ALMOST NILPOTENT LIE ALGEBRAS

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1. Introduction and main results. Throughout we shall consider only finite-dimensional Lie algebras over a field of characteristic zero. In [**3**] it was shown that the classes of solvable and of supersolvable Lie algebras of dimension greater than two are characterised by the structure of their subalgebra lattices. The same is true of the classes of simple and of semisimple Lie algebras of dimension greater than three. However, it is not true of the class of nilpotent Lie algebras. We seek here the smallest class containing all nilpotent Lie algebras which is so characterised.

Let L, M be Lie algebras over the same field F, and let $\mathscr{L}(L)$, $\mathscr{L}(M)$ denote their lattices of subalgebras. By an *l-isomorphism* (lattice isomorphism) of L onto M we mean a bijective map $\theta : \mathscr{L}(L) \to \mathscr{L}(M)$ such that

$$\theta(A \cap B) = \theta(A) \cap \theta(B) \quad \text{and} \quad \theta(A \cup B) = \theta(A) \cup \theta(B)$$

for all subalgebras A, B of L (where $A \cup B$ denotes the subalgebra of L generated by A and B in L). We shall write A^* for $\theta(A)$, the image of $A \in \mathscr{L}(L)$ under the l-isomorphism from L onto $M = L^*$.

If $x_1, \ldots, x_n \in L$ we shall denote by $((x_1, \ldots, x_n))$ (respectively, $\langle x_1, \ldots, x_n \rangle$) the subspace (respectively, subalgebra) of L generated by $x_1, \ldots, x_n$. We shall often write $((x^*))$ for $((x))^*$.

A Lie algebra L is called *almost abelian* if it is a split extension $L = L^2 + ((a))$ with $L^2 \cap ((a)) = 0$ and with *ada* acting as the identity map on the abelian ideal L^2; L is *quasi-abelian* if it is abelian or almost abelian. (Note that quasi-abelian Lie algebras are what the author has previously called semiabelian Lie algebras).

We shall call L *almost nilpotent of index n* if it has a basis

$$\{x; e_{11}, \ldots, e_{1r_1}; \ldots; e_{n1}, \ldots, e_{nr_n}\}$$

such that

$$-e_{ij}x = xe_{ij} = e_{ij} + e_{i+1,j} \quad \text{for} \quad 1 \leqq i \leqq n-1,\ 1 \leqq j \leqq r_i,$$

$$-e_{nj}x = xe_{nj} = e_{nj}, \quad \text{and} \quad r_j \leqq r_{j+1} \quad \text{for} \quad 1 \leqq j \leqq n-1,$$

all other products being zero. We shall refer to this as the *standard basis*. Note that $L = N + ((x))$, where N is an abelian ideal of L (in fact, it is the nilradical), and $N \cap ((x)) = 0$. Then L is *quasi-nilpotent* if it is nilpotent or almost nilpotent.

Finally we shall say that the nilpotent Lie algebra L has *index of nilpotency* (or just *index*) n if $L^n \neq 0$, but $L^{n+1} = 0$.

The main results of this paper are the following theorems.

THEOREM 1. *Let L be a nilpotent Lie algebra of index n and of dimension greater than two for which L^* is not nilpotent. Then L^* is almost nilpotent of index n.*

THEOREM 2. *Every almost-nilpotent Lie algebra is l-isomorphic to a nilpotent Lie algebra.*

COROLLARY 3. *If L is a quasi-nilpotent Lie algebra of index n of dimension greater than two then so is L^*.*

COROLLARY 4. *If L is a nilpotent Lie algebra in which $L^{(3)} = (L^2)^2 \neq 0$ then L^* is also nilpotent.*

2. The proofs. The following lemmas will prove useful.

LEMMA 5. *If L is quasi-abelian then either L^* is quasi-abelian, or L is two-dimensional and L^* is three-dimensional non-split simple.*

Proof. Let L be quasi-abelian. Then L is upper semi-modular (see [**1**], Theorem 2.4). Since this condition is clearly preserved by l-isomorphisms, L^* is upper semi-modular. The result now follows from Theorem 2.4 of [**1**].

