# Lancaster University Management School
# Working Paper
# 2009/041

# Capacitated Dynamic Lot Sizing with Capacity Acquisition

Hongyan Li and Joern Meissner

The Department of Management Science
Lancaster University Management School
Lancaster LA1 4YX
UK



The LUMS Working Papers series can be accessed at http://www.lums.lancs.ac.uk/publications/
LUMS home page: http://www.lums.lancs.ac.uk/

# Capacitated Dynamic Lot Sizing with Capacity Acquisition

Hongyan Li
Aarhus School of Business
Aarhus University, Denmark
hojl@asb.dk

Joern Meissner
Lancaster University
Management School, UK
joe@meiss.com

November 3, 2009

**Abstract**

One of the fundamental problems in operations management is to determine the optimal investment in capacity. Capacity investment consumes resources and the decision is often irreversible. Moreover, the available capacity level affects the action space for production and inventory planning decisions directly. In this paper, we address the joint capacitated lot sizing and capacity acquisition problem. The firm can produce goods in each of the finite periods into which the production season is partitioned. Fixed as well as variable production costs are incurred for each production batch, along with inventory carrying costs. The production per period limited by a capacity restriction. The underlying capacity must be purchased up front for the upcoming season and remains constant over the entire season. We assume that the capacity acquisition cost is smooth and convex. For this situation, we develop a model which combines the complexity of time-varying demand and cost functions and that of scale economies arising from dynamic lot-sizing costs with the purchase cost of capacity. We propose a heuristic algorithm that runs in polynomial time to determine a good capacity level and corresponding lot sizing plan simultaneously. Numerical experiements show that our method is a good trade-off between solution quality and running time.

**Keyword**: supply chain management, lot sizing, capacity, approximation, heuristics

# 1 Introduction

One of the fundamental problems in operations management is to determine the optimal investment in capacity. A firm's capacity determines its maximal potential production per time unit. To acquire capacity is usually costly and time consuming, and once the investment is made, the cost is often partially or completely irreversible, as installed capacity is difficult to adjust in the short term. Moreover, the decision on how much capacity to acquire also strongly influences the action space for future operations planning. Obviously, acquisition of too much capacity wastes investment that could be used for other important operation activities such as new product development and marketing; too little capacity means long waiting times, missed sales opportunities, and lost revenue. Therefore, it is necessary to find an effective and comprehensive method to determine the proper capacity configuration for operations with specific planning horizons.

We consider a single-production facility that produces a single product item to satisfy a known demand. The decision problem for the firms is to determine the optimal capacity and solve a capacitated lot-sizing problem simultaneously. The capacity-acquisition, production, and inventory-holding costs are considered and we formulate the problem as a cost-minimizing Non-Linear Mixed Integer Programming (MIP) model. We conjecture that the problem is impossible to solve using a polynomial time algorithm, thus we develop a heuristics algorithm. The major difference between our study and previous efforts to address capacitated lot-sizing problems, for example, in the well-known papers of Wagner and Whitin (1958) and Zangwill (1968), is that in our model, the capacity level is an internal decision.

The remainder of this paper is organized as follows. We review the relevant literature in Section 2. Section 3 introduces the relevant notation and the basic model. In Section 4 we propose a heuristic to solve this problem. A computational study and numerical results are presented in Section 5. Finally, the conclusions and future research directions are given in Section 6.

## 2 Literature Review

The aim of capacity-acquisition decisions is to select the proper capacity that not only satisfies demand completely, but also reduces overcapacity. The research on capacity acquisition includes two major streams, the traditional mathematical programming models and the economic models.

The flexible capacity investment and management problems was addressed at a relatively early stage with mathematical programming methods. Fine and Freund (1990) introduced a two-stage stochastic programming model and analyzed of the cost-flexibility trade-offs involved in the investment in product-flexible manufacturing capacity for a firm. They addressed the sensitivity of the firm's optimal capacity investment decision to the costs of capacity, demand distribution, and risk level. van Mieghem (1998) studied the optimal investment problem of flexible manufacturing capacity as a function of product prices, investment costs and demand uncertainty for a two-product production environment. He suggested finding the optimal capacity by solving a multi-dimensional news-vendor problem assuming continuous demand and capacity. Netessine et al. (2002) proposed a one-period flexible-service capacity optimization and allocation model taking the capacity acquisition, usage, and shortage costs into account. While each paper mentioned above considered the multiple products and multiple resources problems with demand uncertainties, their focuses were limited to single-period models.

Apart from the studies on flexible capacity investment, many efforts have also been made to solve generalized capacity-investment problems. Harrison and van Mieghem (1999) developed a single-period planning model to incorporate both capacity investment and production decisions for a multiple-product manufacturing firm. This study yielded a multi-dimensional descriptive model generated from the "news-vendor model", and gave qualitative insights into real-world capacity-planning and capital-budgeting practices. Nevertheless, the decisions on optimal capacity investment are highly generalized, and the production plan decisions were not explicitly presented. van Mieghem and Rudi (2002) extended the work of Harrison and van Mieghem (1999) to include an operations environment with multiple products, production processes, storage facilities, and inventory management. Moreover, they investigated how the structural properties of a single period extend to a multi-period setting. They also improved on previous studies by considering some inventory-management issues.

Since the news-vendor model was developed and applied to capacity decision problems, it has been an important analysis technique to model and solve complex capacity-optimization problems under uncertainty. Burnetas and Gilbert (2001) proposed a news-vendor-like characterization of the optimal production policy on capacity under unknown demand and increasing costs within a finite horizon with discrete time periods. The approach focused on the trade-off between increasing production cost and the learning mechanism about demand, neglecting set-up costs and capacity-supplying limitations.

