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Universal routes to spontaneous $\mathcal{PT}$-symmetry breaking in non-Hermitian quantum systems

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$\mathcal{PT}$-symmetric systems can have a real spectrum even when their Hamiltonian is non-Hermitian, but develop a complex spectrum when the degree of non-Hermiticity increases. Here we utilize random-matrix theory to show that this spontaneous $\mathcal{PT}$-symmetry breaking can occur via two distinct mechanisms, whose predominance is associated to different universality classes. Present optical experiments fall into the orthogonal class, where symmetry-induced level crossings render the characteristic absorption rate independent of the coupling strength between the symmetry-related parts of the system.

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In quantum mechanics, the analysis of level crossings provides a powerful tool for the extraction of discrete symmetries, with applications ranging from the Zeeman effect to present-day studies, for example, of the surface states in condensed-matter systems [1,2]; in the absence of crossings, the degree of level repulsion distinguishes principal universality classes of quantum systems [3,4]. Spectral phenomena acquire an additional richness in non-Hermitian quantum systems, which generally have a complex energy spectrum, with imaginary parts of the energies related to decay or amplification rates. However, in the presence of symmetries which ensure that loss and gain are in balance, the spectrum can still be real. One intensely researched route to try and achieve such a balance is to couple two identical systems symmetrically and then induce opposite amounts of gain and loss into the two parts, as illustrated in Fig. 1(a) [5–17]. The Hamiltonian then possesses a combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry, and its secular equation is real. Interestingly, this does not guarantee a real spectrum; as the level of non-Hermiticity (loss and gain) is increased, pairs of complex-conjugate energy levels appear [7–9]. This phenomenon of spontaneous $\mathcal{PT}$-symmetry breaking has gained recent prominence because it leads to optical effects such as double refraction, solitons, and nonreciprocal diffraction patterns, which provide mechanisms for the design of unidirectional couplers and left-right sensors [11,12], concepts that are now being realized experimentally in a variety of optical settings [13]. Over the past months, these systems were proposed for at-threshold lasers [14] and laser-absorbers [15,18]. In turn, these developments have instigated a deeper theoretical understanding of the role of the dynamics (such as the consequences of Anderson localization and wave chaos [16], as well as interactions [17]). In this Rapid Communication, based on an investigation of the symmetries of such systems, we establish distinct universality classes which directly affect the nature of spontaneous $\mathcal{PT}$-symmetry breaking and relate these to the propensity of level crossings.

To do so, we derive random-matrix ensembles where loss and gain in the two parts of the system are implemented by a uniform rate μ, while coupling is established through N channels with transmission probability T; the mean level spacing of the decoupled parts is Δ [19]. We find that the mechanism behind spontaneous $\mathcal{PT}$-symmetry breaking depends on whether the Hermitian limit $\mu = 0$ is time-reversal symmetric or not, amounting to a predominance of symmetry-induced level crossings or level repulsion [see Fig. 1(b)]. This results in different characteristic scales μ_{PT} of amplification or absorption governing the transition from an essentially real to an essentially complex spectrum. Present optical experiments and theoretical studies concern systems without magneto-optical effect (the orthogonal symmetry class), in which the Hermitian limit is time-reversal symmetric. In this case $\mu_{PT} \sim \sqrt{N}\Delta/2\pi \equiv \mu_0$ becomes *fully independent of the coupling strength* as soon as T surpasses a parametrically small threshold $T_c \sim 1/N$, thereby exhibiting a level of universality that goes beyond what is normally encountered in mesoscopic systems. For weak coupling ($T < T_c$), $\mu_{PT} \sim \sqrt{NT}\mu_0$. Adding magneto-optical effects to the system essentially changes the nature of the transition. In this case (the unitary symmetry class), $\mu_{PT} \sim \sqrt{T}\mu_0$ in the full range of weak and strong coupling. These findings are illustrated in Fig. 2.