LEMMA 6. *Let L be a nilpotent Lie algebra of dimension greater than two and let $((x^*, y^*, z^*))$ be a three-dimensional subalgebra of L^*.*
(i) *If $x^*y^* = y^*$, $x^*z^* = \lambda z^*$ and $y^*z^* = 0$ then $\lambda = 1$.*
(ii) *If $x^*y^* = y^* + z^*$, $x^*z^* = \lambda z^*$ and $y^*z^* = 0$ then $\lambda = 1$.*

Proof. (i) It is easy to see from Lemma 5 that $((x, y))$, $((x, z))$ and $((y, z))$ are quasi-abelian.
Since L is nilpotent, they must be abelian; hence L is abelian. It follows that L^* is almost abelian, giving the desired result.
(ii) If $\lambda \neq 1$ then replacing y^* by $y^* + (1/(1-\lambda))z^*$ makes it clear that the subalgebra has the same structure as in (i).

Proof of Theorem 1. First note that L^* is solvable [**3**, Theorem 4.2]. We use induction on the index of nilpotency of L. The result is clear if L is abelian, so suppose that it holds when the index is strictly less than k ($k \geqslant 2$), and let L be a Lie algebra for which $L^{k+1} = 0$, $L^k \neq 0$. By Theorem 5.1 of [**3**], $(L^k)^*$ is an ideal of L^* and L/L^k is l-isomorphic to $L^*/(L^k)^*$. Suppose that $L^*/(L^k)^*$ is nilpotent. Then $L^*/(L^2)^*$ is nilpotent. But L^2 is the Frattini subalgebra of L (see [**2**], for example), and so $(L^2)^*$ is the Frattini subalgebra of L^*. It follows from Theorem 6.1 of [**2**] that L^* is nilpotent, a contradiction. By the inductive hypothesis, therefore, there is a basis for L^* of the form

$$\{x^*; f_{11}^*, \ldots, f_{1r_1}^*; \ldots; f_{k-1,1}^*, \ldots, f_{k-1,r_{k-1}}^*; f_{k1}^*, \ldots, f_{kr_k}^*\}$$

such that $\{f_{k1}^*, \ldots, f_{kr_k}^*\}$ is a basis for $(L^k)^*$,

$$x^*f_{ij}^* = f_{ij}^* + f_{i+1,j}^* + g_{ij}^* \text{ for } 1 \leqq i \leqq k-2,$$

$$x^*f_{k-1,j}^* = f_{k-1,j}^* + g_{k-1,j}^*, \quad \text{where} \quad g_{ij}^* \in (L^k)^*,$$

and any product of the form $f_{ij}^*f_{pq}^*$ belongs to $(L^k)^*$.

Put $N^* = \langle \{f_{ij}^* : 1 \leqq i \leqq k, 1 \leqq j \leqq r_i\} \rangle$ and $F = L^2$. Then $N^* = (L^*)^2$ is nilpotent, and F is the Frattini subalgebra of L. Therefore F^* is the Frattini subalgebra of L^*, and so $F^* = \langle \{f_{ij}^* : 2 \leqq i \leqq k, 1 \leqq j \leqq r_i\} \rangle$ and we have the following situation

$$\begin{array}{ccccccccccccc}
 & & & & & & & & F & & & & \\
 & & & & & & & & \| & & & & \\
0 & — & L^k & — & \cdots & — & L^3 & — & L^2 & — & N & — & L \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0^* & — & (L^k)^* & — & \cdots & — & (L^3)^* & — & F^* & — & N^* & — & L^*
\end{array}$$

Now $\langle x, F \rangle$ is of index at most $k-1$, and so $\langle x^*, F^* \rangle = \langle x, F \rangle^*$ is almost nilpotent, by the inductive hypothesis. Hence its nilradical, which must be F^*, is abelian; that is,

(i) $(F^*)^2 = 0$.

Also, for any $y \in L$, $\langle y, L^k \rangle$ is abelian, so that $\langle y^*, (L^k)^* \rangle$ is quasi-abelian. Now if $y^* \in N^*$, $\langle y^*, (L^k)^* \rangle$ is nilpotent, and therefore must be abelian. We thus have

(ii) $N^*(L^k)^* = 0$.

We now claim that $f_{1i}^* f_{mj}^* = 0$ for $1 \leqq m \leqq k$. The proof proceeds by backwards induction on m. Result (ii) above shows that it holds for $m = k$, so suppose that it holds for $m = t$ and consider the case $m = t-1$. Then

$$\begin{aligned}
x^*(f_{1i}^* f_{t-1,j}^*) &= -f_{1i}^*(f_{t-1,j}^* x^*) - f_{t-1,j}^*(x^* f_{1i}^*) \\
&= f_{1i}^*(f_{t-1,j}^* + f_{tj}^* + g_{t-1,j}^*) - f_{t-1,j}^*(f_{1i}^* + f_{2i}^* + g_{1i}^*) \\
&= 2 f_{1i}^* f_{t-1,j}^*
\end{aligned}$$