Lot sizing has remained a fruitful topic in operations research since the 1950's. Various variants including single-item and multi-item, uncapacitated and capacitated lot-sizing problems have been studied extensively. Over fifty years ago, Wagner and Whitin (1958) develop a forward algorithm for a general dynamic version of the uncapacitated economic lot-sizing model. The zero-inventory ordering theorem is a key contribution in this paper for the uncapacitated cases. Although many alternative algorithms have been presented, the dynamic programming method remains the major approach to solving lot-sizing problems.

More recent studies include Federgruen and Tzur (1991), who considered a dynamic lot-sizing model with general cost structure. The authors present a simple forward algorithm which solves the general dynamic lot-size model in $O(T \log T)$ time and in $O(T)$ under mild assumptions on the cost data. This is an important improvement over the previously recommended well-known shortest path algorithm solution in $O(T^2)$ space. Wagelmans et al. (1992) extended the range of allowable cost data to allow for coefficients that are unrestricted in sign. They developed an alternative algorithm to solve the resulting problem in $O(T \log T)$ time.

The uncapacitated lot-sizing problem is however an ideal case and hardly applicable to real-world operations. Capacity constraints always heavily influence production-plan decision making. Furthermore, the general capacitated lot-sizing problem is $\mathcal{NP}$-hard, see Bitran and Yanasse (1982). For the special case of a constant capacity restriction over the entire planning horizon, a number of efficient algorithms are capable of calculating an optimal production plan. For example, Florian and Klein (1971) presented an algorithm with the computational complexity $O(T^4)$ for the capacitated lot-sizing problem and explored the important properties of an optimal production plan. The optimal plan consists of a sequence of optimal sub-plans. Baker et al. (1978) discovered some important properties of an optimal solution to the problem when the production and inventory-holding costs are constant.

Other studies have tried to relax the strict cost-structure restrictions in the algorithms reviewed above. Kirca (1990) presented a dynamic programming-based algorithm with the computational complexity of $O(T^4)$ and Shaw and Wagelmans (1998) a dynamic programming algorithm for the capacitated lot-size problem with general holding costs and piecewise linear production costs. The algorithm of the latter reduces the computation time to $O(T^2\bar{d})$, where $\bar{d}$ is the average demand when production cost is linear. Many other contributions in this area include van Hoesel and Wagelmans (1996), Chen et al. (1994), and Chung and Lin (1988).

The studies mentioned above all addressed capacity investment or production planning decision problems individually. The implications of combining these problems are, however, rarely discussed. As an exception, Atamturk and Hochbaum (2001) studied a problem on capacity acquisition, subcontracting, and lot sizing. That is the only study we have found which is closely relevant with our studies. Although their approach makes the production plan and capacity acquisition decisions simultaneously, the authors simply discussed some special cases of production and holding-cost structure. Moreover, the study still focused on solving a series of capacitated lot-sizing problems discretely, causing the computational complexity to increase exponentially with the number of planning periods and demands. Additionally, Ahmed and Garcia (2004) studied a dynamic capacity-acquisition and assignment problem in a simplified operations setting to determine the resource capacity and allocation of the resources to tasks. This study actually proposed a capacity-expansion and planning approach without considering inventory carry-over and the determination of production plans.

In summary, while progress has been made on investigating the questions of capacity acquisition decisions and lot sizing separately, few results have been achieved that address joint optimization of capacity acquisition and production decisions under a capacitated lot-sizing cost structure, even when considering only a single firm.

# 3 The Model and Notation

In this section, we analyze the capacity acquisition and lot sizing problem. A firm has to determine the optimal capacity to purchase and set a corresponding lot sizing plan simultaneously.

## 3.1 Formulation

The firm produces an item or product that consumes a common resource during its production. The amount of the resource the firm purchased is assumed to be the capacity limit in a dynamic lot sizing setting. An example for this might be the number of trucks to lease, the work force to hire and other supportive affectivities for production. The firm has to purchase the capacity for the entire planning horizon and can then use the capacity over the planning horizon. The capacity must satisfy the demand constraints and the excess capacity will be disposed of without extra disposal costs.

The production plan will be considered in a planning horizon of $T$ periods. If the firms face a natural sales season introducing a new model or variant in each season, a natural choice of $T$ arises, e.g. $T = 52$ weeks in the automobile manufacturing industry operating with a weekly production and sales schedule. Otherwise $T$ is chosen large enough to ensure that the firms' decisions pertaining to the initial periods of the planning horizon are not affected by this truncation of the planning process.

The firm has a demand stream during the planning horizon, known only to the firm itself and following some predictable seasonality pattern. Thus, let

$d_t$ = the demand faced by firm in period $t$, $t = 1, \ldots, T$

The firm produces goods via a production and distribution process that, in principle, allows for inventory replenishment in each period. As in standard dynamic lot sizing problems, we assume that *fixed* as well as *variable* production costs are incurred as well as inventory carrying costs, which are proportional to each end-of-the-period inventory. We assume that all fixed order costs stay constant over the planning horizon, while all other cost parameters may fluctuate in arbitrary ways. We define

$f$ = the *fixed* setup cost for a production batch produced in any period $t$, $t = 1, \ldots, T$;

$a_t$ = the per unit production cost rate for a production batch delivered in period $t$; $t = 1, \ldots, T$;

$h_t$ = the cost to carry one unit product in inventory at the end of period $t$, $t = 1, \ldots, T$.

We define the following variables:

$x_t$ = the amount of product produced in period $t$, $\quad t = 1, \ldots, T$

$$y_t = \begin{cases} 1 & x_t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$I_t$ = the inventory amount at the end of period $t$, $\quad t = 1, \ldots, T$

$C$ = the capacity acquired by the firm.

