so $(V^*_\lambda : \Lambda)$ is isomorphic to $\mathbf{P}^1$.

(iv) Airault, McKean and Moser showed [2, Corollary 1] that any such flow of potentials has the form

$$u(z,t) = \sum_{j=1}^{m} 2\wp(z - a_j(t)) + c \tag{10.34}$$

where the moving poles $a_j(t)$ lie on the manifold defined by the constraints

$$0 = \sum_{j=1;j\neq k}^{m} \wp'(a_j - a_k) \qquad (k = 1,\dots,m). \tag{10.35}$$

and satisfy the system of nonlinear differential equations

$$\frac{da_k}{dt} = 6 \sum_{j=1;j\neq k}^{m} p(a_j - a_k) \qquad (k = 1,\dots,m) \tag{10.36}$$

In an evident analogy with (10.16), the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{m} p_j^2 + \frac{12}{2}\sum_{j\neq k:j,k=1}^{n} \wp(a_j - a_k)^2 \tag{10.37}$$

gives this system of differential equations for the a_j; see [15].

We can realise $2\wp(x)$ as the potential of a periodic linear system $(-A,B,C;E)$, and hence we can realise $u(z,t)$ as the potential of the periodic linear system

$$\Sigma_t = \bigoplus_{j=1}^{m} \big(-A, e^{a_j(t)A}B, C; e^{a_j(t)A}E\big). \tag{10.38}$$

See [10] for more details of the construction and [15] for further information on the dynamics of the poles under KdV flows.

□

Remark. We leave it as an open problem to characterize all finite gap cases of Hill's equation in terms of periodic linear systems.

11. Differential rings related to the KdV hierarchy

Let $(e^{-xA})_{x\in\mathbf{R}}$ be a bounded C_0 group of operators on H, so A is similar to a skew symmetric operator; then by the spectral theorem, $(e^{-tA^3})_{t\in\mathbf{R}}$ also forms a bounded C_0 group on H. We allow $C : H \to \mathbf{C}$ and $B : \mathbf{C} \to H$ to evolve through time so that $C = C_0e^{-tA^3}$ and $B = e^{-tA^3}B_0$ for some initial $C_0 : H \to \mathbf{C}$ and $B_0 : \mathbf{C} \to H$, and correspondingly $R(x,t) = e^{-tA^3}R_xe^{-tA^3}$. The formulas involving C, B and R are symmetrical with respect to time evolution, since B and C both evolve under the same group. In contrast to Theorem 8.2, we do not assume that A commutes with BC; that BC here will typically have rank one, whereas A will have infinite rank. The operation of $\frac{\partial}{\partial t_j}$ on $\det(I - R_x^2)$ is described by the Lyapunov equation (1.10) in the form of the following commutator identity

$$\left[\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{\partial}{\partial t_j} - \begin{bmatrix}0&A^{2j+1}\\A^{2j+1}&0\end{bmatrix}, \begin{bmatrix}R&0\\0&-R\end{bmatrix}\right] = -2\begin{bmatrix}R&0\\0&-R\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{\partial}{\partial t_j}, \tag{11.1}$$

which is analogous to (19) in [30] and contrasts with Proposition 3.4.

Proposition 11.1 *Suppose that A is bounded and let $F_x = (I + R)^{-1}$. Then*

$$\mathbf{A} = \mathrm{span}_{\mathbf{C}}\Big\{A^{n_1}, A^{n_1}F_xA^{n_2}\dots F_xA^{n_r} : n_1, n_2, \dots, n_r \in \mathbf{N}\Big\} \tag{11.2}$$

is a differential subring of $C^\infty((0,\infty)^2; \mathrm{B}(H))$, and the map $\lfloor\,.\,\rfloor : \mathbf{A} \to C^\infty((0,\infty)^2; \mathbf{C})$

$$\lfloor P\rfloor = Ce^{-xA}F_xPF_xe^{-xA}B \tag{11.3}$$

has range $\lfloor\mathbf{A}\rfloor$, where $\lfloor\mathbf{A}\rfloor$ is a differential ring with pointwise multiplication and derivatives $\partial/\partial x$ and $\partial/\partial t_1$.

