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Value-At-Risk Optimal Policies for Revenue Management Problems

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Abstract

Consider a single-leg dynamic revenue management problem with fare classes controlled by capacity in a risk-averse setting. The revenue management strategy aims at limiting the down-side risk, and in particular, value-at-risk. A value-at-risk optimised policy offers an advantage when considering applications which do not allow for a large number of reiterations. They allow for specifying a confidence level regarding undesired scenarios.

We state the underlying problem as a Markov decision process and provide a computational method for computing policies, which optimise the value-at-risk for a given confidence level. This is achieved by computing dynamic programming solutions for a set of target revenue values and combining the solutions in order to attain the requested multi-stage risk-averse policy. Numerical examples and comparison with other risk-sensitive approaches are discussed.

Keywords: capacity control, revenue management, risk, value-at-risk

1 Introduction

Revenue management deals with controling a revenue stream resulting from selling products using a fixed and perishable resource. The industries which use revenue management are manifold. The most popular representatives are airlines, hotels, rental cars, and advertising. But revenue management is also common in event management, ferry lines, retailing or healthcare, to name a few. Talluri and van Ryzin (2005) and Chiang et al. (2007) provide a comprehensive overview of revenue management.

The firm sells multiple products, each consuming a fixed resource with a limited capacity. In this setting, we consider quantity-based revenue management in which a company offers all or just a subset of all products at each point in time. There is a finite time horizon for selling the products, as at the end of the horizon, the salvage value of the resource is zero.

The most common settings use the assumption of a risk-neutral objective. Thus, the policy of the firm is the maximisation of the expected value of its revenue. Often, such a risk-neutral objective is conducive. As in most applications, such as daily operating ferry lines, this policy is repetitively used. By the law of large numbers, using the expected value as the objective function is then appropriate.

Nevertheless, risk neutrality may not be adequate to other industries, such as event management, that do not support a large number of repetitions of a policy. Several scenarios are known that argue for the considerations of risk-sensitive or risk-averse policies.

Levin et al. (2008) emphasize that, in particular, an event promoter has a high risk, as the promoter cannot count on a large number of reiterations of events. The promoter faces high fixed costs and predominantly has to recover them in order to avoid a possible high loss. Financial and also strategic reasons might not allow running into negative cash, because operational mobility might suffer.

Both Bitran and Caldentey (2003) and Weatherford (2004) provide further examples that risk-neutral considerations are not applied for every real scenario. They report that airline analysts show some natural risk-averse behaviours, and they overrule their revenue management system in situations when the system recommends waiting for high-fare passengers, instead accepting low-fare passengers a few days before flight departure.

That risk-neutral and risk-sensitive policies make a difference is shown in several recent papers. Barz and Waldmann (2007), Huang and Chang (2009), Koenig and Meissner (2009a) and Koenig and Meissner (2009b) analyse both types of policies using the same underlying model that is used in this paper. All four approaches analyse the effects of applying different kinds of risk-sensitive polices, assuming various levels of risk aversion for a decision maker. However, none of these approaches computes an optimal policy for the common risk measures, such as standard deviation, value-at-risk, or conditional-value-at-risk. However, simulations can be run to determine their values for a given policy.

In this paper, we propose a method which computes a value-at-risk optimal policy. The value-at-risk ($V@R$) is a common risk measure often used in finance (cf. Jorion 2006). It measures down-side risk and is determined for a given probability level. With regard to $V@R$, this probability level is often referred to as confidence level. In our context, the $V@R$ is the lowest revenue which exceeds the confidende level, which is often set at 5 or 10%. Basically, it is a quantile of the revenue distribution determined by the given confidence level.

The advantage of using $V@R$ as parameter to be optimised is that it is a well-known risk measure, and it is easily interpreted by practitioners. A desired confidence level is specified, and the $V@R$ is returned in the monetary unit of the revenue. Other risk-sensitive approaches often require an interpretation of an uncommon parameter to adjust the desired level of risk preference.

In order to find a $V@R$ optimal policy, we take advantage of the computing a target level optimal policy proposed by Koenig and Meissner (2009b). The target level optimal policy can be computed for a certain target and gives information about the probability of not achieving this target. This probability is minimised to find the best policy. It defines a confidence level for a fixed target, which is the corresponding $V@R$. Hence, our task is similar to computing a target level optimised policy, but we optimise the treshold value instead of the percentile. We are given with $V@R$ optimal policies and have to determine the policy which is the best one for the desired confidence by their associated confidence levels. We describe in this paper how that can be accomplished in an efficient manner.

The paper is structured as follows. This introduction is followed a brief overview of related work dealing with revenue management models incorporating risk in Section 2. In Section 3, we continue with the description of the revenue model, which builds our basic position. We describe the target level approach and how we use it to efficiently obtain a $V@R$ optimal policy. We discuss different strategies useful for numerical approximation of such a policy. Section 4 gives a detailed overview of the numerical results and studies the effect of numerical approximation methods. Finally, we conclude this paper in Section 5.

2 Related Work

As a starting point for our analysis we use the basic model by Lee and Hersh (1993). They introduce the dynamic capacity control model in a risk-neutral setting. Lautenbacher and Stidham (1999) take this model further and derive a corresponding Markov decision process. This description as a Markov decision process is advantageous for model extensions.

First risk considerations in revenue management models are proposed by Feng and Xiao (1999). Their model considers risk in terms of variance of sales due to changes of prices. To this end, a penalty function reflecting this variance is incorporated in the objective function of the model. Further, Feng and Xiao (2008) integrate expected utility theory into revenue management models in order to support risk-sensitive decisions.