The firm needs to acquire the capacity in question on a spot market prior to the season. We name the acquisition cost A(C) and assume it is smooth and convex. Such an assumption is reasonable, among other explanations, when the purchase of the firm influences the market price. As a simple example, let the market price for the resource be $p = \Lambda + \theta C$ where $\Lambda$ and $\theta$ are positive constants. Hence the acquisition cost for the resource is quadratic in C:

$$A(C) = p \cdot C = p(\Lambda + \theta C) \tag{1}$$

This gives rise to the following formulation of the problem:

$$\min \sum_{t=1}^{T} (a_t x_t + h_t I_t + f y_t) + A(C) \tag{2}$$

subject to

$$I_t = x_t - d_t + I_{t-1}, \quad \forall \quad t = 1, \ldots, T \tag{3a}$$

$$x_t \leq C y_t, \quad \forall \quad t = 1, \ldots, T \tag{3b}$$

$$I_0 = I_T = 0 \tag{3c}$$

$$x_t \geq 0, \quad I_t \geq 0, \quad y_t \in \{0, 1\}, \quad C \geq 0, \quad \forall \quad t = 1, \ldots, T. \tag{3d}$$

where the objective function (2) minimizes the production and inventory-holding costs as well as the acquisition costs of the capacity. Constraints on the problem are: Equation (3a) ensures that inventory is balanced; Production is restricted by (3b); Equation (3c) sets initial and final inventories to zero; and the bounds of the variables are restricted by (3d). Solving the model entails *simultaneously* determining the optimal capacity, order periods, and production amounts in each order period. Capacity is assumed to be a continuous variable, meaning that capacity can be acquired at any non-negative level.

# 4 The Heuristic

## 4.1 Basic idea of the heuristic

The simultaneous calculation of an optimal capacity and an optimal production plan as explained above is a quadratically constrained MIP model. This problem class is generally $\mathcal{NP}$-hard according to Garey and Johnson (1979). While the general capacitated lot sizing problem is $\mathcal{NP}$-hard, since the capacitated lot sizing problem with constant capacity can be solved in polynomial time, the capacity acquisition and lot sizing problem with constant capacity can be solved by discretizing the interval of potential values for the capacities and solving for each of those values. So it is not $\mathcal{NP}$–hard in the strong sense, and can be solved in pseudo–polynomial time.

Solving problems with reasonable sizes by discretizing the solution space for the capacities with CPLEX has shown that, although theoretically satisfactory, the large computational times make such a methodology impractical (see §5 for details). Therefore, in this section, we develop an $O(T^3 \log T)$ heuristic algorithm which improves the computational efficiency dramatically.

To facilitate the presentation of our algorithm, we use the following notation. We define

| | | |
|---|---|---|
| $D(t)$ | = | $\sum_{j=1}^{t} d_j$ to be the cumulative demand in the first $t$ periods, $t = 1, \ldots, T$; |
| $X_t$ | = | $\sum_{j=1}^{t} x_j$ to be the cumulative production level in the first $t$ periods, $t = 1, \ldots, T$; |
| $H(i,j)$ | = | $\sum_{k=i}^{j-1} h_k$ to be the cost of holding a product from period $i$ to period $j$, $\forall \quad 1 \leq i < j \leq T$; |
| $H(i)$ | = | $\sum_{k=i}^{T} h_k$ to be the cost of holding a product from period $i$ to the end of the planning horizon; |
| $C_{min}^n$ | = | to be the minimum capacity that allows a feasible solution with $n$ setups; |
| $\Theta(n)$ | = | $\{1, \ell_2, \ldots, \ell_n\}$ to be a setup strategy with the fixed setup number $n$, $\quad n = 1, \ldots, T$. The orders in periods $1, \ell_2, ..., \ell_n$ obey the assumption that the available capacity is at least $C_{min}^n$. |

In analogy to the algorithm presented by Federgruen and Meissner (2009), who present an algorithm for a combined pricing and *uncapacitated* lot sizing problem, the heuristic developed here considers each possible number of setups $n, \quad n = 1, \ldots, T$ separately and determines the best capacity and production plan. We solve the following problem:

$$\begin{aligned}
\pi^*(C) &= K_n(C) + A(C) && (4)\\
&= \min_n \min_C \left(nf + F_n(C) + A(C)\right) && (5)\\
&= \min_n \min_C \left(nf + F_n(C) + C\left(\Lambda + \theta C\right)\right) && (6)
\end{aligned}$$

The function $F_n(C)$ represents the production and inventory cost for the fixed setup number $n$. For each setup number $n$, our heuristic progresses in the following stages:

1. construct an initial solution with the minimal capacity that allows a feasible solution;
2. update production plan and calculate the cost savings when capacity is increased. This allows us to determine the best capacity choice for each $n$ individually;
3. compare the individual cost of each setup choice and pick the overall best solution.

While this procedure may not be optimal, our computational experiments show that, in many cases, our results are very close to optimality. In the numerical study Section, we compare the heuristics solution with the solution obtained by a full enumeration over the discretized decision space method and using CPLEX 11.0 to solve the individual instances.

## 4.2 Construction of the initial solution

For each number of setups $n$, we first find the minimal capacity that allows a feasible solution to the problem. This minimal capacity $C^n_{min}$ can be calculated as follows:

$$C^n_{min} = \max\left\{\frac{D(T)}{n}, \quad \max_{t=1,\ldots,T}\left\{\frac{D(t)}{t}\right\}\right\} \quad \forall n = 1, \ldots, T \tag{7}$$

After determining this minimal capacity, we find the number of order periods necessary for each fixed setup number $n = 1, 2, \ldots, T$. We start with a solution that places the orders as late as possible under the minimal feasible capacity $C^n_{min}$, and then we improve the solution by shifting the orders forward or backward if this is beneficial. The procedure is fully described in Algorithm 1. While it does not yield the optimal solution in general, in the important case of no prevailing speculative cost motives and $C^n_{min}$ being determined as the average demand per period, it does result in an optimal initial solution:

**Lemma 1** *Assume that there is no speculative cost motive, i.e.* $a(s) + H(s,t) \geq a(t)$ *for all* $1 \leq s < t \leq T$, *and that* $C^n_{min} = \frac{D(T)}{n}$, *then Algorithm 1 results in an optimal solution for the fixed setup number* $n$.