Random-matrix ensembles. To derive the appropriate random-matrix ensembles we formulate a quantization condition based on scattering theory [14,20,21]. The $N \times N$-dimensional scattering matrix,

$$S_L(E;\mu) = 1 - 2iV^\dagger(E - i\mu - H + iVV^\dagger)^{-1}V, \qquad (1)$$

of the left subsystem can be expressed in terms of an $M \times M$-dimensional Hamiltonian H, which is real symmetric (a member of the standard Gaussian orthogonal ensemble) if the Hermitian limit $\mu = 0$ is time-reversal symmetric and complex Hermitian (a member of the Gaussian unitary ensemble) if this is not the case [3]. We assume $M \gg N \gg 1$ and denote the mean level spacing in the energy range of interest by Δ. The $M \times N$ coupling matrix V then fulfills $VV^\dagger = \mathrm{diag}(v_m)$, where N diagonal entries $v_m = \Delta M/\pi$ correspond to fully transparent channels, and $M - N$ entries $v_m = 0$ describe the closed channels [20]. Adopting a basis where the time-reversal operation $\mathcal{T}$ is identical to complex conjugation, $\mathcal{PT}$ symmetry results in the relation [14]

$$\begin{aligned} S_R(E;-\mu) &= [S_L^{-1}(E^*;\mu)]^* \\ &= 1 - 2iV^\dagger(E + i\mu - H^* + iVV^\dagger)^{-1}V \qquad (2) \end{aligned}$$

for the scattering matrix of the right subsystem. The tunnel barrier is described by reflection amplitudes $r = -\sqrt{1-T}$ and transmission amplitudes $t = i\sqrt{T}$. As shown in the lower part of Fig. 1(a), these scattering matrices relate amplitudes

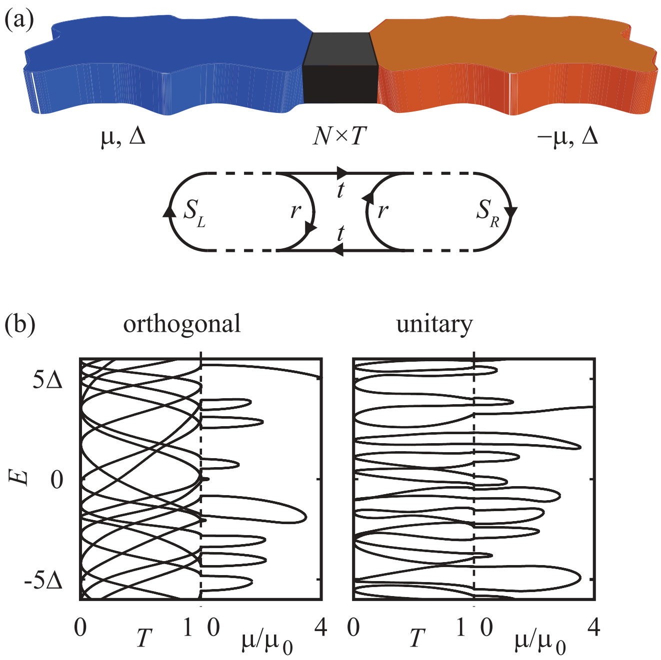

FIG. 1. (Color online) (a) Sketch of a non-Hermitian $\mathcal{PT}$-symmetric system, where a region with absorption rate μ (and mean level spacing Δ, left) is coupled symmetrically via a tunnel barrier (supporting N channels with transmission probability T) to an amplifying region with a matching amplification rate (right). Below this is the scattering description of the system. (b) Two routes to spontaneous $\mathcal{PT}$-symmetry breaking, depending on whether the Hermitian limit $\mu = 0$ is $\mathcal{T}$ symmetric (orthogonal class displaying level crossings, left) or not (unitary class displaying avoided crossings, right). Shown are real eigenvalues of a random Hamiltonian $\mathcal{H}$ [Eq. (4)] as function of T for fixed $\mu = 0$ (left of dashed line), and then as a function of μ for fixed $T = 1$ (right of dashed line). Complex-valued levels (formed by level coalescence at $\mu > 0$) are not shown. Here $\mu_0 = \sqrt{N}\Delta/2\pi$, and we set $N = 10$.

of left- and right-propagating waves at the two interfaces of the tunnel barrier. The requirement of consistency of these relations results in the quantization condition

$$\det\left[\begin{pmatrix} r & t \\ t & r \end{pmatrix}\begin{pmatrix} S_L & 0 \\ 0 & S_R \end{pmatrix} - \mathbb{1}\right] = 0, \tag{3}$$

which can be rearranged into an eigenvalue problem $\det(E - \mathcal{H}) = 0$ with effective Hamiltonian

$$\mathcal{H} = \begin{pmatrix} H - i\mu & \Gamma \\ \Gamma & H^* + i\mu \end{pmatrix}. \tag{4}$$

The positive semidefinite coupling matrix $\Gamma = \mathrm{diag}(\gamma_m)$ now incorporates the finite transmission probability of the barrier; its N nonvanishing entries read $\gamma_m = [\sqrt{T}/(1 + \sqrt{1-T})]\Delta M/\pi \equiv \gamma$ [22].

