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In recent years, many traditional practitioners of revenue management such as airlines or hotels were confronted with aggressive low-cost competition. In order to stay competitive, these firms responded by cutting down fare restrictions that were originally meant to fence off customer segments, that is to prevent high-yield customers from buying down. For the corresponding markets, unrestricted fares were introduced whose essentially only distinctive feature is its price. Some markets, however, were unaffected and here restrictions could be maintained as it was the case for long-haul flights, for example.

We develop choice-based network revenue management approaches for such a mixed fare environment that can handle both the traditional opening or closing of restricted fare classes as well as handling pricing of the unrestricted fares simultaneously. For any such unrestricted fare, we assume a fixed number of price points to choose from. It is natural to ask then how these price points shall be chosen. To that end, we formulate the problem as a dynamic program and approximate it with an efficient mixed integer linear program (MIP) that selects the best n, say, price points out of a potentially large set of price candidates for each unrestricted fare. We show both theoretically and practically that it is advantageous to recompute price points later in the booking horizon using our approach. Furthermore, additional insight is gained from the fact that the dual values associated with MIP provide an upper bound on the value of having an additional price point.

Numerical experiments illustrate the quality of the obtained price structure and that computational effort is relatively low given that we need to tackle the large-scale MIP with column generation techniques where each column pricing problem itself is NP-hard.

Key words: revenue management, restriction-free pricing, network, pricing structure

1. Introduction

Revenue Management (RM) is rooted in the airline industry where it emerged in the mid 1980's as an impressively effective means to fend off the low-cost carriers that entered the US market after its deregulation. Driven by these successful implementations, many other industries such as hotel chains, car rentals or trains (just to name a few) adopted RM practices. The idea was based on effective customer segmentation according to price sensitivity, enabling the firm to offer competitive rates while minimizing cannibalization of sales to less price-sensitive customers. This segmentation occurs as a result of companies' imposing restrictions on discounted tariffs, so for airlines or hotels these might be minimum lengths of stay, mandatory Saturday-night stays, advanced time limits for booking or age-based discounts. However, in recent years an increasing number of firms successfully implemented low-cost business strategies that operate without the complicated tariff structure that combines discounted products with restrictions. Prominent examples are Ryanair or businesses of the easyGroup who advance this concept in many industries, such as airline (easyJet), bus (easyBus) and many more. By advertising unfavourable comparisons between their own tariffs and the ones of the incumbents, many of the latter felt the need of also having to offer at least some unrestricted products in order to counter the negative impressions created by such campaigns. Currently we observe this trend most articulately in the airline business where traditional carriers such as Lufthansa or British Airways experiment with offering both restricted and unrestricted fares, but it begins to manifest also in other areas as the example of easyGroup with their cross-sectoral restriction-free approach shows. Hence it is likely that soon we will frequently face similar topics as there was on the agenda of the practitioner conference “eyefortravel Travel Distribution Summit Europe” in 2005: “The rise of the ‘no frills’ hotel. Could this have the same effect on the hospitality industry as low-cost carriers had on the airline sector?”.

The consequence of this development for RM of incumbent firms who respond to this aggressive competition by offering a mix of restricted and unrestricted products is the partial invalidation of the premises for customer segmentation, though there is still a substantial part of the market where the segmentation works well. However, current RM systems build upon the traditional assumptions of offering only restricted products and thus there is a need for research in the realm of mixed restriction and restriction-free fares as illustrated in the recent practitioner article by Vinod (2006). However, despite these appeals from the practitioners that have to face these issues, little academic research has been carried out to date in the context of mixed fare environments.

In the following, for the sake of illustration we will present our approach in the airline context and use airline terminology to stay in the picture, but we remark that the ideas transfer with few adjustments to other industries that use multiple resources in their products as well. For example, transfer to hotel industry is done by exchanging flight legs by room nights, that means a connecting flight becomes then a multiple night stay etc.

We propose in this paper a revenue maximizing framework tailored to this new fare environment which does without proper customer segmentation. More specifically, the model distinguishes between unrestricted and restricted fares, incorporates a finite set of price points for each unrestricted fare which can be obtained by a preprocessing method, and leads to customer choice-based bid price heuristics by providing opportunity cost estimates. Modelling customer choice is of great importance, in particular since unrestricted fares can be considered for purchase by different segments given that the restrictions meant to fence low fares off have been removed. A choice model suited to this task is the Multinomial Logit (MNL) with overlapping consideration sets, see Bront et al. (2007), for example. Some modifications allow us to use their approach to tackle the problem at hand so that we can pursue the following main research issues:

- A network revenue management optimisation approach that can handle both traditional and unrestricted demand such that it selects which fares to offer and (for the unrestricted ones) at what price?
- How to pre-select price points for the unrestricted products, given that booking systems are often limited in the number of price points they can manage and that it might not be possible to change the available price points during the booking horizon?
- What is the worth of an additional price point?

Our main contributions lie within providing answers to these issues: First, we propose a choice-based RM model from which control policies can be derived that work in mixed fare environments. This model uses a finite set of price points for each unrestricted fare and treats each price points as a separate, “virtual” fare under the condition that at most one such virtual fare may be offered at a time for each unrestricted fare. Since the ability of booking systems to handle many fares is limited, the question arises which price points shall be used. To this end, we contribute in developing a dynamic programming model that represents the optimal policy in both selecting the best price points and in controlling which set of products to offer at any point in time. It is of theoretical interest only due to the curse of dimensionality but yields insights into pricing in this context, namely, the later one commits to price points the better. In order to approximate this intractable dynamic program, we develop a mixed integer linear program that provides us with a good feasible price structure that can even be optimal as illustrated in the numerical experiments. As a by-product we obtain upper bounds on the value of having an additional price points by means of the optimal dual solution of the linear programming relaxation, which is again an interesting feature in testing fare structures.

The paper is organized as follows: In the next section we briefly review the related literature, then we present the modelling framework in Section 3 followed by the mixed fare environment optimization model given fixed prices in Section 4. The related question of how to pre-select price points is discussed in Section 5 including presentation of the underlying dynamic program and the linear mixed integer program approximation. Numerical evidence for the performance of the price point pre-selection is provided in Section 6 before we conclude in Section 7.