*Proof*: Let the initial production strategy from the Algorithm 1 be $\Theta^0 = \{\ell^0_1, \ell^0_2, \ldots, \ell^0_n\}$, and moreover, since $C^n_{min} = \frac{D(T)}{n}$, the production quantity in each setup period has to be $C^n_{min}$ in order to satisfy demands. The proposition will be proved if we show the minimal cost $\pi^* = \pi(C^n_{min}|\Theta^0)$.

Suppose that the strategy $\Theta^0$ is not optimal given the condition described in Lemma 1, there exists another production strategy $\Theta = \{\ell_1, \ell_2, \ldots, \ell_n\}$ which makes $\pi(C^n_{min}|\Theta) \leq \pi(C^n_{min}|\Theta^0)$. According to the algorithm, the setups $\ell^0_i, \quad i = 1, \ldots, n$ cannot be postponed in order to satisfy the feasibility of solution, thus, there must exist at least one $i$, so that $\ell^0_{i-1} < \ell_i < \ell^0_i$. This means that $a(\ell_i) + H(\ell_i, \ell^0_i) \leq a(\ell^0_i)$. It contradicts the assumption of no speculative cost motive, $a(s) + H(s,t) \geq a(t)$ for all $1 \leq s < t \leq T$. Thus, Algorithm 1 results in an optimal solution. □

### 4.3 Update with increased capacity

Having found an initial solution, we update it with increased capacity. We introduce the following additional notation:

$\Omega$ = a list of potential saving opportunities;

$\Xi$ = $\{\xi_i, \quad i = 1, \ldots, T\}$, where $\xi_i = \{0, 1\}$;

$\Phi$ = a list of active savings generated from $\Omega$;

**Algorithm 1** Initialization

```
R = 0
N = n
Require: d, a, H, C^N_min
for t = T : −1 : 1 do
    R = R + d(t);
    if R >= C^n_min then
        x_t = C^n_min
        l_N = t
        y_t = 1
        N = N − 1
        R = R − C^n_min
    end if
end for
for i = 2 : 1 : n do
    V = 0
    B = 0
    for j = l_{i−1} : 1 : l_i − 1 do
        if V > a(j) + H(j, l_i) − a(l_i) then
            V = a(j) + H(j, l_i) − a(l_i);
            B = j;
        end if
    end for
    if V < 0 then
        y_{l_i} = 0
        l_i = B
        y_{l_i} = 1
    end if
end for
R := 0
for t = T : −1 : 1 do
    R = R + d(t);
    if y_t = 1 then
        x_t = min{R, C^n_min}
        R = R − x_t
    end if
end for
```

$\Gamma$ = $(\epsilon_{min}, \text{Savings})$ to be the executive list to update the lot sizing plan in each iteration of computation.

The list of potential saving opportunities $\Omega$ is created first, and then elements of potential savings $\Omega$ are converted to a list of active savings $\Phi$ that we pursue at a given capacity increase. Each time a saving opportunity is exhausted, we check whether another element can be brought from $\Omega$ to $\Phi$. Once $\Omega$ is empty, stop the algorithm. Each element of $\Omega$ is a quadruplet of the form $\{\ell^-, \ell^+, \delta, \epsilon\}$, $\ell^-$ represents the period in which production is to be decreased, $\ell^+$ is the period in which production is to be increased, and $\delta$ is the potential cost saving per unit, and $\epsilon$ denotes the maximum number of units for which the savings opportunity can be exploited.

After finding the initial solution, we update the production and lot sizing plan while the capacity increases. For any given number of order periods, we examine the possibility of improving the solution by using the additional capacity that the company might acquire by comparing the cost of such a change between two adjacent order periods. The two options are either a shift of production to a previous order period or a postponement to a later order period. The first case, shifting the production earlier, creates no problems and can be repeated until the decreasing order period reaches zero. A postponement is potentially problematic, but can be done either until the first decreasing period has reached zero production level or until a further decrease leads to an infeasible solution. The maximum decrease is given by:

$$\epsilon = \min\left\{x_{\ell_i}, \left(\sum_{k=1}^{i} x_{\ell_k} - \sum_{k=1}^{\ell_i - 1} d_k\right)\right\} \tag{8}$$

In Algorithm 2, under the fixed setup number $n$, we compare each pair of sequential setups in period $\ell_i$ and $\ell_{i+1}$, $i = 1, \ldots, n-1$ to determine $\{\ell^-, \ell^+, \delta, \epsilon\}$, and adding it to $\Omega$.