Expected utility theory as tool for risk consideration is recommended by Weatherford (2004), as well. From a practitioner's perspective, he criticizes risk-neutral revenue management and endorses risk-averse models, in particular, the expected marginal seat revenue (EMSR) heuristic by Belobaba [reference].

Barz and Waldmann (2007) base their risk-sensitive model on the Markov decision process of the dynamic capacity model and expected utility theory. They integrate an exponential utility function as the objective function into the Markov decision model. The exponential utility function allows the use of different levels of risk-sensitivity.

Another way of employing expected utility theory in a revenue management context is proposed by Lim and Shanthikumar (2007). They analyse robust and risk-sensitive control with an exponential utility function for dynamic pricing.

Lai and Ng (2005) formulate a robust optimisation model for revenue management in the hotel industry. Their model incorporates mean versus average deviation. Mitra and Wang (2005) look at mean-variance, mean-standard-deviation and mean-conditional-value-at-risk approach for deriving a risk-sensitive objective function with revenue management application in traffic and networks. Koenig and Meissner (2008) demonstrate that risk considerations might lead to different decisions when deciding between a quantity-based or and price-based revenue model.

Also applying risk considerations to the dynamic capacity model, Huang and Chang (2009) show the effect of using a relaxed optimality condition instead of the optimal one. They investigate model behaviour in numerical simulations and discuss results, given as mean and standard deviation and in a ranking based on a Sharpe ratio. A related approach is presented by Koenig and Meissner (2009a), who provide a detailed study of several risk-averse policies for the dynamic capacity model by applying risk measures.

Regarding the use of $V@R$, Lancaster (2003) provides some strong arguments. He demonstrates that risk-neutral revenue management models are vulnerable to the inaccuracy of demand forecasts. Inspired by the $V@R$ metric, he recommends the relative revenue per available seat mile at risk metric. His metric measures the expected maximum of underperformance over a time period for a given confidence level.

Finally, the idea of expanding the state spaces of revenue management models is used by Levin et al. (2008) and Koenig and Meissner (2009b) in order to consider risk in terms of probability for achieving a certain given revenue target. Levin et al. (2008) incorporate risk aversion into a dynamic pricing model of perishable products by integrating constraints into the objective function. Koenig and Meissner (2009a) use the Markov decision model of the dynamic capacity control model and compute optimal policies for revenue targets. Section 3 explains how to find a $V@R$ optimal policy that can employ this model. In a similar manner, finding a $V@R$ optimal policy could also integrate the approach of Levin et al. (2008) for computing probability of achieving a desired target in the associated context.

3 Modelling and Algorithm

In this section, we begin with a brief introduction of a well-known revenue management problem originally stated as risk-neutral formulation by Lee and Hersh (1993). We continue with a short summary of a recently proposed modification of this problem which leads to a risk-sensitive model. The risk-sensitive model optimises the risk of failing a previously defined revenue target and provides a basis for the proposed computational approach focussing on the value-at-risk metric. The value-at-risk metric is explained, and its computation is described in our setting.

3.1 Dynamic Capacity Control Revenue Management Problem

Lee and Hersh (1993) introduce a revenue management model often referred to as the dynamic capacity control model. It was originally formulated for the airline industry, and we too describe it in terms of this industry. Lautenbacher and Stidham (1999) state the problem as a Markov decision process. Using this representation, it is more convenient to derive risk-sensitive policies as done by Barz and Waldmann (2007) for an exponential utility and by Koenig and Meissner (2009b) for a target level. As we are interested in a computational approach for value-at-risk policies, we focus on dynamic programming equations, which are equivalent to their Markov decision process counterparts but more suitable for computation.

The model of Lee and Hersh (1993) divides the booking period for a single-leg flight into N decision periods. The decision periods are assumed small enough so that there is no more than one arrival in that period. The decision periods are represented by $n \in \{0, \ldots, N\}$ and 0 is the period of departure. There are k different fare classes with fares $F_i, F_1 > F_2 > \ldots > F_k$. Further, the probability $p^r_{n,i}$ denotes a request for the fare class i in period n. Probabilities for the last decision period $n = 0$ are zero for all fare classes: $p^r_{0,i} = 0$, meaning the last decision is made at $n = 1$. The probability of no request for any class is given by $p^r_{n,0} = 1 - \sum_{i=1}^{k} p^r_{n,i}$. Initial seat capacity is C, and remaining seats in time period n are given by $c \leq C$. In this model, a policy π is built from the decision rules which decide to accept or reject a booking request given the current capacity and time. The set of all policies is denoted by Π. The optimal risk-neutral policy $\pi^* \in \Pi$ is the policy which achieves the maximal expected revenue $V_n^{\pi^*}(c) = \max_\pi \mathcal{E}\left(\sum_{j=0}^{n} r_j\right)$, where r_n denotes the random variable for the gained revenue at time n when using a policy π. As Lee and Hersh (1993) show, such optimal policy can be computed by a dynamic programming solution:

$$V_n^{\pi^*}(c) = \begin{cases} \sum_{i=0}^{k} p^r_{n,i} \max_{a \in \{0,1\}} \left\{ aF_i + V_{n-1}^{\pi^*}(c-a) \right\}, & n > 0,\ c > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

3.2 Target Level Objective

The risk-sensitive approach proposed by Koenig and Meissner (2009b) builds the basis for calculating a value-at-risk optimised policy. Their approach computes an optimal policy for achieving a given target revenue. Boda and Filar (2006) describe the latter approach as a target-percentile problem, as the percentile for a fixed target is optimised.