2. Literature Review

Naturally, the first to identify the changes necessary in revenue management optimisation with respect mixed fare environments was the practitioner community. A number of publications from airlines, software providers and pricing consultancies appeared since 2003 that analysed the changes in the business environment due to low cost competition. Academia followed with some delay in providing potential answers to the outlined questions. Let us first turn to the practitioner reports to frame the problem: Among the first was Foran (2003) from British Airways (BA), describing their dramatic cut of restrictions at the time to simplify their fare structure. Many traditional airlines had very refined market segmentations in place so that many network products ended up almost never being purchased. BA decided to simplify fares and thereby accepting a loss of ability to segment because the high fare complexity offset potential customers. This customer behaviour was also stressed by Cary (2004), who further added that business customers became unusually price-sensitive as compared to the 90ies. Low cost competition on the short-haul links undermined the traditional carriers' ability to price discriminate as noted by Tretheway (2004) owing to the introduction of cheap one-way fares. Many other companies followed suit in cutting restrictions, for example, GNER and Virgin Trains in 2005 in the United Kingdom. While some firms even replaced their whole revenue management system by a one-way fare structure as, for example, bmi (see Donnelly et al. (2004)), most chose to introduce low-cost fares along with the traditional ones. Westermann (2005) stressed that often unrestricted fare structures need only to be introduced on links facing low cost competition. On other markets, in particular connecting traffic, traditional methods are still working well and should be kept since unrestricted fares usually lead to revenue dilution. The resulting mix of restricted and unrestricted products is a major challenge and should be addressed by the optimisation module in a origin and destination (O&D) mechanism as Westermann (2005) pointed out. The underlying problems of such an approach are touched on in the AGIFORS presentation of Weber and Thiel (2004) who speak of "augmented" optimisation problems since input values such as prices are not fixed any more but rather also decision variables. Models of customer choice become important in the presence of solely price-orientated customers as illustrated by Boyd and Kallesen (2004). They observed that customers tend to ignore fare restrictions and focus mostly on price such that demand is realised at the lowest available fare. The credit crunch in 2008 and the subsequent economic downturn further aggravate the situation in that demand for premium and business fares has broken down. For illustration, in February 2009 demand for such products dropped by 21% relative to the same month the year before, as announced by IATA (2009). Though there is still demand which can be addressed by traditional means, a firm must be aware of this mix of demand types and adjust their forecasting and optimisation systems accordingly, favourably in an O&D model since customer choice is best being modelled in this context. Ratliff and Vinod (2005) and Vinod (2006) identify the issue of optimising in a mixed fare environment as future important problems.

From an academic perspective, not much work has been done yet to address this problem. Gorin and Belobaba (2004) reported results from a simulation study focussing on the effect of a new low-cost competitor on the revenue management of an incumbent network carrier. One of the main findings was that Origin-Destination (O&D) controls are very robust to changes in the competitive environment as compared to leg-level RM. Such an O&D optimisation method was presented at AGIFORS by Fiig et al. (2005) as an extension of the well-known displacement-adjusted virtual nesting (DAVN), and was labelled DAVN-MR. They proposed to split demand into dependent and independent demand, and then to transform dependent demand into independent demand. This would be fed into a linear program that returns displacement costs. Fares are adjusted by subtracting displacement costs to account for the cost of committing capacity, and, in addition, by subtracting price elasticity costs that reflect risk of buy-down. Finally, booking limits are computed using the standard expected marginal seat revenue (EMSR) method. An interesting approach was presented more recently by Gallego et al. (2007) who built their model on DAVN-MR but also included buy-up by using the multinomial-logit choice model, and Gallego et al. (2009), who focus on the static single-leg RM problem. Their work is somewhat related to ours in that they also use the MNL model to address both restricted and unrestricted airfare conditions. However, we investigate dynamic multi-period network problems, and, furthermore, focus on how to optimize the price structure of the unrestricted fares. Our model is based on the work of Bront et al. (2007) who a consider choice-based network RM approach for the MNL model with overlapping segment consideration sets, that means, there may be products that are considered for purchase by more than one customer segment. This feature is exploited in our work to depict choice in mixed fare environments. Meissner and Strauss (2009a) recently extended other RM approaches to allow for overlapping consideration sets and we note that the model developed in this paper can be based on these approaches as well. As our intent is to highlight ways to optimize the price structure in mixed fare environments, we confine ourselves to the simpler model of Bront et al. (2007). The essential ideas would remain the same, only the policy performance can be expected to be better at the cost of significantly higher computational requirements.

3. The Modelling Framework

In this section we first wish to present our ideas of how the above described situation of a mixed fare environment can be modelled where we will stick to the example of an airline network throughout for the sake of simplicity.

In general, our model admits both restricted and/or unrestricted fares on each flight leg or on their combinations. For airline application, the practitioner reports Boyd and Kallesen (2004) and Vinod (2006) suggest that on each flight leg of a traditional carrier that competes with a LCC on this particular leg, the former typically has a fare structure similar to an unrestricted one. For connecting flights, however, demand is little affected so that restrictions can be maintained. Thus the examples given in this article assume that direct flights are only offered as an unrestricted fare for whose price we control, and restricted fares on connecting flights where we control fare availability. Further we assume that connecting flights cannot be substituted by buying tickets for its several flight legs separately. However, the analysis of our model holds without these assumptions as well.

The notation of our model in the network case is geared to the network RM model of van Ryzin and Liu (2008) and Bront et al. (2007) .

Product

We consider a network consisting of m resources, for example, flight legs in the airline application. Each resource i has a fixed capacity of c_i, and the network capacity is given by the corresponding vector $c = [c_1, \ldots, c_m]^T$. The capacity is homogenous, i.e. all seats are perfectly substitutable and do not differ, hence allowing us to accommodate all kind of requests from the given general capacity on a given flight leg. We need to find a common ground for the availability and pricing control, respectively, and achieve this by treating every possible price for the unrestricted fare on a given point-to-point flight i as a separate product. Hence, for each point-to-point flight i there is a set of unrestricted "virtual fare products" U_i, each such product $j \in U_i$ in lieu for a specific price out of a discrete price set. The entity of virtual fare products is denoted by $\mathcal{U} := \bigcup_i U_i$. A restricted product consists of a seat on one or several flight legs in combination with a fare class and departure date. The set of restricted products is denoted by $\mathcal{R}$, accordingly $N := \mathcal{R} \cup \mathcal{U}$ is the set of all $n = |N|$ products in the network. Every product $j \in N$ has an associated revenue r_j. By defining $a_{ij} = 1$ if resource i is used by product j, and $a_{ij} = 0$ otherwise, we obtain the incidence matrix $A = (a_{ij}) \in \{0,1\}^{m \times n}$ whose columns shall be denoted by A_j. Each column A_j gives us information about which resources product j uses. Accordingly we write $i \in A_j$ if resource i is being used by product j. The state of the system is given by the vector of unused capacity $x = [x_1, \ldots, x_m]^T$, and selling product j changes x to $x - A_j$. Defining A to be a binary matrix entails the implicit assumption that no group requests are allowed. We emphasize that allowing $a_{ij} > 1$ does not change the analysis, hence it is straightforward to include group requests in our model.

Customers and Choice Model

Customers arrive at random in the system (for example, on the website), subsequently decide what to product to purchase depending on the available alternatives, or potentially do not buy at all. The (non-)purchase decision is made on the basis of a choice model that we explain in the following paragraph.

There are L customer segments in total, and each segment $l \in \{1, \ldots, L\}$ has a certain set $C_l \subset N$ of products that they consider for purchase. For all products $j \in C_l$, customers of this segment have a preference value v_{lj}. These values are derived by means of a random utility model, for and introduction see, for example, Section 7.2.2 in Talluri and van Ryzin (2004). Utility for a certain product j is the sum of a deterministic part u_{lj} and a random error term ζ_j. A potential way to estimate u_{lj} would be to define a number of attributes that affect utility like price, service, scheduled departure and arrival times etc. Sensitivities of segment l to these explanatory attributes are represented in a vector β_l, and $u_{lj} := \beta_l^T \tilde{x}_j$, where $\tilde{x}$ denotes the vector of attribute values. We assume that customers choose according to the Multinomial Logit (MNL) choice model that is constructed by assuming that the random terms ζ_j are i.i.d. random variables with a Gumbel distribution with zero mean and variance $(\tilde{\mu}\pi)^2/6$ with some scaling parameter $\tilde{\mu}$. Preference values are finally being obtained via $v_{lj} := \exp(u_{lj}/\tilde{\mu})$, and we set v_{lj} equal to zero if the product j is not in the consideration set C_l.