Based on the saving opportunities matrix generated from the Algorithm 2, we sort the potential savings candidates $\Omega$. Next, the Algorithm 3 moves to realize the savings. In order to keep the linear decrease of lot sizing cost, we consider the capacity increases in

**Algorithm 2** Build sorted list of potential savings opportunities $\Omega$

Given: Set of order periods $\Theta(n) = \{1, \ell_2, \ldots, \ell_n\}$
new list $\Omega$
**for** $i = 1:1:n-1$ **do**
  **if** $a(l_i) + H(l_i, l_{i+1}) < a(l_{i+1})$ **then**
    Insert new element in $\Omega$: $(l_{i+1}, l_i, a(l_{i+1}) - a(l_i) - H(l_i, l_{i+1}), x(l_{i+1}))$
  **else**
    **if** $a(l_i) + H(l_i, l_{i+1}) > a(l_{i+1})$ **then**
      **if** $X(l_i) - D(l_{i+1} - 1) < x(n, l_i)$ **then**
        Insert new element in $\Omega$: $\{l_i, l_{i+1}, a(l_i) + H(l_i, l_{i+1}) - a(l_{i+1}), x(l_i)\}$
      **else**
        Insert new element in $\Omega$: $\{l_i, l_{i+1}, a(l_i) + H(l_i, l_{i+1}) - a(l_{i+1}), X(l_i) - D(l_{i+1} - 1)\}$
      **end if**
    **end if**
  **end if**
**end for**

a variable step size which is the minimum value of $\epsilon$ in the active savings candidate list $\Phi$. The value of the current capacity adding a step size will be a breakpoint of capacity increasing. Upon reaching one of the breakpoints, the savings opportunity has been exhausted and is removed from the calculation. We have two options, either we stop when one order period had reached zero, with the reasoning that we can reach a similar solution in a run with $n-1$ setups or, since there is no harm from the point of view of complexity, we can also proceed until our list is empty.

According to the heuristic procedure described above, we have Lemma 2 below. The Proposition 1 is resulted from the heuristic procedure. A outline of proof is provided to help illustrate the algorithm and proposition.

**Lemma 2** *For a fixed setup number $n$, the lot sizing cost function $K_n(C)$ is piecewise-linear non-increasing in capacity and convex.*

*Proof*: The lot sizing cost function is $K_n(C) = F_n(C) + nf$. Since fixed setup cost is constant, if the total production and inventory cost function $F_n(C)$ is piecewise-linear decreasing in capacity. Given a production plan $\left\{x_{\ell_1}, \ldots, x_{\ell_n}\right\}$, the production and inventory cost function is

**Algorithm 3** Calculation of cost function with increased capacity

```
M = size (Ω)
new binary array Ξ[T] := 0
for i = 1 : 1 : M do
   if Ξ(Ω[i] → ℓ⁻) = 1 then
      delete element Ω[i]
      M := M − 1
   else
      Ξ(Ω[i] → ℓ⁺) := 1
      Ξ(Ω[i] → ℓ⁻) := 1
   end if
end for
delete Ξ[T] := 0
N = 0
new binary array Ξ[T] := 0
new list Φ
new variable Savings := 0
for i = 1 : 1 : M do
   if Ξ(Ω[i] → ℓ⁺) ≠ 1 then
      Φ = Φ ∪ Ω[i]
      Savings = Savings + Ω[i] → δ
      Ω = Ω\Ω[i]
      Ξ(Ω[i] → ℓ⁺) = 1
      N = N + 1
   end if
end for
M = M - N
new list Γ
repeat
   ϵ_min = min_{i=1,...,N} Φ[i] → ϵ
   Append element to Γ : (ϵ_min, Savings)
   Update {x_t, y_t, I_t}
   for i = 1 : 1 : N do
      Φ[i] → ϵ = Φ[i] → ϵ − ϵ_min
      if Φ[i] → ϵ = 0 then
         Savings = Savings - Φ[i] → δ
         for j = 1 : 1 : M do
            if Ω[j] → ℓ⁺ = Φ[i] → ℓ⁺ then
               Φ = Φ ∪ Ω[i]
               Savings = Savings + Ω[i] → δ
               Ω = Ω\Ω[i]
            end if
         end for
         Savings = Savings - Φ[i] → δ
         Φ = Φ\Φ[i]
      end if
   end for
until Φ = ∅
```

$$F_n(C) = \sum_{i=1}^{n} \left( a_{\ell_i} x_{\ell_i} + \sum_{j=\ell_i+1}^{\ell_{i+1}} h_j \left( X(j) - D(j) \right) \right). \tag{9}$$

In order to prove that $F_n(C)$ is piecewise-linear decreasing in capacity, the following three properties of the function need to be proved respectively (all discussion below is based on a fixed setup number $n$):

(1) $F_n(C)$ is non-increasing in capacity;

If capacity increases to be $C^{'}$, the production plan $\{x_{\ell_1}, \ldots, x_{\ell_n}\}$ is still feasible, and the decision space is broader, therefore, we have at least $F_n(C^{'}) \le F_n(C)$.

(2) $F_n(C)$ is piecewise–linear in capacity;

According to the initial solution from Algorithm 1 and Algorithm 2, search all cost saving opportunities and record them in array Ω which allows the capacity to vary in the range $[C^n_{min}, D(T)]$.

Furthermore, by Algorithm 3, we deal with the saving opportunities array Ω. According to the heuristic procedure, the computation includes a finite number of iterations based on different capacity levels.

In each iteration, we define and calculate an active cost saving array $\Phi = \{\phi_m, \quad m = 1, 2, \ldots, M\}$, where $\phi_m = \{\ell_-, \ell_+, \delta_m, \varepsilon_m\}$. For the detailed steps please refer to the algorithm.

From the array Φ, we determine a capacity increase quantity $\Delta C = \min \epsilon_m$ with the unit cost saving $\sum_{m=1}^{M} \delta_m$. Thus, cost function $F_n(C)$ is linear non-increasing in capacity interval $(C, C + \Delta C]$. Capacity level $(C + \Delta C)$ is a new breakpoint of capacity increase.

(3) $F_n(C)$ is continuous.