by the inductive hypothesis. Suppose that $f_{1i}^* f_{t-1,j}^* \neq 0$. Then $\langle x^*, (L^k)^* \rangle$ is almost abelian, and $x^* g^* = 2g^*$ for all $g^* \in (L^k)^*$. But if $g_{k-1,1}^* = 0$ then $((x^*, f_{k-1,1}^*, f_{1i}^* f_{t-1,j}^*))$ contradicts Lemma 6(i), and if $g_{k-1,1}^* \neq 0$ then $((x^*, f_{k-1,j}^*, g_{k-1,1}^*))$ contradicts Lemma 6(ii). The result follows by induction. We have now shown

(iii) $(N^*)^2 = 0$.

As was remarked in the paragraph preceding (ii), $\langle x^*, (L^k)^* \rangle$ is quasi-abelian. Suppose it is abelian. Then, if $g_{k-1,1}^* = 0$, the subalgebra $((x^*, f_{k-1,1}^*, g^*))$ (for any $g^* \in (L^k)^*$) contradicts Lemma 6(i), and, if $g_{k-1,1}^* \neq 0$, the subalgebra $((x^*, f_{k-1,1}^*, g_{k-1,1}^*))$ contradicts Lemma 6(ii). It follows that $\langle x^*, (L^k)^* \rangle$ is almost abelian, and hence that $x^* g^* = g^*$ for every $g^* \in (L^k)^*$.

Put $e_{ij}^* = f_{1j}^*$, $e_{2j}^* = f_{2j}^* + g_{1j}^*, \ldots, e_{k-1,j}^* = f_{k-1,j}^* + g_{k-2,j}^*$, $e_{kj}^* = g_{k-1,j}^*$. It is easy to check that the multiplication table for these elements is that given for an almost nilpotent Lie algebra, that $\{e_{mj}^* : 1 \leqq m \leqq k-1, 1 \leqq j \leqq r_m\}$ is a linearly independent set, and that $(L^k)^*$ is spanned by $\{e_{kj}^* : 1 \leqq j \leqq r_k\}$. The only problem is that the latter set may be linearly dependent.

Suppose that $\sum_{j=1}^{r_k} \lambda_j e_{kj}^* = 0$ and that $\lambda_{r_k} \neq 0$. Then

$$x^*\left(\sum_{j=1}^{r_k} \lambda_j e_{k-1,j}^*\right) = \sum_{j=1}^{r_k} \lambda_j e_{k-1,j}^*$$

and

$$x^*\left(\sum_{j=1}^{r_k} \lambda_j e_{mj}^*\right) = \sum_{j=1}^{r_k} \lambda_j (e_{mj}^* + e_{m+1,j}^*) \qquad (1 \leqq m \leqq k-2),$$

so replace e_{mj}^* by $\sum_{j=1}^{r_k} \lambda_j e_{mj}^*$ for each $1 \leqq m \leqq k-1$. Continuing in this way produces the desired basis.

Proof of Theorem 2. Let L be almost nilpotent of index n and denote the standard basis as in the definition. We claim that L is l-isomorphic to the nilpotent Lie algebra L^* with the same basis elements (though we shall add a 'star' when referring to them as elements of L^*) and multiplication given by

$$x^* e_{ij}^* = -e_{ij}^* x^* = e_{i+1,j}^* \quad \text{for} \quad 1 \leqq i \leqq n-1, \quad 1 \leqq j \leqq r_{i+1}$$

(all other products being zero). Then the obvious non-singular linear transformation from L to L^* (namely, $x \mapsto x^*$, $e_{ij} \mapsto e_{ij}^*$) is a lattice isomorphism.

To prove this we use induction on the dimension of L. The result is clear if L is one dimensional, so suppose that it holds for Lie algebras of dimension strictly less than dim L which are almost nilpotent. It suffices to show that U is a subalgebra of L if and only if U^* is a subalgebra of L^*.

This clearly holds if U is a maximal subalgebra of L, since $L^*/F(L^*)$ is abelian, and $L/F(L)$ is almost abelian and of the same dimension as $L^*/F(L^*)$. Now the maximal subalgebras of L are either abelian (and equal to N, the nilradical) or are almost nilpotent, whilst the maximal subalgebras of L^* are either abelian (and equal to N^*) or are of the same form as L. So the case where U is not maximal is dealt with by the inductive hypothesis.

The proofs of Corollaries 3 and 4 are straightforward.

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