Despite some shortcomings such as the independence from irrelevant alternatives property, the MNL model is very popular in practical applications because it is easy to use and very flexible. A particular advantage is that we can allow consideration sets to overlap, reflecting the lacking means of segmentation. Furthermore, we can adjust preferences for products according to the extent that restrictions are being imposed on them, and any other attribute affecting customers' perceived utility. On the downside, note that we need to estimate preference values for all price points for unrestricted fares, including those that might have never been offered before. To that end, we refer to the literature on calibrating the MNL model, for example, the recent work of Ratliff et al. (2008) or Vulcano et al. (2008).

The booking horizon is divided into T periods that are small enough such that there is at most one customer arrival according to a time-homogeneous Poisson process with arrival rate λ. Time-varying arrivals can also be captured by our models in that we first partition the time horizon into subintervals on which arrivals can be assumed time-homogenous, and then carry out the same analysis for each subinterval. Decisions on which products to offer must be made at the beginning of each time period.

Since the consideration sets overlap, the firm cannot distinguish with certainty between different segments. Therefore, we can only attach a probability p_l, $\sum_l p_l = 1$ to the event that a customer belongs to segment l. We define Poisson processes with rate $\lambda_l := p_l \lambda$ for every segment, so taken together we have $\lambda = \sum_l \lambda_l$. The probability that a segment l customer purchases product j when the fare set S is offered is given by

$$P_{lj}(S) = \frac{v_{lj}}{\sum_{\iota \in C_l \cap S} v_{l\iota} + v_{l0}} \text{ for } S \subset N, \forall i : |S \cap U_i| \leq 1,$$

where v_{l0} is the preference for not buying anything. We remark that the latter quantity v_{l0} can also be used to include the influence of competition on the decision in that it may reflect the attractiveness of competitive products. The condition $|S \cap U_i| \leq 1$ for all direct flights i means that at most one price for the unrestricted fare can be offered at a time. A major advantage of this model is that every restricted or unrestricted fare which is considered by some segment l can be compared to the others in consideration set C_l, and intuitive probabilities can be derived that reflect preferences and offer set. That is, the segment's preference vector essentially has the function of shifting the purchase probabilities according to the offered set of fares.

Finally, the purchase probability for product j given the arrival of a customer is defined by

$$P_j(S) = \sum_{l=1}^{L} p_l P_{lj}(S).$$

4. Optimise price values given the price levels

In this section, we derive control policies that can be used in a mixed fare environment given a finite set of fixed price points for each unrestricted fare. We begin with stating the optimal policy in terms of a dynamic programming (DP) formulation. Unfortunately, the dimension of the state space forces us to approximate the DP's value function in some way. To that end, we draw on the choice-based deterministic linear programming model (CDLP) in the version as investigated by Bront et al. (2007). After some adjustments, CDLP can be used in a mixed fare environment essentially because it allows segment consideration sets to overlap. Meissner and Strauss (2009a) likewise extended other approaches to allow overlapping consideration sets, so these approximations could likewise be used to model mixed fares. We stick to the simpler CDLP for the sake of clearer illustration of the main ideas, and note the extension to other approximations is likely to yield increased policy results at the cost of higher computational expense.

Given a set of price levels U_i for each unrestricted fare i, we wish to optimize the price values along with optimal availability of restricted fare products. This problem can be formulated as the following dynamic programme, where $v_t(x)$ denotes the expected revenue from having uncommited network capacity vector x at time t:

$$\begin{aligned} v_t(x) &= \max_{S\subset N, \forall i: |S\cap U_i|\le 1} \sum_{j\in S} \lambda P_j(S)\Big[r_j + v_{t+1}(x-A_j)\Big] + \Big[1-\lambda+\lambda P_0(S)\Big] v_{t+1}(x) \\ &= \max_{S\subset N, \forall i: |S\cap U_i|\le 1} \left\{ \sum_{j\in S} \lambda P_j(S)\Big[r_j - \big(v_{t+1}(x) - v_{t+1}(x-A_j)\big)\Big]\right\} + v_{t+1}(x), \qquad \forall\, t, x. \end{aligned}$$

The boundary conditions are given by $v_t(0)=0$ for $t=1,\dots,T$, and $v_{T+1}(x)=0$ for all inventory states x. Theoretically, it is possible to solve this problem quite easily via backward dynamic programming, but the size of the state space makes it intractable for practical implementation. Thus we need methods to approximate the optimal value function but which reduce the computational load.

As announced earlier, we approximate the above dynamic programme using the approach of Bront et al. (2007). Let us define the expected revenue from offering set S by $R(S) := \sum_{j\in S} r_j P_j(S)$, the expected consumption of resource i by $Q_i(S) := \sum_{j\in S} a_{ij} P_j(S)$, and $Q(S) := [Q_1(S),\dots,Q_m(S)]^T$. The modified CDLP is given by

$$\begin{aligned} z_{CDLP} = \max & \sum_{S\subset N, \forall i: |S\cap U_i|\le 1} \lambda R(S) t(S) \\ & \sum_{S\subset N, \forall k: |S\cap U_k|\le 1} \lambda Q(S) t(S) \le c, \\ & \sum_{S\subset N, \forall k: |S\cap U_k|\le 1} t(S) = T, \\ & t(S) \ge 0, \quad \forall S\subset N, \forall k: |S\cap U_k| \le 1. \end{aligned}$$

The real-valued variables $t(S)$ represent the total length of time that product set S should be offered; under the assumption of time-homogeneous arrivals and choice probabilities, only total duration is of importance and not in when it should be offered. Expected resource consumption is constrained by capacity vector c, and we can only offer products throughout the length T of the booking horizon. CDLP is identical to the one considered by Bront et al. (2007), the only necessary adjustment relates to the fact that we may offer at most one price point for each unrestricted fare at a time.

CDLP can be solved via column generation and yields the optimal dual values π_i^* to the capacity constraints as static estimates for the marginal opportunity cost of each resource i. With this information one could define the approximation $v_t(x) \approx \sum_i \pi_i^* x_k$, but since it is static, we choose the classic dynamic programming decomposition by the flight legs is used to refine the value function approximation and to introduce time- and capacity-dependence as proposed by van Ryzin and Liu (2008). The network is decomposed by the resource and the value function is approximated by $v_t(x) \approx v_t^i(x_i) + \sum_{k \neq i} \pi_k^* x_k$, where $v_t^i(x_i)$ is computed by the single resource dynamic program

$$v_t^i(x_i) = \max_{S \subseteq N, \forall i: |S \cap U_i| \leq 1} \sum_{j \in S} \lambda P_j(S) \left[r_j - \left(v_{t+1}^i(x_i) - v_{t+1}^i(x_i - 1) - \pi_i^* \right) \mathbf{1}_{\{i \in A_j\}} - \sum_{k \in A_j} \pi_k^* \right] + v_{t+1}^i(x_i), \qquad \forall t, x_i,$$

with $v_{T+1}^i(x_i) = 0$ for all x_i and $v_t^i(0) = 0$ for all t on the boundary. Bront et al. (2007) proposed a linear mixed integer program to solve the dynamic programming subproblem and, alternatively, a polynomial time heuristic. Once we have obtained all functions $v_t^i(\cdot)$ we approximate the value function with $v_t(x) \approx \sum_{i=1}^m v_t^i(x_i)$.

