ON CONJUGACY OF MAXIMAL SUBALGEBRAS OF SOLVABLE LIE ALGEBRAS

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Abstract

The purpose of this paper is to consider when two maximal subalgebras of a finite-dimensional solvable Lie algebra L are conjugate, and to investigate their intersection.
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Throughout L will denote a finite-dimensional solvable Lie algebra over a field F. Let $x \in L$ and let $\operatorname{ad} x$ be the corresponding inner derivation of L. If F has characteristic zero suppose that $(\operatorname{ad} x)^n = 0$ for some n; if F has characteristic p suppose that $x \in I$ where I is a nilpotent ideal of L of class less than p. Put

$$\exp(\operatorname{ad} x) = \sum_{r=0}^{\infty} \frac{1}{r!} (\operatorname{ad} x)^r.$$

Then $\exp(\operatorname{ad} x)$ is an automorphism of L. We shall call the group $\mathcal{I}(L)$ generated by all such automorphisms the group of *inner automorphisms* of

L. More generally, if B is a subalgebra of L we denote by $\mathcal{I}(L : B)$ the group of automorphisms of L generated by the $\exp(\operatorname{ad} x)$ with $x \in B$. Two subsets U, V are *conjugate under* $\mathcal{I}(L : B)$ if $U = \alpha(V)$ for some $\alpha \in \mathcal{I}(L : B)$; they are *conjugate in* L if they are conjugate under $\mathcal{I}(L) = \mathcal{I}(L : L)$.

If U is a subalgebra of L, the *centraliser* of U in L is the set $C_L(U) = \{x \in L : [x, u] = 0\}$. In [1] Barnes showed that if A is a minimal ideal of L that is equal to its own centraliser in L, then A is complemented in L and all complements are conjugate under $\mathcal{I}(L : A)$. In [4] Stitzinger extended this result by finding necessary and sufficient conditions for two complements of an arbitrary minimal ideal of L to be conjugate.

Theorem 0.1 *([4, Theorem 1]) Let A be a minimal ideal of the solvable Lie algebra L. Then there is a bijection between the set $\mathcal{M}$ of conjugacy classes of complements to A under $\mathcal{I}(L : A)$ and the set $\mathcal{N}$ of complements to A in $C_L(A)$ that are ideals of L.*

Corollary 0.2 *([4, Corollary]) Suppose that L is a solvable Lie algebra and let M, K be complements to a minimal ideal A of L. Then M and K are conjugate under $\mathcal{I}(L : A)$ if and only if $M \cap C_L(A) = K \cap C_L(A)$.*

Clearly, such complements are maximal subalgebras of L. The purpose of this paper is to consider further when two maximal subalgebras of L are conjugate, and to investigate their intersection.

Lemma 0.3 *Let L be a solvable Lie algebra, and let M, K be two core-free maximal subalgebras of L. Then M, K are conjugate under $\exp(\operatorname{ad} a) = 1 + \operatorname{ad} a$ for some $a \in A$; in particular, they are conjugate in L.*

Proof. Let A be a minimal abelian ideal of L. Then $L = A \oplus M = A \oplus K$, $C_L(A) = A$ and A is the unique minimal ideal of L, by [5, Lemma 1.3]. The result, therefore, follows from [3, Theorem 1.1]. □

If U is a subalgebra of L, its *core*, U_L, is the largest ideal of L contained in U.

Theorem 0.4 *Suppose that L is a solvable Lie algebra over a field F. If F has characteristic p suppose further that L^2 has nilpotency class less than p. Let M, K be maximal subalgebras of L. Then M is conjugate to K in L if and only if $M_L = K_L$.*

Proof. Suppose first that M, K are conjugate in L, so that $K = \alpha(M)$ for some $\alpha \in \mathcal{I}(L)$. Then it is easy to see that $\exp(\operatorname{ad} x)(M_L) = M_L$ whenever $\exp(\operatorname{ad} x)$ is an automorphism of L, whence $K_L = \alpha(M_L) = M_L$.

Conversely, suppose that $M_L = K_L$. Then $M/M_L, K/M_L$ are core-free maximal subalgebras of L/M_L, and so are conjugate under $\mathcal{I}(L/M_L : (L/M_L)^2)$, by Lemma 0.3. But now M and K are conjugate under $\mathcal{I}(L : L^2)$ by [2, Lemma 5], and so are conjugate in L. $\square$

The above result does not hold for all solvable Lie algebras, as the following example shows.

Example **0.1** *Let F be a field of characteristic p and consider the Lie algebra $L = (\oplus_{i=0}^{p-1} Fe_i) \dot{+} Fx \dot{+} Fy$ with $[e_i, x] = e_{i+1}$ for $i = 0, \dots, p-2$, $[e_{p-1}, x] = e_0$, $[e_i, y] = ie_i$ for $i = 0, \dots, p-1$, $[x, y] = x$, and all other products zero. Let C be a faithful completely reducible L-module. Since L is monolithic with monolith $A = \oplus_{i=0}^{p-1} Fe_i$, C has a faithful irreducible submodule B. Let X be the split extension of B by L. Then $A + Fx + Fy$ and $A + F(x + e_1) + Fy$ are maximal subalgebras of X, both of which have A as their core. However, B is the unique minimal ideal of L and these subalgebras are not conjugate under inner automorphisms of the form $1+ \operatorname{ad} b$, $b \in B$. Since B is the nilradical of X, defining other inner automorphisms is problematic.*

Let $0 = L_0 < L_1 < \ldots < L_n = L$ be a chief series for L and let M be a maximal subalgebra of L. Then there exists k with $0 \leq k \leq n-1$ such that $L_k \subseteq M$ but $L_{k+1} \not\subseteq M$. Clearly $L = M + L_{k+1}$ and $M \cap L_{k+1} = L_k$; we say that the chief factor L_{k+1}/L_k is *complemented* by M.