To this end, they follow a method described by White (1988), Wu and Lin (1999) and Boda and Filar (2006). First, the objective function is the probability of failing the given target revenue. Thus, the objective function has to be minimised in order to derive the risk-sensitive policy. Second, the Markov decision process is augmented by a further state representing the currently remaining target to be achieved in latter time steps.

We use the same notation as before and introduce a few more variables. The recent target revenue is denoted by x_n and the given target value to be achieved at N time steps to go is x_N. The value function $W_n^\pi(c, x_n) := \mathcal{P}^\pi\left((\sum_{j=0}^{n} r_j) \leq x_n\right)$ stands for the probability of failing a target x_n, applying a policy $\pi \in \Pi$ in n remaining time steps and with remaining capacity c. The optimal policy $\tilde{\pi}^* = \operatorname{argmin}_\pi W_N^\pi$ minimises the risk of not attaining the target x_N. The dynamic programming solution computing this policy is given as:

$$W_0^{\tilde{\pi}^*}(c, x_0) = \begin{cases} 1 & x_0 > 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$W_n^{\tilde{\pi}^*}(c, x_n) = \sum_{i=0}^{k} p_{n,i}^r \min_{a \in \{0,1\}} \left\{ W_{n-1}^{\tilde{\pi}^*}(c - a, x_n - aF_i) \right\}. \tag{2}$$

For a target level x_N, we have to consider all possible realisations ending at the final time step 0. With each ongoing time step, a part of the target value can be achieved according to the decision made. The new target revenue x_{n-1} of the next time step $n - 1$ is given by the current target value minus the fare achieved in the current time step $x_n - aF_i$.

The border conditions for time step 0 are initialised with 1 for all positive targets and 0 otherwise. For all fares F_i attainable in the previous time step 1, the probability of failing is less than 1, so their probabilities can be excluded while summing up. Computing $W_N^{\tilde{\pi}^*}$ starts with initialising time step 0 and proceeds to time step N.

The optimal decision rule looks for the minimal probability between accepting a seat request while reducing the target with its fare and rejecting a request while retaining the current target.

However, the computation of the dynamic programming solution as described requires the computation of all cumulative rewards up the specified target x_N. As this computation of the complete solution is very inconvenient, a more suitable way is using a grid as discussed by Boda et al. (2004). In particular, the state space dimension which represents the target levels is reduced.

To this end, the complete range of all cumulative rewards is discretised. The interval between 0 and the target x_N is separated into m smaller intervals. Each interval spans a width of $\frac{x_N}{m}$. We use $y_i, i \in \{0, \ldots, m\}$ as variables for interval boundaries, and the intervals are $[y_0, y_1] := [0, \frac{x_N}{m}], [y_1, y_2] := [\frac{x_N}{m}, \frac{2x_N}{m}], \ldots, [y_{m-1}, y_m] := [\frac{x_N(m-1)}{m}, x_N]$. Instead of computing for each possible cumulative reward target x, only the upper boundaries are taken as targets. A target value inside an interval $y \in (y_i, y_{i+1}]$ is rounded to the upper interval boundary y_{i+1}. This boundary value y_{i+1} is used while approximately computing the dynamic programming solution.

The computation of $W_n^{\tilde{\pi}^*}$ is done only with value pairs of targets y_i and probabilities $W_n^{\tilde{\pi}^*}(c, y_i)$. We obtain a grid of values $\{(y_0, W_n^{\tilde{\pi}^*}(c, y_0)), \ldots, (y_m, W_n^{\tilde{\pi}^*}(c, y_m))\}$ for $c \in \{0, \ldots, C\}$. Using the dynamic programming, the probability values of the grid can be updated in various ways.

The simplest method is rounding occurring target values to the upper value, thus $W_n^{\tilde{\pi}^*}(c, y) = W_n^{\tilde{\pi}^*}(c, y_{j+1}) \forall y \in (y_j, y_{j+1}]$. However, this is very inaccurate. Using nearest neighbour or linear interpolation offers a more accurate way. Nearest neighbour approximation selects the value nearest to the actual required target value y. If the inequality $|y_{j+1} - y| < |y_j - y|$ is valid, the upper value on the grid is taken $W_n^{\tilde{\pi}^*}(c, y) = W_n^{\tilde{\pi}^*}(c, y_{j+1})$ else the lower value $W_n^{\tilde{\pi}^*}(c, y) = W_n^{\tilde{\pi}^*}(c, y_j)$ is taken. Linear interpolation computes weights according to the distances between actual value and grid values. These weights are combined for computing a value for $W_n^{\tilde{\pi}^*}(c, y) = \frac{|y_{j+1} - y|}{y_{j+1} + y_j} W_n^{\tilde{\pi}^*}(c, y_j) + \frac{|y_j - y|}{y_{j+1} + y_j} W_n^{\tilde{\pi}^*}(c, y_{j+1})$.

The dynamic program of Equation 2 is indifferent for equality when taking the minimum. There can be several ways for achieving the same minimum, and one of these ways should be selected. As a decision might be indifferent for minimising the probability, it might be beneficial for increasing the revenue. Thus, one strategy is to accept a request instead of rejecting it in such cases.

3.3 Value-at-Risk

The target level approach provides us with the means for computing a value-at-risk policy. We explain the value-at-risk metric first and move then to the computation of a value-at-risk optimal policy.