We use a policy that relies on this estimate of the value function and seeks to maximize the (approximately) displacement adjusted revenue within the given time period by

$$\max_{S \subset N(x), |S \cap U_i| \leq 1 \forall i} \left[\sum_{j \in S} \lambda P_j(S) (r_j - \sum_i \Delta v_{t+1}^i a_{ij}) \right], \tag{1}$$

where $\Delta v_{t+1}^i := v_{t+1}^i(x_i) - v_{t+1}^i(x_i - 1)$ is the marginal value of resource i in time $t+1$. The problem (1) has a similar structure like the column generation subproblems that arise in solving CDLP, hence again the problem can again be solved either by a mixed integer linear program or by a greedy heuristic.

5. Pre-Selection of Price Points

In practice, the number of potential price points is often limited by technical constraints of the booking system. Consequently, a natural question to ask is which price points out of a finite set would be the best to include in our price structure given a constraint on the total number of price points per unrestricted fare. Furthermore, practitioners are interested in the value of an additional price point, in particular, which price point should be included if we could implement one additional price point.

This section proposes methods that seek to optimize the pricing structure. First, we consider a dynamic programming formulation that represents an optimal policy and is of interest from a theoretical point of view, but that is again computationally intractable. For practical purposes, we propose a heuristic in the form of a linear mixed integer program that is shown to provide an upper bound on the optimal expected revenue over all feasible pricing structures.

5.1. Dynamic Programming Formulation

Let us denote the maximum expected revenue to be obtained over time period t up to the end of the booking horizon when we have capacity x still uncommitted by $V(t,x,y)$, where y is a binary vector that indicates whether a price point $j \in \mathcal{U}$ of an unrestricted fare is in the price structure, that is, $y_j = 1$ in this case. The set of all feasible states $\mathcal{S}$ is defined by

$$\mathcal{S} := \left\{(t,x,y) : \sum_{j\in U_k} y_j \leq L_k\right\},$$

namely the set of all states where no more than the limit L_k price points have been offered for each unrestricted fare k. The transition function for the y state is defined for all $j \in \mathcal{U}$ by

$$\hat{y}_j(S,y) := \begin{cases} y_j & \text{if } j \notin S, \\ 1 & \text{if } j \in S. \end{cases}$$

The dynamic program that can determine the optimal price points to pre-select can be stated as follows:

$$\begin{aligned} V(t,x,y) = \max_{S:|S\cap U_k|\leq 1 \forall k} & \sum_{j\in S} \lambda P_j(S)\left[r_j + V(t+1, x - A_j, \hat{y}(S,y))\right] \\ & + (1-\lambda+\lambda P_0(S))V(t+1,x,\hat{y}(S,y)) && \forall (t,x,y)\in\mathcal{S}, \qquad (2) \\ & V(t,0,y) = 0 && \forall (t,0,y)\in\mathcal{S}, \\ & V(T+1,x,y) = 0 && \forall (T+1,x,y)\in\mathcal{S}, \\ & V(t,x,y) = -\infty && \forall (t,x,y)\notin\mathcal{S}. \end{aligned}$$

We are interested in a vector y that maximizes $V(1,c,y)$ such that, for any unrestricted fare k, we are only using L_k price points, but direct solution is again not computationally intractable. The following theoretical result shows that up-front commitment to specific price points potentially reduces the expected revenue compared to a situation where we only need to commit to price points once we offer them.

Proposition 1 *For any y^1, $y^2 \in \{0,1\}^{|\mathcal{U}|}$ with $y^1 \preceq y^2$, it holds that $V(t,x,y^1) \geq V(t,x,y^2)$ for all t and x.*

L et y^1, $y^2 \in \{0,1\}^{|\mathcal{U}|}$ with $y^1 \preceq y^2$. It is clear from the boundary condition that $V(t,x,y^1) \geq V(t,x,y^2)$ holds for $t = T+1$ since the only possible values are either both 0 in the case that y^1 and y^2 are feasible, both $-\infty$ in case that both are infeasible or $V(t,x,y^1) = 0$ and $V(t,x,y^2) = -\infty$ in case that only y^2 is infeasible.

Suppose now $t \leq T$ and the assertion holds for $t+1$. For any offer set S we have $\hat{y}(S,y^1) \preceq \hat{y}(S,y^2)$

by definition of the transition function $\hat{y}$. It follows that $V(t+1, x-A_j, \hat{y}(S,y^1)) \geq V(t+1, x-A_j, \hat{y}(S,y^2))$ and $V(t+1, x, \hat{y}(S,y^1)) \geq V(t+1, x, \hat{y}(S,y^2))$. Using the Bellman equation (2) for $V(t,x,y^1)$ and exploiting the latter inequalities yields the desired result.

Essentially, by fixing the price points at the outset we restrict our pricing flexibility over the remaining time horizon. The result indicates that it would be beneficial for the firm to re-optimize their price structure to account for the demand information that has become available in the meantime. While it might not be possible to implement more than a certain number of price points in a booking system, it might be possible to change the price points available in the system at least once or twice during the booking horizon.

Suppose now we could increase the limit on the number of price points of an unrestricted fare. As we increase the limit, the marginal gain in expected revenue decreases, that is, the function mapping the number of price points into the maximum expected revenue from these many price points is concave.

5.2. Linear Programming Approach

The main idea for the construction of a heuristic to tackle the DP (2) is that the objective of CDLP is an upper bound on the optimal expected revenue for a fixed pricing structure, and therefore can be used as a measure of its quality. Though we do not know how close the bound is to the optimal value, we still can expect from numerical evidence that an increase in the bound reflects an increase in optimal expected revenue. Essentially, we maximize this upper bound over all feasible price point combinations. This idea gives rise to the following linear mixed integer program, where $\mathcal{N} := \{S \subset N : |S \cap U_k| \leq 1 \forall k\}$:

$$\textbf{(MIP)} \quad \max_{t,z} \sum_{S \in \mathcal{N}} \lambda R(S) t(S) \tag{3}$$

$$\sum_{S \in \mathcal{N}} t(S) = T, \tag{4}$$

$$\sum_{S \in \mathcal{N}} \lambda Q(S) t(S) \leq c, \tag{5}$$

$$\sum_{j \in U_k} z_j \leq L_k \qquad \forall k, \tag{6}$$

$$\sum_{S \in \mathcal{N} : j \in S} t(S) \leq T z_j \qquad \forall j \in U_k \forall k, \tag{7}$$

$$z_j \in \{0,1\} \qquad \forall j \in U_k \forall k, \tag{8}$$

$$t(S) \geq 0 \qquad \forall S \in \mathcal{N}. \tag{9}$$

The linear program (3,4,5,9) is identical to the one proposed by van Ryzin and Liu (2008). We introduced an additional binary variable z_j for every price point $j \in \mathcal{U}$ of any unrestricted fare k

which indicates whether j is being used or not. Constraint (6) forces the total number of used price points to be less or equal to the prescribed limit for the corresponding unrestricted fare k, and constraints (7) ensure that $z_j = 1$ as soon as price point j is being used for any positive amount of time; note that $\sum_{S\subset\mathcal{N}:j\in S} t(S)$ represents the length of time that j is offered.