Proof. As in Lemma 3.2, the basic relations are

$$\frac{\partial}{\partial x}\lfloor P\rfloor = \Big\lfloor A(I-2F_x)P + \frac{\partial}{\partial x}P + P(I-2F_x)A\Big\rfloor, \tag{11.4}$$

$$\frac{\partial}{\partial t_1}\lfloor P\rfloor = \Big\lfloor A^3(I-2F_x)P + \frac{\partial}{\partial t}P + P(I-2F_x)A^3\Big\rfloor,$$

$$\lfloor P\rfloor\lfloor Q\rfloor = \lfloor P(AF_x + F_xA - 2F_xAF_x)Q\rfloor. \tag{11.5}$$

Indeed it follows from the Lyapunov equation (1.8) that F_x satisfies the differential equations

$$\frac{\partial F_x}{\partial x} = AF_x + F_xA - 2F_xAF_x, \tag{11.6}$$

$$\frac{\partial F_x}{\partial t_1} = A^3F_x + F_xA^3 - 2F_xA^3F_x \tag{11.7}$$

and hence the derivatives from the first and last factors in (11.10) satisfy

$$\frac{\partial}{\partial x}Ce^{-xA}F_x = Ce^{-xA}F_xA(I-2F_x), \quad \frac{\partial}{\partial x}F_xe^{-xA}B = (I-2F_x)AF_xe^{-xA}B; \tag{11.8}$$

$$\frac{\partial}{\partial t}Ce^{-xA}F_x = Ce^{-xA}F_xA^3(I-2F_x); \quad \frac{\partial}{\partial t}F_xe^{-xA}B = (I-2F_x)A^3F_xe^{-xA}B. \tag{11.9}$$

By applying Leibniz's rule, we deduce that $\lfloor \mathbf{A} \rfloor$ is closed under $\partial/\partial x$ and $\partial/\partial t_1$. Furthermore

$$F_xe^{-xA}BCe^{-xA}F_x = AF_x + F_xA - 2F_xAF_x, \tag{11.10}$$

so $\lfloor \mathbf{A} \rfloor$ is closed under multiplication, and the product rule (11.8) holds.

The pole divisor $\mathbf{z}_\lambda(t)$ is determined by $\{z_n(t) : \psi_{BA}(z_n(t), t; \lambda) = 0\}$ and is associated with the potential $u_\lambda(x;t) = -2\frac{d^2}{dx^2}\log \tau_\lambda(x,t)$. In this section, we introduce dynamical systems on $\mathcal{T}^\infty_{\mathbf{R}}$ such that $u_\lambda(x,t)$ undergoes the nonlinear evolution associated with the KdV hierarchy. To obtain $KdV(2n+1)$, we vary t_n while fixing t_j for $j \neq n$.

Lemma 11.2 *Suppose that $C_0A^4 : H \to \mathbf{C}$ and $A^4B_0 : \mathbf{C} \to H$ are bounded.*

(i) Then the scattering function $\phi(x;t_1) = C_0e^{-2t_1A^3-xA}B_0$ satisfies the linearized Korteweg –de Vries equation

$$\frac{\partial \phi}{\partial t_1} = 2\frac{\partial^3 \phi}{\partial x^3}. \tag{11.11}$$

(ii) Let $v(x,t_1)$ be as in (2.2), so that

$$v(x,t) = -C_0e^{-xA-t_1A^3}(I+R)^{-1}e^{-xA-t_1A^3}B_0; \tag{11.12}$$

and let $u(x,t_1) = -2\frac{\partial v}{\partial x}$. Then

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\det(I+R) \tag{11.13}$$

belongs to $\lfloor \mathbf{A} \rfloor$ and satisfies the KdV equation

$$4\frac{\partial u}{\partial t_1} = \frac{\partial^3 u}{\partial x^3} + 12u\frac{\partial u}{\partial x}. \tag{11.14}$$

Proof. (i) This follows from a simple computation.