In the heuristic algorithms, a new breakpoint of capacity increase is always calculated based on the capacity level of the previous one iteration. In addition, the solution of the production plan of an iteration is always the initial solution of the next iteration. Therefore, we see that the cost function $F_n(C)$ is continuous. Moreover, since the slope of the function $F_n(C)$ results from picking various elements from Ω, the convexity is a direct result of our picking elements in decreasing order of their savings. □

For an illustration of Lemma 2, see Figure 1 selected from a numerical example discussed in Section 5.

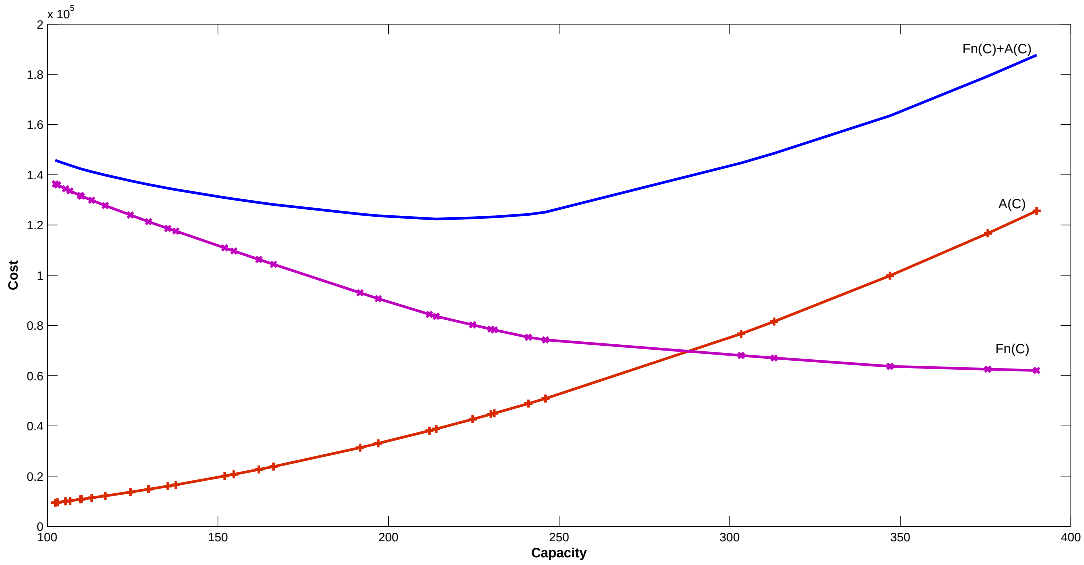


Figure 1: An example of cost variation with the capacity increase under a fixed setup number

## 4.4 Calculation of the optimal capacity

Upon obtaining the piecewise-linear functions for each individual setup number $n$, we calculate the optimal capacity to acquire by finding the appropriate breakpoint. According to the Lemma 2, and finite possible setup numbers, the optimal solution is obtained by comparing the minimal costs of all possible setup numbers. The optimal capacity corresponds to the optimal setup number. To this stage, the entire heuristic procedure is completed in polynomial time as shown in the following section.

## 4.5 Complexity

To assess the complexity of the algorithm, first, the individual complexity of the major algorithm steps are described. Note that the steps taken to solve the problem have to be repeated $T$ times, once for each potential setup $n$. Then in each iteration under fixed setup $n$, finding the minimal feasible capacity can be done in O(T); next, finding the initial solution takes O(T) for the first phase and O(T) for the final initial solution; Third, in the update procedure of initial solution with the capacity increase, finding potential savings again takes O(T), since $n$ pairs at most have to be evaluated. After obtaining the list of potential savings, this list has to be sorted once, which takes O(TlogT) using Quicksort or a similar algorithm. Finally, searching the list for potential savings to determine each breakpoint can be done in O(T). Given that there are at most $n$ breakpoints, this leaves us with a complexity of O(T) to update the solution from the previous lower capacity level.

Considering the relationships (paralleled or hierarchical) between the steps, the integrated algorithm complexity is as follows. For each setup, we find the minimum capacity, find the initial solution, optimize the initial solution, and finally compare the optimal solution for each setup number which cause a complexity of $O(T^3)$. Based on each initial solution under $C^n_{min}$, the improvement procedure including the determination of Ω with a complexity of $O(T)$, the sorting of potential savings adding another $O(TlogT)$ and updating of solution adding $O(T)$ again. Doing this totally introduces a complexity $O(T^logT)$. The final comparison of each solution of each setup number gives a complexity of $O(T^2)$. Taking everything into account, we have a complexity of $O(T^3) + O(T^3logT) + O(T^2)$. Without loss of the generality, the overall heuristic algorithm terminates in $O(T^3logT)$.

# 5 Numerical Example

In this section, we present computational examples for our heuristic. Using the heuristic algorithm we developed for the best response problem, capacity acquisition and lot sizing problem, a numerical study is carried out first to show the robust performance of the algorithm. It is assumed that the firm faces a planning horizon of $T = 54$ periods with varying seasonal demand. The demand behaves according to:

$$d_t = \beta_t * (\bar{d}) * U[0.5; 1.5] \tag{10}$$

We consider six different seasonality patterns $\{\beta_t : t = 1, \ldots, 54\}$ as follows:

(I) Time-invariant demand functions: $\beta_t = 1 \quad ; \quad t = 1, \ldots, 54$

(II) Linear Growth: $\beta_t = 0.25 + 1.5\frac{(t-1)}{53} \quad ; \quad t = 1, \ldots, 54$

(III) Linear Decline: $\beta_t = 1.75 - 1.5\frac{(t-1)}{53} \quad ; \quad t = 1, \ldots, 54$