In order to solve this large-scale linear mixed integer program we employ column generation. To that end, we consider the dual of the relaxation of (**MIP**) and derive the reduced cost formula for the column corresponding to $t(S)$ from it. In our experiments, we generated all column belonging to z_j at the outset so that we focus only generating $t(S)$ columns only. We associate Lagrangian multipliers σ, π_i, μ_k, ξ_j and $\o_j$ with the constraints (4), (5), (6), (7) and $z_j \leq 1 \forall j$. The dual is given by

$$
\begin{aligned}
(\mathbf{D}) \quad & \min_{\sigma,\pi,\mu,\xi,\o} T\sigma + c^T\pi + L^T\mu + \sum_{j\in U_k \forall k} \o_j \\
& \lambda Q(S)^T\pi + \sigma + \sum_{j\in U_k\forall k} \xi_j \mathbf{1}\{j\in S\} \geq \lambda R(S) \qquad \forall S\in\mathcal{N}, \\
& \mu_k - T\xi_j + \o_j \geq 0 \qquad \forall j\in U_k \forall k, \\
& \sigma \text{ free}, \pi, \mu, \xi, \o \geq 0.
\end{aligned}
$$

The reduced cost of the column corresponding to $t(S)$ is therefore $\lambda R(S) - \lambda Q(S)^T\pi - \sigma - \sum_{j\in U_k\forall k} \xi_j \mathbf{1}\{j\in S\}$. Starting from a pool of columns, we would like to know which column next to generate and to add to the master problem. We select them in a greedy fashion by maximizing the reduced cost over all feasible offer sets, that is

$$
\begin{aligned}
& \max_{u\in\{0,1\}^n} \sum_{j\in\mathcal{U}} \Big[(r_j - A_j^T\pi)P_j(u) - \xi_j\Big] u_j + \sum_{j\in\mathcal{R}} \Big[(r_j - A_j^T\pi)P_j(u)\Big] u_j - \sigma \\
& \sum_{j\in U_k} u_j \leq 1 \qquad \forall k. \qquad (10)
\end{aligned}
$$

Constraints (10) ensure that each unrestricted fare k can be offered at most at one price point. The term $P_j(u)$ stands for the probability that a customer arrives and purchases product j if we offer products as indicated by the binary vector u, and is given by $P_j(u) := \sum_l \lambda_l v_{lj}/(\sum_{\iota\in C_l} v_{l\iota}u_\iota + v_{l0})$ as discussed earlier. This column generation subproblem is similar to the one solved by van Ryzin and Liu (2008) and can be tackled with the same ideas. In particular, we can reformulate this problem as a mixed integer linear program that can be solved either directly, or by using a greedy heuristic as presented for a similar problem in Bront et al. (2007).

Another interesting feature of our approach is that it yields the value of a price point in the form of the Lagrangian multipliers ξ_j corresponding to the constraints (7). Suppose we have an optimal solution (t, z) to (**MIP**), and $z_j = 0$ for some j. If $\xi_j > 0$, increasing the right-hand side of the constraint (7) by one time unit would enable us to offer price point j for one time period and increase our revenue by ξ_j. It is for this reason that we can interpret ξ_j as the marginal value of a price point with respect to time. We can also state an upper bound for the value of having the limit on the number of price points of an unrestricted fare relaxed by 1: The dual value μ_k of constraint (6) gives us the increase in revenue due to this enhanced flexibility, however, it is an upper bound and not the exact value of revenue increase because we consider the relaxed linear program.

So far, we assumed arrivals and customer preferences to be time-homogeneous. In reality, however, time dependent purchase behavior has a great impact on which prices to offer. As indicated earlier, we can approach this more general situation by dividing the booking horizon into sufficiently small parts where we can assume time-homogenety. We illustrate how to optimize the pricing structure with the following example.

Example 1 *For the sake of simplicity, suppose arrivals and preferences are homogenous throughout the first three quarters of the time horizon and then only change once, that is, we have Poisson processes with rates λ_1 and λ_2 for the first and second part of the booking horizon, respectively. Likewise, expected revenue $R_1(S)$ and expected resource consumption $Q_1(S)$ change to $R_2(S)$ and $Q_2(S)$ at time period $3/4T$. The mixed integer linear problem is then:*

$$
\begin{aligned}
(\mathbf{MIP})' \quad & \max_{t,z} \sum_{S\in\mathcal{N}} \left[\lambda_1 R_1(S) t_1(S) + \lambda_1 R_2(S) t_2(S)\right] \\
& \sum_{S\in\mathcal{N}} t_1(S) = \frac{3}{4}T, \\
& \sum_{S\in\mathcal{N}} t_2(S) = \frac{1}{4}T, \\
& \sum_{S\in\mathcal{N}} \lambda_1 Q_1(S) t_1(S) \le c, \\
& \sum_{S\in\mathcal{N}} \lambda_2 Q_2(S) t_2(S) \le c - \sum_{S\in\mathcal{N}} \lambda_1 Q_1(S) t_1(S), \\
& \sum_{j\in U_k} z_j \le L_k \qquad \forall k, \\
& \sum_{S\in\mathcal{N}: j\in S} \big(t_1(S) + t_2(S)\big) \le T z_j \qquad \forall j \in U_k \forall k, \\
& z_j \in \{0,1\} \qquad \forall j \in U_k \forall k, \\
& t_1(S), t_2(S) \ge 0 \qquad \forall S \in \mathcal{N}.
\end{aligned}
$$

*Note that this problem is not considerably more difficult to solve than **(MIP)** because there are only $m+1$ more constraints. Clearly, each additional inhomogeneity will result in additional $m+1$ constraints. We conclude that incorporating time-dependence is possible, though the more the booking horizon needs to be split up the more run time the computations will require.*

5.3. Stochastic Program

Formulating the stochastic program that underlies the model is insightfull because it can be shown that (**MIP**) constitutes an upper bound on the optimal expected revenue. Note that optimality is over all policies that admit no more than the given limit L_k for each unrestricted fare k. This result is not surprising since CDLP provides an upper bound for a fixed feasible pricing structure, and (**MIP**) essentially maximizes the CDLP objective over all feasible pricing structures.

We need some additional notation in order to formulate the stochastic program. Denote by $S_\nu(t,\omega)$ the control under policy ν at time t for a sample path ω. The n-dimensional binary random vector $N(S_\nu(t,\omega))$ indicates which product is purchased in time t under policy ν and sample path ω.