Numerical evaluation. Before engaging in an analytical discussion of the different routes to spontaneous $\mathcal{PT}$-symmetry breaking [illustrated in Fig. 1(b)], we put forward numerical results which illustrate the physical consequences of the points to be made below. These results, presented as (color) gradient plots in Fig. 2, concern the fraction $f(\mu,T)$ of complex-valued energy levels within a range where the mean level spacing can be assumed constant. We fix $M = 1000$, which ensures a large number of levels within the range in question, and $N = 50$ [23].

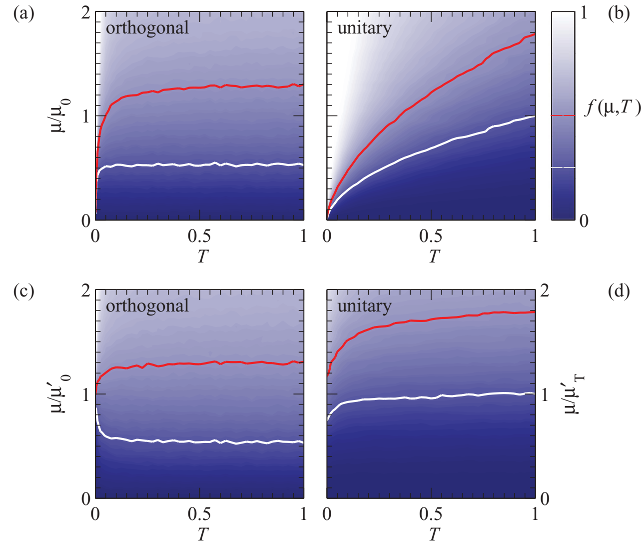

FIG. 2. (Color online) Gradient plots of the ensemble-averaged fraction $f(\mu,T)$ of complex energy levels (among all levels within a range of energies over which the mean spacing Δ can be assumed constant), for the orthogonal class (left) and the unitary class (right). In (a) and (b), μ is scaled to μ_0. The bottom panels show same data with μ scaled to $\mu_0' = \mu_0/\sqrt{1+1/NT}$ (c) and $\mu_T' = \sqrt{T}\mu_0$ (d). Numerical results with $N = 50$.

Figure 2(a) shows results for the orthogonal ensemble, with μ scaled to $\mu_0 = \sqrt{N}\Delta/2\pi$. We see that above a small threshold T_c (to be determined below as $T_c \sim 1/N$), the transition from the real spectrum ($f = 0$, obtained for $\mu = 0$) to a spectrum which is partially real and partially complex spectrum ($f \sim 1/2$) indeed occurs on the scale $\mu_{PT} \sim \mu_0$ and then is independent of the value of T. Only for $T < T_c$, $\mu_{PT} \sim \sqrt{NT}\mu_0 \equiv \mu_T$ is coupling dependent. In order to get a unified view over both regimes, we plot in panel (c) the same data, but with μ scaled to $\mu_0' = \mu_0/\sqrt{1+1/NT}$, which interpolates between μ_T for $T \ll T_c$ and μ_0 for $T \gg T_c$. The convergence of gradient lines for $T \to 0$ indicates that for weak coupling the transition becomes more abrupt.

Figure 2(b) shows the corresponding results for the unitary ensemble, where μ is again scaled to μ_0. Here we find that a systematic T dependence persists across the full range of coupling strengths. As shown in panel (d), this dependence takes the form $\mu_{PT} \sim \sqrt{T}\mu_0 \equiv \mu_T'$. Now the only difference between the strong and weak coupling regimes is a factor of order 1. In further contrast to the orthogonal case, for weak coupling the transition remains smooth; however, since $\mu_T' = \mu_T/\sqrt{N} \ll \mu_T$, it then occurs at a far smaller deviation from Hermiticity.