Given a predefined fixed confidence level, the value-at-risk metric computes the maximum loss that one might be exposed to. The confidence level $\alpha \in [0, \dots, 1]$ specifies the level of risk as probability level and its associated α-quantile is the value-at-risk. There is some inconsistency in the nomenclature of value-at-risk in the literature (cf. Pflug and Römisch 2007a, p57). We use the following definition of the value-at-risk:

$$V@R_\alpha(Y) = \inf\{u : \mathcal{P}(Y \leq u) \geq \alpha\},$$

where Y is a random variable and $\mathcal{P}$ denotes a probability measure. Using this definition, common values for α are 5 or 10 percent.

Applying the $V@R_\alpha$ metric to our model, we use the gained revenue r_n as the random variable and get

$$V@R_\alpha^\pi\left(\sum_{j=0}^{n} r_j\right) = \inf\left\{u : \mathcal{P}^\pi\left(\sum_{j=0}^{n} r_j \leq u\right) \geq \alpha\right\} = \inf\{u : W_n^\pi(c, u) \geq \alpha\}, \qquad (3)$$

with a policy π, remaining time steps n and remaining capacity c.

As we are dealing with revenue, we are interested in finding the policy $\bar{\pi}^*$, which has the maximal $V@R_\alpha$ of all policies Π given confidence α. In other words, we are looking for the policy $\bar{\pi}^*$ which has the highest revenue target of all policies Π given the quantile α. Thus, α fixes the probability of failing a target searched for every policy $\pi \in \Pi$. The best policy $\bar{\pi}^*$ fails with same probability α as other policies but achieves a higher target.

The results of Wu and Lin (1999) show, that $W_N^{\bar{\pi}^*}(c, x_N)$ as computed by Equation 2 has the property of a cumulative distribution function of variable x_N. Thus, we can employ Equation 2 for computing the policies which optimise the target quantiles of a range of targets. We can find the $V@R_\alpha$ by using a look-up table or in a similar way by a binary search.

The targets x_N and their associated confidence level $W_N^{\bar{\pi}^*}(c, x_N)$ can be stored in a look-up table. The table is filled by such value pairs, whereby the accuracy of the result depends on the used step size and the interval boundaries used for the various values for x_N. This enables us to look up the target which achieves a quantile equal to confidence α.

Binary search looks up in a sorted sequence for an element by continually splitting the sequence by its median and retaining only the part where the element must be contained in. We can search the $V@R_\alpha$ in a similar way, as $W_N^{\bar{\pi}^*}(c, x_N)$ is a increasing function in x_N. We start with a arbitrary target x_N and decrease or increase it depending on $W_N^{\bar{\pi}^*}(c, x_N)$. Again, the accuracy depends on the increment respectively to the decrement when searching the $V@R_\alpha$.

4 Numerical Results and Discussion

We evaluated the proposed computation method by the same model introduced by Lee and Hersh (1993). Their model serves as an example in various recent papers, cf. Barz and Waldmann (2007), Huang and Chang (2009), Koenig and Meissner (2009a), Koenig and Meissner (2009b). Hence, it provides a basis for a comparison of different policies.

4.1 Experiment Setup

The parameters of this model use $N = 30$ time periods to go before departure. At this point in time, there is a capacity $C = 10$ of seats left. Four fare classes are given with the prices $F_1 = 200, F_2 = 150, F_3 = 120, F_4 = 80$. The probability of an arriving customer requesting a distinct fare in the remaining periods are given in Table 1.

i	F_i	$1 \leq n \leq 4$	$5 \leq n \leq 11$	$12 \leq n \leq 18$	$19 \leq n \leq 25$	$26 \leq n \leq 30$
1	200	0.15	0.14	0.10	0.06	0.08
2	150	0.15	0.14	0.10	0.06	0.08
3	120	0	0.16	0.10	0.14	0.14
4	80	0	0.16	0.10	0.14	0.14

Table 1: Fares and probabilities of an arriving customer requesting fare class i in time period n.

4.2 $V@R$ Computation and Evaluation

We demonstrate a computational approach for finding optimal $V@R$ policies for $\alpha = 5\%$ and $\alpha = 10\%$ as described in the previous section. In this way, we get an achievable $V@R$ value, as well as its corresponding optimal policy.

Table 2 shows the results of computing for a range of possible targets the probabilities of failing them. The underlying computation is based on computing the probability of not achieving a target for every possible target. Thus, no grid which combined ranges of values was used in this case. The first row of Table 2 shows each possible target in the range between 1100 and 1250. This range is just an extract of the overall range of achievable targets. The second row shows the probability of not achieving the target. The next three rows are the simulation results evaluating the policy computed for a target.

We evaluated a policy by using its decision rules in a simulation applying random arrivals according to the probabilities of Table 1. Each simulation result was based on 1000 random runs, whereby for each set of runs, the same random values were used. We used the decision rule of accepting a request, if the decision had no effect on the probability. Further, we switched to the risk-neutral policy, if the $V@R$ was attained in a simulation run.

Table 2 shows the fraction of runs which failed the corresponding target, the average and the standard deviation over all achieved revenues. Comparison of the computed probability and the fraction of simulation runs of not reaching the target were plausible within numerical errors.