Definition 1 *A policy ν is called* admissible *if it is non-anticipating, the capacity constraints are almost surely not violated, $S_\nu(t,\omega)$ does not allow more than one price point per unrestricted fare k and uses at most L_k price points in total for all k. The set of admissible policies $\mathcal{M}$ is*

$$\mathcal{M} := \Big\{\nu : \nu \text{ non-anticipating}, \quad \sum_{t=1}^{T} AN(S_\nu(t,\omega)) \le c \text{ (a.s.)}, \\ S_\nu(t,\omega) \in \mathcal{N}, \quad \sum_{j\in U_k} \mathbf{1}\{\exists t,\omega : j \in S_\nu(t,\omega)\} \le L_k \forall k. \Big\}$$

The stochastic program is:

$$V^S = \max_{\nu\in\mathcal{M}} \mathrm{E}\Big[\sum_{t=1}^{T} r^T N(S_\nu(t))\Big].$$

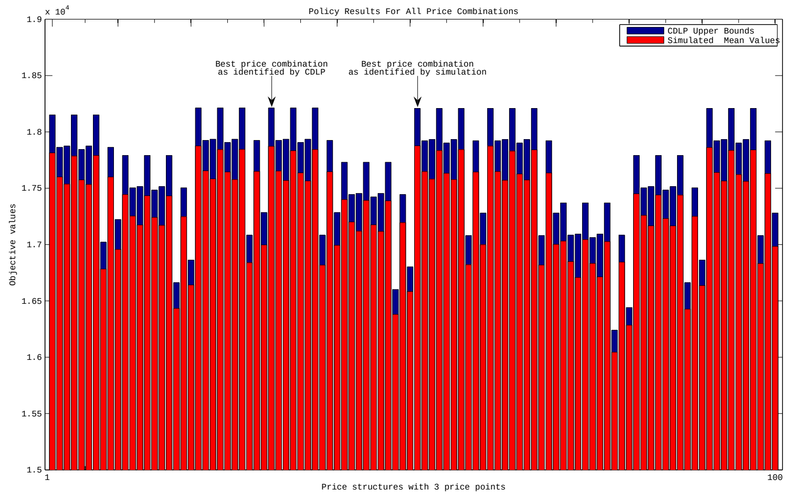
Proposition 2 $V^S \le V^{MIP}$

L et $S_{\nu^*}(t,\omega)$ be the optimal control for the stochastic problem where ω represents dependence on the sample path. As shown in Proposition 1 in van Ryzin and Liu (2008), a vector $t(S)$ can be defined by $t(S) := \mathrm{E}\Big[\sum_{t=1}^{T} \mathbf{1}\{S_{\nu^*}(t,\omega) = S\}\Big]$ that satisfies the capacity constraints (5) and that yields the same objective value. With z_j defined as $z_j := \mathbf{1}\{\exists t,\omega : j \in S_{\nu^*}(t,\omega)\}$, the solution (t,z) is feasible to (**MIP**) because z satisfies the price point constraint (6) on the basis that $\nu^* \in \mathcal{M}$ is an admissible policy, and the remaining constraints are likewise easily verfied.

6. Numerical Results

We test our new method for pricing structure optimization in mixed fare environments on several problem instances that shall illustrate the method's performance with respect to quality and run-time. By quality we refer to the closeness of the expected revenue obtainable under the pricing structure resulting from (**MIP**) compared to the best-possible expected revenue over all feasible

Figure 1 Identification of the best price structure.



pricing structure, where feasibility is with respect to the number of price points not violating the upper limits and them being chosen from the finite set of "virtual fares". As an evaluation of the dynamic program is too expensive even for small instances, we instead consider in a network pricing problem that is small enough to allow for full enumeration and testing of all price constellations by means of simulation. In this paper, we are more interested in the good choice of the pricing structure than in the choice of a good policy: We chose to use the CDLP based dynamic programming decomposition policy of van Ryzin and Liu (2008) for all simulations because it is the generally accepted current benchmark. Keeping this policy fixed we can investigate the impact of altering the pricing structure, however, we remark that recently other policies have been proposed that can achieve higher revenues at higher computational expense, see, for example, Zhang and Adelman (2007), Meissner and Strauss (2009b), Kunnumkal and Topaloglu (2008) or Zhang (2009). These approaches can combined with our pricing method with accordingly improved revenue results.

Naturally, the run-time required to solve (**MIP**) calls for an investigation since we face a mixed integer program with exponentially many columns for which it has been shown that the column generation subproblem is already NP-hard (see Bront et al. (2007)). To that end, we solve (**MIP**) on a set of hub and spoke network instances that correspond to the largest ones used in the aforementioned work.

6.1. Small Network Example

Let us first consider the quality of the price selection via (**MIP**): We run the method on a network with two flights only, as depicted in Figure 2, that is small enough to allow us to run simulations for each feasible price combination so as to provide us with the optimal price set. In this example, we assume that the firm offers an unrestricted fare U for short-haul (direct) flights and traditional fares for the long-haul (connecting) traffic with fare classes Y, M and Q. For both direct flights we have 5 potential price points, however, we are limited to only 3 price points each that may form our price structure. Each origin-destination combination has two segments associated with it, one with high and the other with low price sensitivity. Restrictions on the traditional fares effectively fence off the lower fares for the connecting traffic, however, on the direct flights business customers are able to buy down, resulting in overlapping segments. We summarize the product and segment definitions in Table 1 and 3. In the following, when we refer to a price point $j = 3$, for example, we mean the price point that is described by the virtual product 3 in Table 1. The capacity of leg 1 and 2 is 50 and 70, respectively, and we consider a time horizon of 1000 time periods.

For each direct flight, there are $5!/(2!3!) = 10$ possible sets with three price points, so totally 100 price combinations in the network (note that restricted fares are always included in the pricing structure). Of course, we do not need to consider subsets with less than three price points per direct flight since we may choose never to offer an unrestricted fare at a certain price point. For each one out of the 100 pricing structures, we solve the CDLP as presented in Section 4 and run simulations using the dynamic programming decomposition policy based on CDLP's dual values of the capacity constraints to obtain a close approximation of the optimal expected revenue. The simulation is stopped once the relative error is less than 0.7% with 95% confidence, which is usually reached after about 200 simulations of the booking process for this problem. We report the results in Figure 1 that clearly demonstrates that the upper bound provided by the CDLP can reflect the relative behavior of the simulated mean revenue very well. This is encouraging because (**MIP**) essentially maximizes CDLP over all potential price combinations subject to the price point limits. For this network our method provides price points {3,4} for leg 1 and {7,8} for leg 2 as the optimal solution. Note that we allowed for three price points per direct flight, however, (**MIP**) asserts that these two price points per flight are sufficient which might be somewhat counterintuitive since more price points would give us potentially more flexibility. However, when looking up the 15 price combinations that maximize the simulated mean revenue (listed in Table 2) we observe that indeed these two price points are included with any possible third point on each leg. We ran the simulation with the price points proposed by (**MIP**) and obtained the same mean revenue, that is, the third price point can indeed be chosen arbitrary and does not further increase revenue.

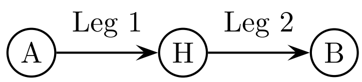
Therefore, an optimal pricing structure has been identified by our method and, furthermore, the dual values of the constraints on the number of price points indicated that the worth of an additional price point is zero. We conclude that the quality achievable by our method being can be very good indeed.