(ii) We have the following table of derivatives

$$\begin{aligned}
\frac{\partial v}{\partial x} &= 2\lfloor A \rfloor; \\
\frac{\partial^2 v}{\partial x^2} &= 4\lfloor A^2 \rfloor - 8\lfloor AF_xA \rfloor; \\
\frac{\partial^3 v}{\partial x^3} &= 8\lfloor A^3 \rfloor - 24\lfloor A^2F_xA + AF_xA^2 \rfloor + 48\lfloor AF_xAF_xA \rfloor;
\end{aligned} \tag{11.15}$$

We shall prove that

$$4\frac{\partial v}{\partial t_1} = \frac{\partial^3 v}{\partial x^3} + 6\Big(\frac{\partial v}{\partial x}\Big)^2, \tag{11.16}$$

which leads to the result for u. By (11.18)

$$\left(\frac{\partial v}{\partial x}\right)^2 = \lfloor 4A^2FA + 4AFA^2 - 8AFAFA \rfloor. \tag{11.17}$$

For comparison we have $\frac{\partial v}{\partial t} = 2\lfloor A^3 \rfloor$; hence we obtain (11.16).

Moreover by Lemma 11.3, $\lfloor \mathbf{A} \rfloor$ contains $u(x,t_1) = 2Ce^{-xA}FAFe^{-xA}B$ and all its derivatives. Observe that

$$-2\frac{\partial}{\partial x}v(x,t_1) = -4\lfloor A \rfloor = u(x,t_1) \tag{11.18}$$

belongs to $\lfloor \mathbf{A} \rfloor$ and satisfies the identity (11.13); moreover, all the partial derivatives of u also belong to the differential ring $\lfloor \mathbf{A} \rfloor$.

□

We now point out some particular solutions which are realised via Lemma 11.3, some of which were also noted by Pöppe [39]. Let λ_j be distinct complex numbers for $j = 1, \dots, m$, such that $\Re\lambda_j > 0$, and let $H = \mathrm{span}\{x^j e^{-\lambda_\ell x} : j = 0, \dots, n_\ell - 1; \ell = 1, \dots, m\}$, which we view as a subspace of $L^2(0,\infty)$, and let $A = -\frac{d}{dx}$ on H.

Corollary 11.3 (Solitons) *(i) Then $(e^{-sA})_{s\in\mathbf{R}}$ defines a C_0 group of operators on H such that $\|e^{-sA}\| < 1$ for $s > 0$, and $\phi(x;t_1)$ satisfies $\frac{\partial \phi}{\partial t_1} = 2\frac{\partial^3\phi}{\partial x^3}$, and $u(x;t_1) \in \mathbf{C}(x, t_1, e^{-\lambda_j x}, e^{-2\lambda_j^3 t_1})$ satisfies KdV.*

(ii) In particular, suppose that A has distinct and simple eigenvalues, and that $B_0 = (b_j) \in \mathbf{C}^{n\times 1}$ and $C_0 = (c_j) \in \mathbf{C}^{1\times n}$. Then

$$\det(I + \mu R_x) = \sum_{m=0}^{N} \mu^m \sum_{\sigma \subseteq \{1,\dots,N\}, \sharp\sigma = m} \prod_{j\in\sigma} b_j c_j e^{-2\lambda_j^3 t_1 - 2\lambda_j x} \prod_{j,k\in\sigma : j\neq k} \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k} \tag{11.19}$$

Proof. (i) The group e^{-sA} operates as translations $e^{-sA}f(x) = f(x+s)$, and hence e^{-sA} is a strict contraction on the finite dimensional space $H = \mathbf{C}^n$ for $s > 0$. In effect, we have returned to the setting of Proposition 2.2. The generator is $-A = d/dx$, and can introduce $A^3 = -d^3/dx^3$ and the group $e^{-t_1 A^3}$ which is associated with the linearized Korteweg de Vries equation. By Lemma 11.3, u satisfies the KdV equation (11.23), and by Theorem 3.1, u is rational in the basic variables.

(ii) Apply Proposition 2.4 and Lemma 11.3.