A possible target represents the $V@R_\alpha$ value and the associated probability, its α value. We find the searched $V@R_{5\%}$ for by looking up the α nearest to 5%, the same way it is done for $\alpha = 10\%$. This determined values-at-risk are highlighted in bold face in Table 2. As the possible targets were not a continuous but a discrete domain, there were also no continuous values for α. Thus, there is no $V@R_{10\%}$ but a $V@R_{10.1\%}$,

Target value	1100	1110	1120	**1130**	1140	1150	1160	1170
$W_N^{\tilde{\pi}^*}$	0.039	0.044	0.047	**0.050**	0.054	0.060	0.065	0.068
Simulation of target value policies								
Failed target	0.041	0.046	0.048	0.049	0.055	0.059	0.068	0.071
Rev. (average)	1327	1327	1326	1326	1326	1326	1327	1327
Rev.(std. dev.)	166	164	162	161	161	159	158	157

Target value	1180	1190	1200	1210	**1220**	1230	1240	1250
$W_N^{\tilde{\pi}^*}$	0.074	0.082	0.088	0.093	**0.101**	0.111	0.120	0.126
Simulation of target value policies								
Failed target	0.072	0.077	0.082	0.089	0.101	0.121	0.125	0.131
Rev. (average)	1328	1329	1329	1330	1331	1333	1335	1336
Rev.(std. dev.)	155	154	153	151	152	152	152	151

Table 2: Extract of the look-up table for finding the $V@R$ nearest to desired values 0.05 and 0.10 for α. Target levels, theoretical percentiles, achieved percentiles (failed target), averages and standard deviation of revenues are shown. The results of the simulations are generated by applying the corresponding policy.

which is nearest to 10% confidence. This is the same for $\alpha = 5\%$, respectively, but the difference is smaller and not visible in the table.

The effect of applying a grid is demonstrated in Table 3. The target level dimension of the state space is reduced by lowering the grid resolution. Results of grid resolutions of $m = 10$, $m = 20$, $m = 40$, $m = 80$ and $m = 166$, which is the highest grid resolution as all possible 166 targets are considered, are compared using exemplarily the confidence value $\alpha = 10\%$. The result of applying the risk-neutral policy is given for comparison. We selected the policy with $W_N^{\tilde{\pi}^*}$, which is nearest and greater or equal to the desired α. The results in Table 3 show that the inaccuracy increases with decreasing grid resolution. A lower grid resolution results in a lower accuracy of $W_N^{\tilde{\pi}^*}$, and the determined policies π^* do achieve their objective more imprecisely. The standard deviations, which increase with decreasing grid resolution, emphasis this.

Further, the simulation results demonstrated that policies which were computed by linear interpolation with a grid are more suitable for finding a $V@R$ optimal policy for a desired α confidence than policies computed by the nearest neighbour method. Taking into consideration that the state space was strongly reduced, the policies computed by linear interpolation worked quite well with grid sizes down to $m = 20$.

	$\alpha := W_N^{\tilde{\pi}^*}$	$V@R_\alpha$	Simulation of policy π^* $V@R_{10\%}$	Rev. (avg.)	Rev. (std.)
Risk-neutral	-	-	1130	1408	203
All targets $m = 166$	0.101	1220	1210	1331	152
Linear interpolation					
$m = 80$	0.105	1225	1180	1336	151
$m = 40$	0.126	1250	1140	1346	154
$m = 20$	0.105	1200	1200	1361	157
$m = 10$	0.160	1200	1140	1398	188
Nearest neighbour					
$m = 80$	0.119	1250	1150	1337	152
$m = 40$	0.118	1110	1130	1322	159
$m = 20$	0.199	1200	1070	1309	174
$m = 10$	0.100	1400	1130	1334	162

Table 3: Comparison of approximation methods by using a grid with different resolution and interpolation. Simulation results were generated by applying the determined $V@R$ optimal policy.

We take a closer look at the different effects of using nearest neighbour selection or linear interpolation in Figure 1 and 2. Both figures show on axis of abscissae the $V@R_\alpha$ and on the axis of ordinates, the corresponding confidence level α. Each depicted graph represents the computed best α for a $V@R_\alpha$ or vice visa. The several graphs in the figures show the effect of using different grid resolutions with nearest neighbour and linear interpolation. Grid resolutions were the same as in Table 3: $166, 80, 40, 20, 10$, and for 166, no grid approximation was necessary.

Figure 1 makes obvious how the accuracy decreased along with decreasing grid resolution when using the nearest neighbour approximation. The graphs of $m = 80$ and $m = 40$ deviated only a little from the accurate graph of $m = 166$. However, the graphs of $m = 20$ and $m = 10$ deviated significantly from the accurate graph and thus, did not longer provide reasonable results. This was quite different from the use of linear interpolation.

As shown in Figure 2, linear interpolation provided better approximation results than the nearest neighbour selection. The graph of $m = 80$ nearly matched the graph for accurate resolution, and the graph of $m = 40$ deviated only slightly from it. The first obvious deviation came with the graph $m = 20$ which might be an acceptable approximation. The graph of $m = 10$ deviated strongly and might no longer be a