Table 1 Products, Segments and Preference Values for Small Network.

Product	Resources	OD	Class	Fare
1	1	A → H	U	100
2	1	”	U	120
3	1	”	U	140
4	1	”	U	160
5	1	”	U	180
6	2	H → B	U	100
7	2	”	U	120
8	2	”	U	140
9	2	”	U	160
10	2	”	U	180
11	1,2	A → B	Q	300
12	1,2	”	M	350
13	1,2	”	Y	500

"Resources" indicates the resources which the respective product utilizes.

Figure 2 Small network example.



6.2. Hub & Spoke Network

Solving (**MIP**) is not a trivial task since it is a mixed integer program with $1+2m+|\mathcal{U}|$ constraints and an exponentially growing number of variables, where m is the number of flight legs the network and $|\mathcal{U}|$ is the total number of price points in the network belonging to unrestricted fares. We assume that there is exactly one unrestricted fare for each direct flight that is to be priced at one out of p price points, giving a total of $|\mathcal{U}| = mp$ price points, while the airline can maintain restrictions on connecing traffic. We demonstrate that (**MIP**) can be solved efficiently using standard optimization software such as CPLEX. To that end, in the following we solve (**MIP**) for the largest network example of Bront et al. (2007) and analyze run time with respect to network capacity and number of price points.

The Hub & Spoke Network Example consists of eight flights as depicted in Figure 3, each with capacity 200 that we scale up or down with a parameter $\alpha \in \{0.6, 0.8, 1, 1.2, 1.4\}$ to account for different load factors. Products are defined in Table 6: There are 48 restricted fares for connecting

Table 2 List of all 15 pricing structures that maximize simulated mean revenue.

Leg 1			Leg 2		
1	2	3	6	7	8
1	2	3	7	8	9
1	2	3	7	8	10
1	3	4	6	7	8
1	3	4	7	8	9
1	3	4	7	8	10
2	3	4	6	7	8
2	3	4	7	8	9
2	3	4	7	8	10
1	3	5	6	7	8
1	3	5	7	8	9
1	3	5	7	8	10
2	3	5	6	7	8
2	3	5	7	8	9
2	3	5	7	8	10
3	4	5	6	7	8
3	4	5	7	8	9
3	4	5	7	8	10

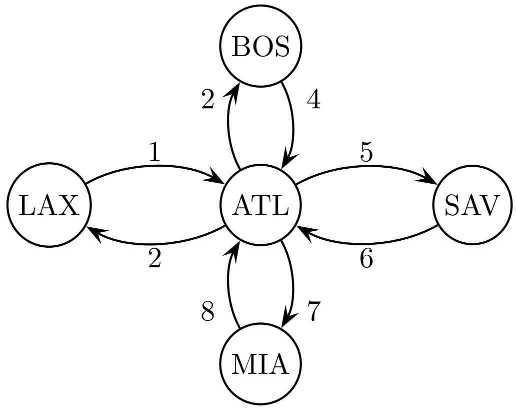
The restricted products are always in the structure and therefore have been omitted.

Table 3 Segments and consideration sets for Small Network Example.

#	Segment	Consideration set	Pref. vector	λ_l (%)	v_{l0}
1	A → H, high price sensitivity	{1,2,3}	[6,4,2]	15	10
2	A → H, low price sensitivity	{1,2,3,4,5}	[5,4,3,2,1]	6	10
3	H → B, high price sensitivity	{6,7,8}	[6,4,2]	15	10
4	H → B, low price sensitivity	{6,7,8,9,10}	[5,4,3,2,1]	6	10
5	A → B, high price sensitivity	{11,12}	[5,3]	3	10
6	A → B, low price sensitivity	{13}	[5]	2	10

traffic, and one unrestricted fare for each direct flight. For example, Product 1 is a ticket BOS to LAX in class Y for \$575 using legs 2 and 4, Product 4 is BOS to LAX in class Q for \$139 and Product 5 is LAX to BOS in class Y using legs 1 and 3. The restricted products are identical to those in Example 3 in Bront et al. (2007), for the restricted products on direct flights, however, we substituted in unrestricted fares that shall be priced at one out of maximal $L_k = 4$ price points for all legs k.We chose this limit because in the restricted environment we have four fare classes,

Figure 3 Hub & Spoke Network example.



so for technical reasons (regarding the booking system) there might be only four "price slots" available to which we need to commit at the beginning of the booking horizon. The model (**MIP**) needs to choose the best four prices out of a set of p prices for each flight on a uniform grid defined over the interval given in Table 6. For example, the candidate price points for ATLBOS are $\{69, 69+\Delta, \ldots, 310\}$ with the price step $\Delta = (310-69)/(p-1)$.

We have two customer segments per origin-destination combination, a high-yield (H) and a low-yield (L) one, the former being less price sensitive than the latter. Preference values for the prices of Y, M, Q and B class similar to those in Bront et al. (2007) were used to inter- and extrapolate those on the uniform grid with cubic splines, the related information being given in Table 4 and 5. The underlying rationale is that customers increasingly ignore restrictions, particularly on short-haul flights, and focus on price instead (see, e.g., Boyd and Kallesen (2004)). Hence we interpret the preference values in the restricted context as being purely motivated by price, giving rise to the idea of extrapolation to other price points in this example to obtain a mixed fare environment under similar customer behavior. Despite the fact that our method can also be used to compare policies in restricted versus mixed fare environments, our purpose is here to illustrate the computational performance. The tests were carried out under the assumption that we seek to identify four price points out of a uniform grid with $p \in \{2, 4, 8, 16, 32, 64\}$ candidates for each direct flight simulataneously. For each p, we vary the scaling parameter α to reflect different load factors.

All computations were done in Matlab with CPLEX using the Tomlab interface on a 3GHz PC. At each iteration in the column generation procedure, we solve the LP relaxation to obtain dual variables that are needed for the column pricing. We used the MILP formulation of the column pricing problem which was acceptable for this problem size. For considerably larger networks, a heuristic similar to the one in Bront et al. (2007) or Meissner and Strauss (2009a) is not difficult to extend to this case and will reduce run times. The algorithm was terminated when the reduced cost was smaller than (current objective + reduced cost) times 10^{-6}.

We report CPU times for solving the master problems and column pricing problems associated with solving an instance of (**MIP**) along with the number of generated columns in Table 7. Run times are very small for the cases of high capacity ($\alpha \in \{1.2, 1.4\}$) since the capacity does not contrain the problem any more. But even for the more interesting cases of tight capacity it took in the worst scenario less than 9 minutes to solve (**MIP**), a problem with $1 + 16 + 8 \cdot 64 = 529$ constraints and $2^{48} \cdot (64 + 1)^8$ real and $8 \cdot 64 = 512$ binary variables. Given that we only need to solve this problem once at the outset of the booking horizon, this seems acceptable for practical implementation.