□

Let $H = L^2(-\infty,\infty)$ and as in section 5, we can take $Af(x) = -f'(x)$ and we note that e^{-tA^3} is the Airy group

$$e^{-tA^3}f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-it\xi^3 + iy\xi}\, d\xi. \tag{11.20}$$

Then with $g \in \mathcal{D}(A^4)$ we choose $B_0 : \alpha \mapsto g(y)\alpha$ and $C_0 : f \mapsto f(0)$, and let

$$\gamma_n = (-1)^n \int_{-\infty}^{\infty} \hat{g}(\xi)\frac{(i\xi+1)^n}{(-i\xi+1)^{n+1}}\frac{d\xi}{\pi}. \tag{11.21}$$

Corollary 11.4 (Non solitons) *Let $g \in \mathcal{D}(A^4)$ have $\sum_{n=0}^{\infty}(1+n)|\gamma_n| < 1$. Then $\phi(x;t)$ satisfies (11.17) and $u(x;t)$ satisfies ().*

Proof. By Plancherel's formula we identify $\mathcal{D}(A^4) = \{g \in L^2 : \int_{-\infty}^{\infty}(1+\xi^8)|\hat{g}(\xi)|^2 d\xi < \infty\}$, so the maps are well defined. By a simple calculation, have

$$R_x f(y) = \int_x^\infty g(y+s)f(s)\,ds, \qquad (f \in L^2(0,\infty)) \tag{11.22}$$

so in particular R_0 is the Hankel integral operator with kernel $g(y+s)$. Hence R_0 is unitarily equivalent to $[\gamma_{j+k}]_{j,k=0}^{\infty}$ on ℓ^2, which by the hypotheses is a trace class operator; likewise, R_x is trace class. Furthermore, $I + R_x$ is invertible, and the inverse F is given by a Neumann series. Given these facts, we can apply Lemma 11.3.

□

For $m \geq 4$, We can choose $g(x) = \mathbf{I}_{(0,\infty)}(x)x^m e^{-x}$ in Corollary 11.6. Whereas the choice of $g(y) = \delta_0$ is technically inadmissible, the resulting expression $\phi(x;t) = t^{-1/3}\mathrm{Ai}(-x/(6t)^{1/3})$ does give a solution of (11.14).

Proposition 11.5 *Suppose that $C_0A^6 : H \to \mathbf{C}$ and $A^5B_0 : \mathbf{C} \to H$ are bounded.*

(i) Then the scattering function $\phi(x;t_2) = C_0e^{-2t_2A^5-xA}B_0$ satisfies

$$\frac{\partial \phi}{\partial t_2} = 2\frac{\partial^5\phi}{\partial x^5}. \tag{11.21}$$

(ii) Let $v(x) = T(x,x)$, so that

$$v(x,t) = -C_0e^{-xA-tA^5}(I+R)^{-1}e^{-xA-t_2A^5}B_0. \tag{11.22}$$

Then $u(x,t_2) = \frac{\partial v}{\partial x}$ satisfies the KdV(5) equation

$$16\frac{\partial u}{\partial t_2} = \frac{\partial^5 u}{\partial x^5} + 10u\frac{\partial^3 u}{\partial x^3} + 20\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 30u^2\frac{\partial u}{\partial x}. \tag{11.23}$$

Proof. We shall prove that

$$16\frac{\partial v}{\partial t_2} = \frac{\partial^5 v}{\partial x^5} + 10\frac{\partial^3 v}{\partial x^3}\frac{\partial v}{\partial x} + 5\Big(\frac{\partial^2 v}{\partial x^2}\Big)^2 + 20\Big(\frac{\partial v}{\partial x}\Big)^6. \tag{11.24}$$