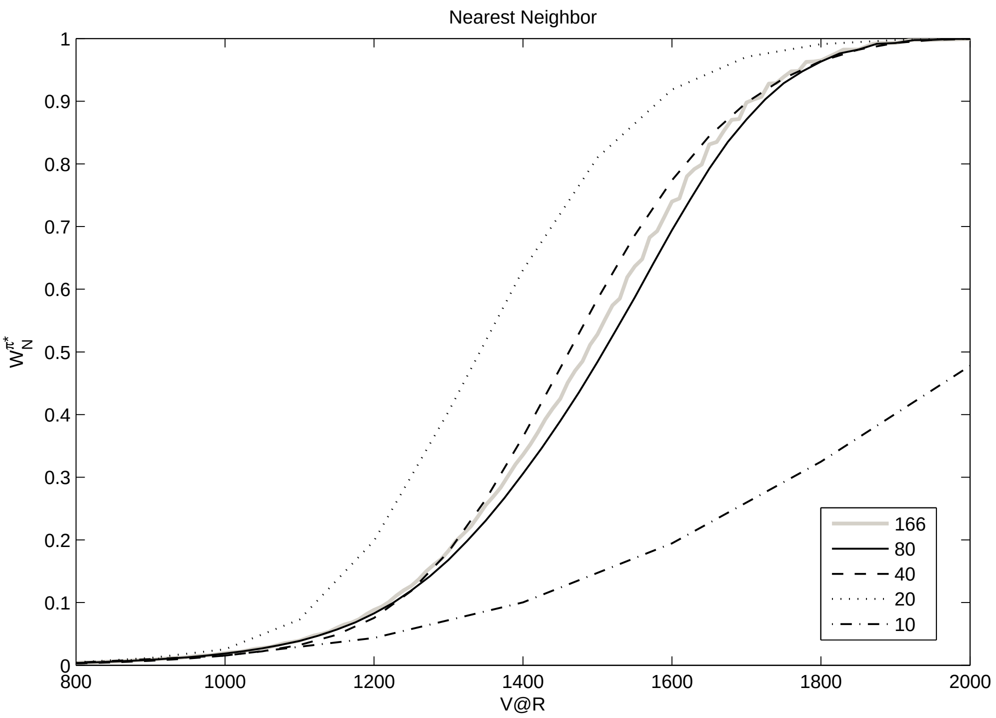


Figure 1: Increasing inaccuracy as effect by decreasing grid size when using nearest neighbour.

useful approximation in practice. However, linear interpolation was significantly more accurate than nearest neighbour and provided reasonable resolution down to ca. 1/8 of the original and accurate resolution.

As linear interpolation was a more accurate approximation than the nearest neighbour selection, we focused on linear interpolation for a further investigation of the impact of grid resolution. Figure 3 displays revenue results from 1000 simulation runs. Using different grid sizes as before, the determined policy for $\alpha = 10\%$ was computed and applied for each simulation run. The axis of the abscissae is the achieved revenue, and the axis of the ordinates is the number of counts the associated revenue was achieved. A histogram shows for a policy of a certain grid resolution the revenue distribution. Further, the results achieved by a risk-neutral policy is given for the purpose of comparison.

Comparing the histograms, we can see that the shape of the revenue distribution of the risk-neutral policy differs from those of the risk-sensitive $V@R_\alpha$ policies for $\alpha = 10\%$. We distinguish between the $V@R_\alpha$ used for finding a policy for $\alpha = 10\%$ and the resulting $V@R_{10\%}$ measurement of the simulation runs. The histograms of

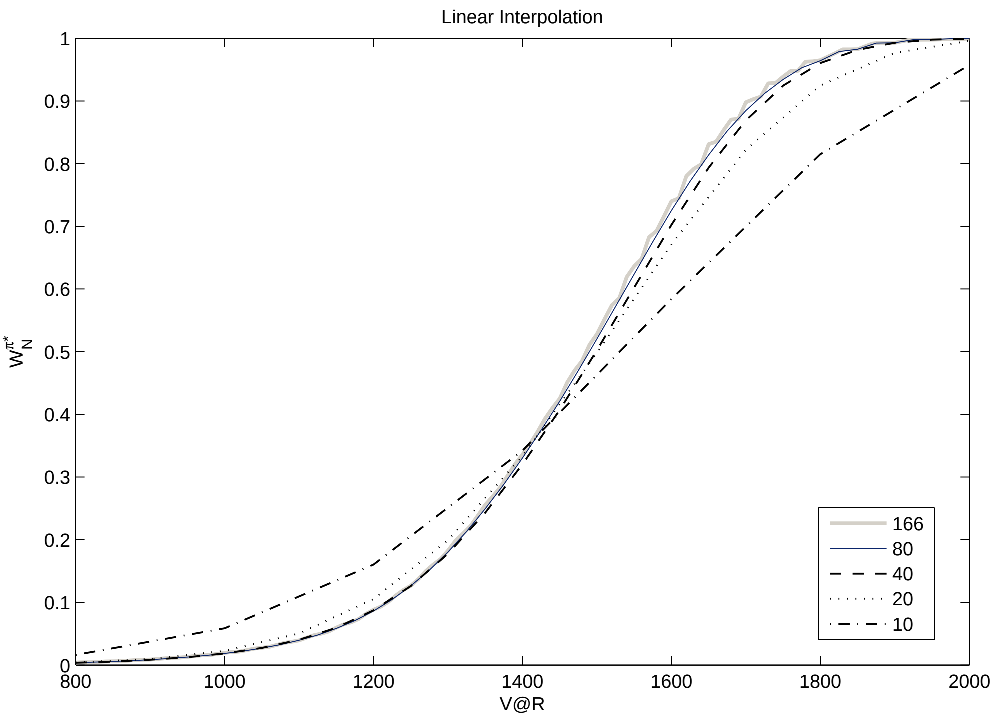


Figure 2: Increasing inaccuracy as effect by decreasing grid size when using linear interpolation.

the results of the policies of grid resolutions $m = 166$, $m = 80$ and $m = 40$ look very similar in their general shape. We note the peak at revenue of approximately 1200. This was expected as the polices were optimised by 'moving' the $V@R_\alpha$ to the highest revenue (the right hand side of the distribution) while limiting revenues which are lower the $V@R_\alpha$ (the left hand side of the distribution). However, the policies did not 'consider' the shape of the distribution on either side of the $V@R_\alpha$. This resulted in the appearance of the peak near the $V@R_\alpha$.