Theorem 0.5 *Suppose that L is a solvable Lie algebra over a field F. If F has characteristic p suppose further that L^2 has nilpotency class less than p. Let A/B be a chief factor of L that is complemented by a maximal subalgebra M of L. If K is conjugate to M in L then $K = exp(ad\, a)(M)$ for some $a \in A$ and $M \cap K = \{m \in M : [m, a] \in M\}$.*

Proof. We have that $L = A + M$ with $A^2 \subseteq M_L$, $M^2 \subseteq M_L$, $B \subseteq A \cap M_L$, and $M_L = K_L$. Clearly then $B \subseteq A \cap M_L \subseteq A$. Moreover, $A \neq A \cap M_L$ since $A \not\subseteq M$. It follows that $B = A \cap M_L$ because A/B is a chief factor. Thus

$$\frac{A + M_L}{M_L} \cong \frac{A}{A \cap M_L} = \frac{A}{B},$$

whence $(A+M_L)/M_L$ is a minimal abelian ideal of L/M_L. Lemma 0.3 implies that $K/M_L = \exp(\mathrm{ad}\,(a+M_L))(M/M_L)$ for some $a \in A$.

Now $[L,A] \subseteq B$ or $[L,A] = A$. The former implies that $[L,A] \subseteq M_L$, contradicting the fact that $(A+M_L)/M_L$ is self-centralising in L/M_L, by [5, Lemma 1.4]. Hence $A = [L,A] \subseteq L^2$, and so $\exp(\mathrm{ad}\, a)$ is defined. If $x \in \exp(\mathrm{ad}\, a)(M)$ then $x + M_L = \exp(\mathrm{ad}\, a)(m) = m + [m,a] + M_L \in K/M_L$ for some $m \in M$, whence $x \in K$. Since $\exp(\mathrm{ad}\, a)$ is an automorphism of L we must have $K = \exp(\mathrm{ad}\, a)(M)$.

Finally we have $(M \cap K)/M_L = (M/M_L) \cap (K/M_L) = C_{(M/M_L)}(a+M_L)$ by [5, Lemma 1.5]. We infer that $m \in M \cap K \Leftrightarrow [m,a] \in M_L \Leftrightarrow [m,a] \in M$. □

Theorem 0.5 gave a characterisation of the intersection of two conjugate maximal subalgebras of L. Finally we consider the intersection of two non-conjugate maximal subalgebras of L.

Theorem 0.6 *Suppose that L is a solvable Lie algebra over a field F. Let M, K be maximal subalgebras of L, and suppose that $K_L \not\subseteq M_L$. Then $M \cap K$ is a maximal subalgebra of M.*

Proof. We have that $K_L \not\subseteq M$, so $L = M + K_L = M + K$. If $K = K_L$ then $L/K \cong M/(M \cap K)$ and the result is clear. So suppose that $K \neq K_L$. Let A/K_L be a minimal ideal of L/K_L. Then $L/K_L = A/K_L \oplus K/K_L$, from [5, Lemma 1.4], giving $A \cap K = K_L$. Also, $A = A \cap (M + K_L) = A \cap M + K_L$, whence

$$\frac{A}{K_L} = \frac{A \cap M + K_L}{K_L} \cong \frac{A \cap M}{K_L \cap M},$$

showing that $(A \cap M)/(K_L \cap M)$ is a minimal ideal of $M/(K_L \cap M) (\cong L/K_L)$ and $A \cap M$ is a minmal ideal of M. Now

$$\dim\left(\frac{M}{M \cap K}\right) \geq \dim\left(\frac{A \cap M + M \cap K}{M \cap K}\right) = \dim\left(\frac{A \cap M}{K_L \cap M}\right)$$

$$= \dim\left(\frac{A}{K_L}\right) = \dim\left(\frac{L}{K}\right) = \dim\left(\frac{M+K}{K}\right) = \dim\left(\frac{M}{M \cap K}\right).$$

It follows that $M = A \cap M + M \cap K$ which yields the result. □

Corollary 0.7 *Suppose that L is a solvable Lie algebra over a field F. If F has characteristic p suppose further that L^2 has nilpotency class less than p. Let M, K be maximal subalgebras of L that are not conjugate in L. Then $M \cap K$ is a maximal subalgebra of at least one of M, K.*

Corollary 0.8 *Suppose that L is a solvable Lie algebra and let M, K be complements to a minimal ideal A of L that are not conjugate in L. Then $M \cap K$ is a maximal subalgebra of both M and K.*

Proof. By Theorem 0.1, both M_L and K_L are complements to A in $C_L(A)$. Since $M_L \neq K_L$ we have $M_L \not\subseteq K_L$ and $K_L \not\subseteq M_L$. The result now follows from Theorem 0.6. □

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