(IV) Holiday Season at the Beginning of the Planning Horizon:

$$\beta_t = \begin{cases} \frac{54}{114} + \frac{540}{570}(t-1) & , t = 1, \ldots, 6 \\ \frac{594}{114} - \frac{540}{570}(t-7) & , t = 7, \ldots, 12 \\ \frac{54}{114} & , t = 13, \ldots, 54 \end{cases} \tag{11}$$

(V) Holiday Season at the End of the Planning Horizon:

$$\beta_t = \begin{cases} \frac{54}{114} & , t = 1, \ldots, 42 \\ \frac{54}{114} + \frac{540}{570}(t-43) & , t = 43, \ldots, 48 \\ \frac{594}{114} - \frac{540}{570}(t-49) & , t = 49, \ldots, 54 \end{cases} \tag{12}$$

(VI) Cyclical Pattern:

$$\beta_t = \begin{cases} 0.25 + 0.75(t-1) & ,t = 1,\ldots,3 \\ 1.75 - 0.75(t-4) & ,t = 4,\ldots,6 \\ \beta_{t\mathrm{mod}6} & ,t = 7,\ldots,54 \end{cases} \tag{13}$$

where $t$mod6 denotes t modulo 6. The first pattern reflects a situation where demand functions are time-invariant and the second (third) pattern one with linear growth (decline). The fourth and fifth patterns represent a planning horizon with a single season of peak demands either at the beginning or at the end of the planning horizon. The last pattern (VI) is cyclical with a cycle length of six periods, such that demands in the two middle periods of each cycle are 7 times their value in the first and last period, while $\beta_t = 1$ in the remaining two periods of the cycle.

We pick $c_t = 15; \quad h_t = 5$ and do our analysis for three different setup cost levels considering the assumption of no speculative inventory in firms. In addition, in order to calculate the capacity acquisition cost, we choose constants $\Lambda = 200$ and $\theta = 1$. We determine the fixed setup cost *indirectly* by first choosing the EOQ-cycle time "Time-between-Orders (TBO)" $= \sqrt{\frac{2\kappa}{hd}}$ and determine the $\kappa$ value from this identity. The TBO value is generated from a *uniform* distribution on the interval [1,3] for low TBO values, the interval [2,6] for *medium* TBO values and [5, 10] for *high* TBO values.

Using different combinations of demand pattern, TBO, and an average demand of 50 units, we generate a number (18) of hypothetical test problems. The heuristic algorithm are coded using Matlab 7.0. The optimal (benchmark) solutions are obtained by calling CPLEX 9.0 solver using Tomlab classes in Matlab environment. The problem instances are solved on a Pentium 4 PC with 512 RAM. Applying the cost and demand data described above and running the code 10 times, we calculate the average gaps between the heuristic and optimal solutions. The results are presented in Table 1.

The results indicate that our heuristic algorithm performs quite well. First, the gaps between the heuristic and the optimal solutions are very small with an overall average gap of 2.71%; this is acceptable given the extremely short computational times of around

| Demand Pattern | TBO=Low Gap | TBO=Low CPU (s) Opt | TBO=Low CPU (s) Heur | TBO=Medium Gap | TBO=Medium CPU(s) Opt | TBO=Medium CPU(s) Heur | TBO=High Gap | TBO=High CPU(s) Opt | TBO=High CPU(s) Heur | Average Gap |
|---|---|---|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) |
| DP1 | 1.01% | 1590 | 1.2 | 0.76% | 4012 | 1.1 | 0.88% | 3013 | 1.1 | 0.88% |
| DP2 | 1.50% | 1214 | 1.2 | 1.59% | 3620 | 1.2 | 2.35% | 3969 | 1.2 | 1.82% |
| DP3 | 1.79% | 1164 | 0.8 | 1.69% | 3247 | 0.8 | 1.05% | 1669 | 0.8 | 1.51% |
| DP4 | 2.46% | 3154 | 0.6 | 5.23% | 3815 | 0.6 | 6.11% | 1766 | 0.6 | 4.60% |
| DP5 | 4.76% | 2264 | 1.3 | 5.09% | 4140 | 1.2 | 4.47% | 3896 | 1.3 | 4.77% |
| DP6 | 2.05% | 388 | 1.1 | 3.56% | 1326 | 1.1 | 2.49% | 1333 | 1.3 | 2.70% |
| Average | 2.26% | 1629 | 1.0 | 2.99% | 3360 | 1.0 | 2.89% | 2607 | 1.0 | 2.71% |

Table 1: The average gaps between the heuristics costs and optimal costs and CPU computation times

| Test Problems | TBO | Demand Pattern | Heuristic Solutions Costs | Heuristic Solutions Setups | Heuristic Solutions Capacity | Optimal Solutions Costs | Optimal Solutions Setups | Optimal Solutions Capacity |
|---|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| 1 | | DP1 | 58588 | 33 | 110 | 58061 | 34 | 106 |
| 2 | | DP2 | 58801 | 35 | 128 | 57881 | 33 | 114 |
| 3 | Low | DP3 | 53000 | 45 | 127 | 52956 | 45 | 127 |
| 4 | | DP4 | 66140 | 28 | 380 | 65156 | 26 | 285 |
| 5 | | DP5 | 66025 | 24 | 268 | 62548 | 23 | 231 |
| 6 | | DP6 | 69273 | 27 | 143 | 68209 | 26 | 151 |
| 7 | | DP1 | 73425 | 21 | 173 | 73028 | 23 | 145 |
| 8 | | DP2 | 103880 | 14 | 239 | 102550 | 13 | 314 |
| 9 | Medium | DP3 | 108210 | 11 | 304 | 105170 | 11 | 322 |
| 10 | | DP4 | 66507 | 24 | 227 | 65720 | 22 | 220 |
| 11 | | DP5 | 95820 | 13 | 306 | 92865 | 12 | 327 |
| 12 | | DP6 | 79911 | 14 | 234 | 78049 | 15 | 234 |
| 13 | | DP1 | 131600 | 9 | 325 | 130140 | 9 | 355 |
| 14 | | DP2 | 176880 | 6 | 490 | 172980 | 7 | 487 |
| 15 | High | DP3 | 174200 | 6 | 504 | 173640 | 6 | 504 |
| 16 | | DP4 | 135420 | 6 | 536 | 127760 | 6 | 562 |
| 17 | | DP5 | 135840 | 7 | 441 | 130370 | 7 | 495 |
| 18 | | DP6 | 152530 | 9 | 353 | 151600 | 9 | 349 |