The basic identities required follow from (), namely

$$\begin{aligned}\frac{\partial^4 v}{\partial x^4} &= 16\lfloor A^4\rfloor - 64\lfloor A^3F_xA + AF_xA^3\rfloor - 96\lfloor A^2F_xA^2\rfloor\\ &\quad + 112\lfloor A^2F_xAF_xA + AF_xA^2F_xA + AF_xAF_xA^2\rfloor - 384\lfloor AF_xAF_xAF_xA\rfloor; \qquad (11.25)\\ \frac{\partial^5 v}{\partial x^5} &= 32\lfloor A^5\rfloor - 160\lfloor A^4F_xA + AF_xA^4\rfloor - 320\lfloor A^3F_xA^2 + A^2F_xA^3\rfloor\\ &\quad + 640\lfloor A^3F_xAF_xA + AF_xA^3F_xA + AF_xAF_xA^3\rfloor\\ &\quad + 960\lfloor A^2F_xA^2F_xA + A^2F_xAF_xA^2 + AF_xA^2F_xA^2\rfloor\\ &\quad - 1920\lfloor A^2F_xAF_xAF_xA + AF_xA^2F_xAF_xA + AF_xAF_xA^2F_xA + AF_xAF_xAF_xA^2\rfloor\\ &\quad + 3840\lfloor AF_xAF_xAF_xAF_xA\rfloor. \qquad (11.26)\end{aligned}$$

Using these, one checks that () holds.

□

Suppose that $\lfloor P\rfloor = CU(t)e^{-xA}F_xPF_xe^{-xA}U(t)B$ and that F_x and A commute. Then

$$-4\frac{\partial}{\partial t}\lfloor A\rfloor + \frac{\partial^3}{\partial x^3}\lfloor A\rfloor - 8\big(\frac{\partial x}{\partial}\lfloor A\rfloor\Big)\lfloor A^{2m+1}\rfloor + 16\frac{\partial}{\partial x}\Big(\lfloor A\rfloor\lfloor A^{2m+1}\rfloor\Big) = 0. \tag{11.27}$$

Proof. We can obtain the following identities by repeatedly using the basic calculus rules

$$\frac{\partial}{\partial t}\lfloor A\rfloor = \lfloor 2A^{2m+4} - 2A^{2m+3}FA - 2AFA^{2m+3}\rfloor; \tag{11.28}$$

$$\begin{aligned}\lfloor A\rfloor\frac{\partial}{\partial x}\lfloor A^{2m+1}\rfloor = \lfloor &2AFA^{2m+3} + 2A^2FA^{2m+2} - 2AFA^{2m+2}FA\\ &- 4AFAFA^{2m+2} - 2A^2FAFA^{2m+1} - 2AFA^2FA^{2m+1}\\ &- 2A^2FA^{2m+1}FA + 4AFAFAFA^{2m+1} + 4AFAFA^{2m+1}FA\rfloor;\end{aligned} \tag{11.29}$$

$$\begin{aligned}\Big(\frac{\partial}{\partial x}\lfloor A\rfloor\Big)\lfloor A^{2m+1}\rfloor = \lfloor &2A^3FA^{2m+1} + 2A^2FA^{2m+2} - 4A^2FAFA^{2m+1} - 4AFA^2FA^{2m+1}\\ &- 4AFAFA^{2m+2} + 8AFAFAFA^{2m+1}\rfloor;\end{aligned} \tag{11.30}$$

$$\begin{aligned}\frac{\partial^3}{\partial x^3}\lfloor A^{2m+1}\rfloor = \lfloor &8A^{2m+4} - 24A^{2m+3}FA - 24AFA^{2m+3} - 24A^{2m+2}FA^2 - 24A^2FA^{2m+2}\\ &+ 48A^{2m+2}FAFA + 48AFA^{2m+2}FA + 48AFAFA^{2m+2} - 8A^{2m+1}FA^3 - 8A^3FA^{2m+1}\\ &- 8A^3FA^{2m+1} + 24A^{2m+1}FA^2FA + 24A^{2m+1}FAFA^2 + 24A^2FA^{2m+1}FA\\ &+ 24AFA^{2m+1}FA^2 + 24A^2FAFA^{2m+1} + 24AFA^2FA^{2m+1}\\ &- 48A^{2m+1}FAFAFA - 48AFA^{2m+1}FAFA - 48AFAFA^{2m+1}FA\\ &- 48AFAFAFA^{2m+1}\rfloor\end{aligned} \tag{11.31}$$

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