Underlying mechanisms. We now show that the features reported above originate from two distinct mechanisms of spontaneous $\mathcal{PT}$-symmetry breaking. We first consider the orthogonal class and start in a regime which can be treated perturbatively. For $\mu = 0$, the effective Hamiltonian $\mathcal{H}$ [Eq. (4)] is real symmetric, and all its eigenvalues are real. For $T = 0$ ($\Gamma = 0$), on the other hand, the spectrum is a superposition of two level sequences $E_k = \varepsilon_k \pm i\mu$, which are all complex if $\mu \neq 0$; here ε_k are the eigenvalues of H. Therefore, in regard to the question of how many levels are complex, the limits $T, \mu \to 0$ do not commute. Nonetheless, for $T = \mu = 0$ the spectrum reduces to the superposition of two degenerate level sequences ε_k, so that quasidegenerate perturbation theory applies. Denote by $\psi_m^{(k)}$ the wave function of H corresponding to eigenvalue ε_k; in random-matrix theory, this is a random normalized vector with real entries. Reduced to the symmetric and antisymmetric extension of this wave function across the whole system, the effective Hamiltonian takes the form

$$\mathcal{H}' = \begin{pmatrix} \varepsilon_k - i\mu & \sum_m [\psi_m^{(k)}]^2 \gamma_m \\ \sum_m [\psi_m^{(k)}]^{*2} \gamma_m & \varepsilon_k + i\mu \end{pmatrix}, \tag{5}$$

whose eigenvalues become complex for $\overline{\mu_{PT}^2} = \{\sum_m [\psi_m^{(k)}]^2 \gamma_m\}^2$. Therefore, using the self-average $\overline{[\psi_m^{(k)}]^2} = 1/M$, as well as $\gamma \sim \sqrt{T}\Delta M/2\pi$ for $T \ll 1$,

$$\mu_{PT} \sim N\sqrt{T}\Delta/2\pi = \mu_T \quad (\text{orthogonal}, T \lesssim 1/N), \tag{6}$$

which indeed recovers the numerical scale in the weak coupling regime of the orthogonal class.

As indicated, this analysis is restricted to small values of T, corresponding to small tunnel splittings, so that the levels of the originally degenerate sequence ε_k from the two subsystems do not cross. Consequently, at larger values of T, a second route to $\mathcal{PT}$-symmetry breaking becomes available, which involves two energy levels that are *nondegenerate* for $T = 0$. In order to describe this case we reformulate the problem by starting with $\mu = 0$ and exploit the thus-emerging $\mathcal{P}$ symmetry to transform the effective Hamiltonian to

$$\mathcal{H}_\mathcal{P} = \begin{pmatrix} H + \Gamma & i\mu \\ i\mu & H - \Gamma \end{pmatrix}. \tag{7}$$

We denote by $\varepsilon_k^\pm$ the two level sequences of $H \pm \Gamma$. Since Γ is positive semidefinite these sequences arise from the sequence ε_k by an oppositive shift which is approximately rigid. From the resulting combined sequence, consider two levels ε_k^+ and ε_l^- which lie adjacent to each other; the corresponding eigenvectors are $\psi^{(k+)}$ and $\psi^{(l-)}$. Finite μ mixes these levels, which is embodied in the reduced Hamiltonian

$$\mathcal{H}'' = \begin{pmatrix} \varepsilon_k^+ & i\mu\langle\psi^{(k+)}|\psi^{(l-)}\rangle \\ i\mu\langle\psi^{(l-)}|\psi^{(k+)}\rangle & \varepsilon_l^- \end{pmatrix}. \tag{8}$$

Now, treating 2Γ as a perturbation which connects the $+$ and $-$ sequence, $\langle\psi^{(k+)}|\psi^{(l-)}\rangle \approx \frac{\langle\psi^{(k+)}|2\Gamma|\psi^{(l+)}\rangle}{\varepsilon_k^+ - \varepsilon_l^+}$. Because $|\varepsilon_k^+ - \varepsilon_l^-| = O(\Delta)$ is small compared to the shift due to the coupling, the denominator can be estimated as $\varepsilon_k^+ - \varepsilon_l^+ \approx \varepsilon_l^- - \varepsilon_l^+ \approx -\langle\psi^{(l+)}|2\Gamma|\psi^{(l+)}\rangle$. The coupling strength drops out, and on average $|\langle\psi^{(l-)}|\psi^{(k+)}\rangle|^2 \sim 1/N$; that is, the mixing is small. As a result, the level pair in question becomes complex for $\mu^2|\langle\psi^{(l-)}|\psi^{(k+)}\rangle|^2 \sim (\varepsilon_k^+ - \varepsilon_l^-)^2 \sim \Delta^2$; that is,

$$\mu_{PT} \sim \sqrt{N}\Delta/2\pi = \mu_0 \quad (\text{orthogonal}, T \gtrsim 1/N). \tag{9}$$

As indicated, comparison with Eq. (6) implies that this mechanism becomes favorable around $T = T_c \sim 1/N$.