The results of grid size of $m = 20$ and $m = 10$ were quite interesting. The limited grid resolution seemed no longer possible to 'shift' revenues above the $V@R_\alpha$ and the shape of the revenue distributions became similar to that of the risk-neutral policy results. We can see that the histogram of the results from the risk-neutral policy has the largest similarity with that of the results of the policy using $m = 10$. There were differences which yielded consequent different mean revenue and attained $V@R_{10\%}$ of the simulations. The shape of the histogram of the results of the $m = 166$ policy and the shape of the histogram of the risk-neutral policy can be considered as two extremes. By decreasing the grid resolution, the shape of a histogram alters from the one extreme to the other. Thus, the shape of the histogram of the results of the $m = 20$ policy looks like the two extreme shapes merged together.

However, the achieved $V@R_{10\%}$ of each experiment has to be assessed with the data of Table 3. The policies were the results of an approximation which did not allowed every possible α. The table shows that for $m = 166$, $m = 80$ and $m = 20$ only, the values of $W_N^{\tilde{\pi}^*}$ were 0.101, 0.105 and 0.105, respectively, and thus close to the desired value of $\alpha = 10\%$. Taking this into account, the results of the simulations were consonant with the expected behavior of the policies.

Hence, only a grid resolution m approximating a policy $\tilde{\pi}^*$ should be chosen which predicts a value $W_N^{\tilde{\pi}^*}$, which has a small difference to the desired confidence level α.

4.3 Comparison with Exponential Utility Employing Policies

Finally, we present a comparison of the $V@R_\alpha$ policies with another risk-sensitive policy, in particular, policies which employ an exponential utility function for implementing risk-aversion. We only used this further kind of risk-sensitive policies, because the results of Koenig and Meissner (2009a) show that there are only small differences between the several risk-sensitive policies, including exponential utility-based ones. We refer to the paper of Koenig and Meissner (2009a) for a detailed explanation of the other risk-sensitive policies.

A comparison between the presented $V@R_\alpha$ approach and the exponential utility based one is difficult due to their different objectives. Whereas an $V@R_\alpha$ policy maximises the revenue for a certain confidence level α, the exponential utility policy maximises the utility defined by a parameter specified by a level of risk aversion. In order to arrive at comparable experiments, we had to find the level of risk aversion which matches the confidence level. To this end, we ran simulations of utility based policies for a range of levels of risk aversion. For comparison, we chose the utility-based policy which achieved the best $V@R_\alpha$ and highest mean revenue. We ran the experiment exemplarily with $\alpha = 10\%$.

Figure 4 shows the difference of the results achieved by both types of policies. The policy which was based on an exponential utility and achieved the highest $V@R_{10\%}$ had a lower $V@R_{10\%}$ than the $V@R_\alpha$ optimised policy. The mean of the results of the utility based policy was higher and its standard deviation was lower than those of the results of the $V@R_\alpha$ based policy. The histograms illustrate these statistical comparison. The histogram shape of the utility policy that results is broader than that of the $V@R_\alpha$ policy results. Furthermore, the left hand side histogram is more skewed to the left, while the right hand side histogram is more skewed to the right. This shows that the

$V@R_\alpha$ policy 'shifted' the revenue above the $V@R_\alpha$, whereas the utility policy optimises an utility function and achieved a more balanced distribution.

We show also the conditional-value-at-risk measure ($CV@R_\alpha$) which is actually the mean of the revenue values lower than the $V@R_\alpha$. The $V@R_\alpha$ policy did not perform better than the utility-based, regarding the $CV@R_\alpha$. This was an important result, as it illustrated one more time that a $V@R_\alpha$ policy does not guarantee the most beneficial lower tail distribution.

5 Conclusions

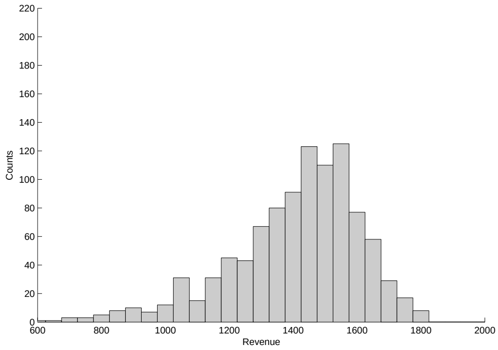
We have developed a computational approach for finding and approximating the optimal value-at-risk policy for a revenue management problem. The used dynamic capacity control model is one of the quantity-based revenue management models.

Given a confidence level specifying the value-at-risk, the proposed method computes possible value-at-risk results leveraging target level computation and selects the best result fitting the confidence level. In order to reduce computational effort, an approximation method for finding an approximate optimal value-at-risk policy has been proposed.

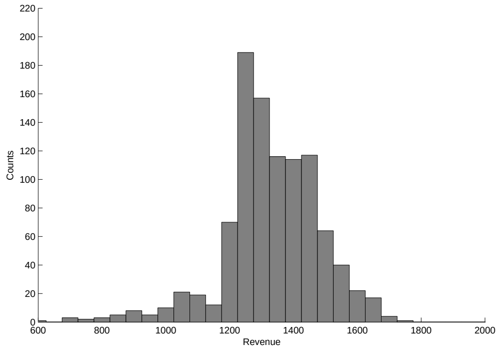
We have evaluated the proposed approach by computing policies in numerical experiments. A comparison with another risk-sensitive method for the same revenue management problem has been conducted.