Table 2: The comparison of the heuristic solutions and the optimal solutions

one second. Second, under the different demand patterns, the average gap does not vary dramatically. For the time-varying demand scenario, the average gap is the least, about 1% or less. For the holiday demand scenarios, the average gap remains below 5% (see column 11 in Table 1). Additionally, comparing with the optimal solutions, the heuristic solutions are reasonable since the heuristic algorithm also suggests similar setup numbers and capacity levels (see table 2).

## 6 Conclusion

In this paper we consider a single resource acquisition and lot sizing problem considering capacity competition. We solve this problem by a comprehensive and efficient heuristic algorithm. The algorithm solves the capacity acquisition, production, inventory decisions simultaneously with a similar computation complexity of $O(T^3 \log T)$ as the classical single-item capacitated lot sizing problem. This is an important improvement on existing methods. Our numerical study shows that our heuristic performs well while using substantially less time compared to a solution where the potential capacity space is discretized, while losing only a modest amount of accuracy.

While this study solves the capacity acquisition and lot sizing problem effectively, it is based on the deterministic demand and constant capacity assumptions. Considering demand uncertainty and time varying capacity would be important directions to extend current results. In addition, it would also be very interesting to look into the capacity acquisition and lot sizing problem under a competition or coordination operations environment as future research.

## References

Ahmed, S., R. Garcia, 2004. Dynamic capacity acquisition and assignment under uncertainty. *Annals of Operations Research* 124(1-4) 267–283.

Atamturk, A., D. S. Hochbaum, 2001. Capacity acquisition, subcontracting, and lot sizing. *Management Science* 47(8) 1081–1100.

Baker, K. R., P. Dixon, M. J. Magazine, E. A. Silver, 1978. An algorithm for the dynamic lot-size problem with time-varying production capacity constraints. *Management Science* 24(16) 1710–1720.

Bitran, G. R., H. H. Yanasse, 1982. Computational complexity of the capacitated lot size problem. *Management Science* 28(10) 1174–1186.

Burnetas, A., S. Gilbert, 2001. Future capacity procurements under unknown demand and increasing costs. *Management Science* 47(7) 979–992.

Chen, H. D., D. W. Hearn, C. Y. Lee, 1994. A new dynamic programming algorithm for the single item capacitated dynamic lot size model. *Journal of Global Optimization* 4 285–300.

Chung, C. S., C. H. M. Lin, 1988. An $\mathrm{O}(\mathrm{T}^2)$ algorithm for the ni/g/ni/nd capacitated lot size problem. *Management Science* 34(3) 420–426.

Federgruen, A., J. Meissner, 2009. Competition under time-varying demands and dynamic lot sizing costs. *Naval Research Logistics* 67(1) 57–73.

Federgruen, A., M. Tzur, 1991. A simple forward algorithm to solve general dynamic lot sizing models with n periods in $o(n \log n)$ or $o(n)$ time. *Management Science* 37(8) 909–925.

Fine, C. H., R. M. Freund, 1990. Optimal investment in production-flexible manufacturing capacity. *Management Science* 36(4) 449–446.

Florian, M., M. Klein, 1971. Deterministic production planning with concave costs and capacity constraints. *Management Science* 18(1) 12–20.

Garey, M. R., D. S. Johnson, 1979. *Computers and intractability—A guide to the theory of NP-Completeness.* W H Freeman and Company.

Harrison, J. M., J. A. van Mieghem, 1999. Multi-resource investment strategies: operational hedging under demand uncertainty. *European Journal of Operational Research* 113(1) 17–29.

Kirca, O., 1990. An efficient algorithm for the capacitated single item dynamic lot size problem. *European Journal of Operational Research* 45(1) 15–24.

Netessine, S., G. Dobson, R. A. Shumsky, 2002. Flexible service capacity: optimal investment and the impact of demand correlation. *Operations Research* 50(2) 375–388.

Shaw, D. X., A. P. M. Wagelmans, 1998. An algorithm for single-item capacitated economic lot sizing with piecewise linear production costs and general holding costs. *Management Science* 44(6) 831–838.

van Hoesel, C. P. M., A. P. M. Wagelmans, 1996. An $O(T^3)$ algorithm for the economic lot-sizing problem with constant capacities. *Management Science* 42(1) 142–150.

van Mieghem, J. A., 1998. Investment strategies for flexible resources. *Management Science* 44(7) 1071–1078.

van Mieghem, J. A., N. Rudi, 2002. Newsvendor networks: inventory management and capacity investment with discretionary activities. *Manufacturing & Service Operations Management* 4(4) 313–335.

Wagelmans, A., S. van Hoesel, A. Kolen, 1992. Economic lot sizing: an $o(n \log n)$algorithm that runs in linear time in the wagner-whitin case. *Operations Research* 40(S1) S145–S156.

Wagner, H. M., T. M. Whitin, 1958. Dynamic version of the economic lot size model. *Management Science* 5(1) 89–96.

Zangwill, W. I., 1968. Minimum concave cost flows in certain networks. *Management Science* 14(7) 429–450.