We now turn to the unitary class, and start again in the perturbative regime (we find that this now indeed extends to $T = 1$). The reduced Hamiltonian is still of the form (5), but the components of the vector $\psi_m^{(k)}$ now are complex. Therefore, $\overline{[\psi_m^{(k)}]^2} = 0$, and the average $\overline{[\psi_m^{(k)}]^2[\psi_n^{(k)}]^{*2}} \sim \delta_{mn}/M^2$ appearing in the expression of μ_{PT}^2 involves an additional contraction over channel indices m and n. As a result, the perturbative crossover scale is now given by

$$\mu_{PT} \sim \sqrt{NT}\Delta/2\pi = \mu_T' \quad (\text{unitary}). \tag{10}$$

Compared to the perturbative expression (6) in the orthogonal class, μ_{PT} is, therefore, reduced by the parametrically large factor $\sim\sqrt{N}$. Physically, this amounts to vastly reduced tunnel splittings. Therefore, the perturbative regime now extends to $T = 1$, considering that only at this value does it cross over to μ_0. A related difference to the orthogonal class is revealed in the $\mathcal{P}$ basis, where in place of Eq. (7) we now have

$$\mathcal{H}_\mathcal{P} = \begin{pmatrix} \mathrm{Re}H + \Gamma & i\mathrm{Im}H + i\mu \\ i\mathrm{Im}H + i\mu & \mathrm{Re}H - \Gamma \end{pmatrix}. \tag{11}$$

Consequently, finite coupling now results in a direct mixing of levels in the individual sequences. Therefore, instead of level crossings one encounters level repulsion. This difference is illustrated in Fig. 1(b), which shows the evolution of energy levels as T is increased from 0 to 1 (while $\mu = 0$), and the subsequent fate of real levels as μ is increased from 0 to $4\mu_0$ (while $T = 1$); pairwise coalescing levels become complex and then are no longer shown. In the orthogonal class, such pairs trace back to well-separated levels ε_k^+, ε_l^- from the two different sequences (which are distinguished by the opposite slopes of the levels for increasing coupling). In contrast, in the unitary class the coalescing levels trace back to originally closely spaced or degenerate levels, even when the coupling is strong.

Conclusions. In summary, we identified two routes to the formation of complex energy levels in non-Hermitian quantum systems with $\mathcal{PT}$ symmetry (spontaneous $\mathcal{PT}$-symmetry breaking). The predominant mechanism depends on whether or not the Hermitian limit possesses time-reversal symmetry (orthogonal or unitary universality class, respectively). Present optical experiments fall into the orthogonal class, where level crossings result in a characteristic absorption and amplification rate $\mu_{\mathcal{PT}}$ which is independent of the coupling between the symmetry-related parts of the system (unless the coupling is very weak). The unitary class features strong level repulsion, which reduces $\mu_{\mathcal{PT}}$ and makes it coupling dependent.

While we employed random-matrix theory to obtain specific expressions, the concept of discrete unitary and antiunitary symmetries like $\mathcal{P}$ and $\mathcal{T}$ and their relation to universality and level crossings are broader [3,4]. Therefore, the different routes identified here also apply to alternative models [19], and indeed to individual systems. In particular, in optical or microwave experiments, the two routes could be verified in a fixed geometry as illustrated in Fig. 1(a), where one varies the transparency of a semitransparent mirror to change the coupling between the left and the right cavities. Instead of amplification, it suffices to implement different rates μ_L, μ_R of absorption, which results in complex resonance frequencies ω_n on the line $\mathrm{Im}\omega_n = -(\mu_L + \mu_R)/2$, as well as pairs of resonances symmetrical to this line. In such settings, time-reversal symmetry can be broken via magneto-optical effects [24,25].

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$$\mathcal{H} = \begin{pmatrix} H - i\mu & \Gamma \\ \Gamma & H + i\mu \end{pmatrix},$$

even when time-reversal symmetry is broken. Compared to the $\mathcal{PT}$-symmetric case, this requires to invert the magneto-optical effects in one part of the system (i.e., the orientation of the effective magnetic field). In the parity basis, the Hamiltonian then takes the form of Eq. (7) even for unitary symmetry. Coupling now induces level crossings, and the transition to the complex spectrum is governed by the same characteristic scales μ_T and μ_0 as encountered in the orthogonal symmetry class of $\mathcal{PT}$-symmetric systems.