The presented methods allow for a fast computation of a good approximation of value-at-risk optimal policies. They provide a basis for applying risk-sensitivity in revenue management. However, such policies optimise for value-at-risk but, as often, on costs of other measures. This should be borne in mind when applying such policies in practice.

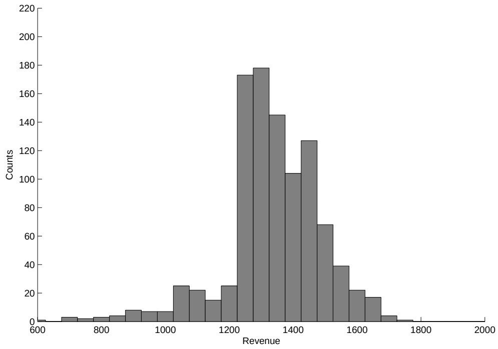
Finally, the presented computation approach aiming at value-at-risk optimal policies could also be used for other revenue management models, such as dynamic pricing, if the target level optimal policy is already known.



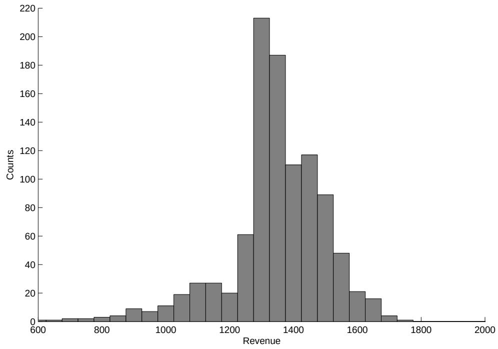
Risk neutral, $V@R_{10\%}$: 1130



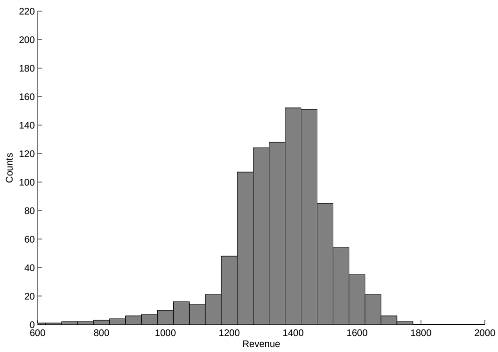
Grid 166, $V@R_{10\%}$: 1210



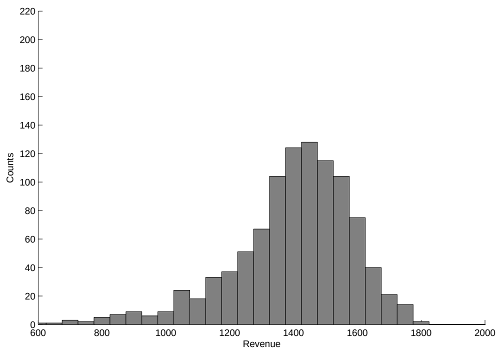
Grid: 80, $V@R_{10\%}$: 1180



Grid: 40, $V@R_{10\%}$: 1140

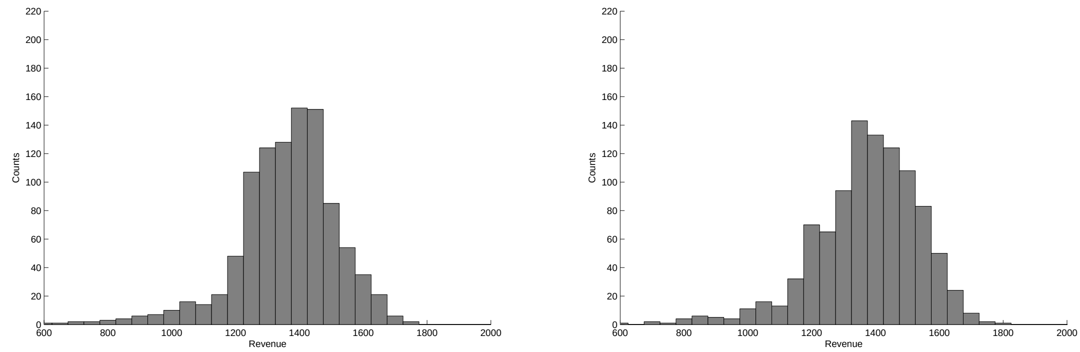


Grid: 20, $V@R_{10\%}$: 1200



Grid: 10, $V@R_{10\%}$: 1140

Figure 3: The histograms show the effect of grid resolution on the revenue distribution of numerical simulation. The $V@R_{10\%}$ is given for 1000 simulation runs applying the computed best policy for $\alpha = 10\%$.



$V@R_{10\%}$	1200
Rev. (avg)	1361
Rev. (std.)	157
$CV@R_{10\%}$	1035

$V@R_\alpha$ with grid $m = 20$

$V@R_{10\%}$	1180
Rev. (avg)	1374
Rev. (std.)	164
$CV@R_{10\%}$	1039

Exponential utility

Figure 4: Statistics and histograms of revenue distribution comparing results of $V@R_\alpha$ policy with $\alpha = 10\%$ and results of exponential utility based policy with best $V@R_{10\%}$.

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