7. Conclusion

We proposed a choice-based network revenue management model that can be used to optimize the pricing structure in unrestricted or mixed restricted/unrestricted fare environments. In addition, the model provides upper bounds on the value of an additional price point. It was also shown on a small network example that the method can identify good solutions (indeed, the optimal pricing structure had been identified), and runs quick despite the complexity having to solve a large-scale mixed integer linear program as tested on larger hub & spoke networks. An optimal solution can be obtained by a dynamic programming formulation which, though being computationally intractable, is of theoretical interest. For example, we can derive from it the insight that late commitment to price points can potentially increase expected revenues. This could be exploited by resolving our proposed model several times throughout the booking horizon, and changing the pricing structure accordingly. Naturally, this will be constrained by the cost of price changes and technical obstacles.

As for future research, our model could be used to perform simulation studies to examine under which circumstances entirely unrestricted product structures to be preferred over mixed ones, or how the pricing structure changes in response to changes in the customer's purchase behavior. The pre-selection of price points can also be paired with recent achievements in tightening the upper bound on the optimal expected revenue, see, for example, Talluri (2008). Such an approach can be expected to yield potentially even better results because we use the upper bound as the objective to maximize over all possible pricing structures, and accordingly a tighter bound should yield a more accurate objective.

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Table 4 Segments and consideration sets for the Hub & Spoke Network Example.

Segment	C_l	v_l	λ_l	Segment	C_l	v_l	λ_l
BOSLAX H	{1,2,3,4}	{5,5,7,10}	0.01	MIASAV H	{41,42,43,44}	{6,7,8,10}	0.01
BOSLAX L	{3,4}	{9.10}	0.032	MIASAV L	{43,44}	{9,10}	0.025
LAXBOS H	{5,6,7,8}	{5,5,7,10}	0.01	SAVMIA H	{45,46,47,48}	{6,7,8,10}	0.01
LAXBOS L	{7,8}	{9,10}	0.032	SAVMIA L	{47,48}	{9.10}	0.025
BOSMIA H	{9,10,11,12}	{6,7,10,10}	0.008	ATLBOS H	$\{49,\ldots,48+p\}$	interp	0.015
BOSMIA L	{11,12}	{8,10}	0.03	ATLBOS L	$\{49,\ldots,48+p\}$	interp	0.035
MIABOS H	{13,14,15,16}	{6,7,10,10}	0.008	BOSATL H	$\{49+p,\ldots,48+2p\}$	interp	0.015
MIABOS L	{15,16}	{8,10}	0.03	BOSATL L	$\{49+p,\ldots,48+2p\}$	interp	0.035
BOSSAV H	{17,18,19,20}	{5,6,9,10}	0.01	ATLLAX H	$\{49+2p,\ldots,48+3p\}$	interp	0.01
BOSSAV L	{19,20}	{8,10}	0.035	ATLLAX L	$\{49+2p,\ldots,48+3p\}$	interp	0.04
SAVBOS H	{21,22,23,24}	{5,6,9,10}	0.01	LAXATL H	$\{49+3p,\ldots,48+4p\}$	interp	0.01
SAVBOS L	{23,24}	{8,10}	0.035	LAXATL L	$\{49+3p,\ldots,48+4p\}$	interp	0.04
LAXMIA H	{25,26,27,28}	{5,6,10,10}	0.012	ATLMIA H	$\{49+4p,\ldots,48+5p\}$	interp	0.012
LAXMIA L	{27,28}	{9,10}	0.028	ATLMIA L	$\{49+4p,\ldots,48+5p\}$	interp	0.035
MIALAX H	{29,30,31,32}	{5,6,10,10}	0.012	MIAATL H	$\{49+5p,\ldots,48+6p\}$	interp	0.012
MIALAX L	{31,32}	{9,10}	0.028	MIAATL L	$\{49+5p,\ldots,48+6p\}$	interp	0.035
LAXSAV H	{33,34,35,36}	{6,7,10,10}	0.016	ATLSAV H	$\{49+6p,\ldots,48+7p\}$	interp	0.01
LAXSAV L	{35,36}	{9,10}	0.03	ATLSAV L	$\{49+6p,\ldots,48+7p\}$	interp	0.03
SAVLAX H	{37,38,39,40}	{6,7,10,10}	0.016	SAVATL H	$\{49+7p,\ldots,48+8p\}$	interp	0.01
SAVLAX L	{39,40}	{9,10}	0.03	SAVATL L	$\{49+7p,\ldots,48+8p\}$	interp	0.03

p is the number of potential price points per leg, *interp* indicates that the preference values have been interpolated based on the data in Table 5.

Table 5 Preference values for at given prices that were used for inter-/extrapolation over the uniform price grid.

Segment	Prices	Preferences
ATLBOS/BOSATL H	[310,290,95,69]	[6,7,9,10]
ATLBOS/BOSATL L	[95,69]	[8,10]
ATLLAX/LAXATL H	[455,391,142,122]	[5,6,9,10]
ATLLAX/LAXATL L	[142,122]	[9,10]]
ATLMIA/MIAATL H	[280,209,94,59]	[5,5,10,10]
ATLMIA/MIAATL L	[94,59]	[8,10]
ATLSAV/SAVATL H	[159,140,64,49]	[4,5,8,9]
ATLSAV/SAVATL L	[64,49]	[7,10]

Table 6 Product definition for the Hub & Spoke Network Example.

O-D Market	Legs	Revenue				
		Y	M	B	Q	U
BOSLAX/LAXBOS	4,2/1,3	575	380	159	139	-
BOSMIA/MIABOS	4,7/8,3	403	314	124	89	-
BOSSAV/SAVBOS	4,5/6,3	319	250	109	69	-
LAXMIA/MIALAX	1,7/8,2	477	239	139	119	-
LAXSAV/SAVLAX	1,5/6,2	502	450	154	134	-
MIASAV/SAVMIA	8,5/6,7	226	168	84	59	-
ATLBOS/BOSATL	3/4	-	-	-	-	[69,310]
ATLLAX/LAXATL	2/1	-	-	-	-	[122,455]
ATLMIA/MIAATL	7/8	-	-	-	-	[59,280]
ATLSAV/SAVATL	5/6	-	-	-	-	[49,159]

Table 7 Computational times (in seconds) for for Hub & Spoke Network Example.

p	α	Master	ColGen	#GC	p	α	Master	ColGen	#GC
	0.6	2.2	13.5	30		0.6	2.9	59.2	40
	0.8	1.8	10.3	21		0.8	2.2	44.2	29
2	1.0	1.7	8.8	15	16	1.0	1.7	15.1	10
	1.2	1.3	1.2	2		1.2	1.4	2.8	1
	1.4	1.3	1.2	2		1.4	1.4	2.8	1
	0.6	3.1	47.2	71		0.6	4.2	138.2	40
	0.8	2.4	28.2	40		0.8	2.7	100.3	29
4	1.0	2.0	15.6	24	32	1.0	1.8	30.5	9
	1.2	1.6	6.7	10		1.2	1.4	5.2	1
	1.4	1.6	6.5	10		1.4	1.4	5.2	1
	0.6	2.6	34.1	42		0.6	26.4	512.2	41
	0.8	2.3	27.4	31		0.8	5.0	359.4	31
8	1.0	1.7	8.5	10	64	1.0	2.0	86.2	9
	1.2	1.3	1.5	1		1.2	1.4	13.9	1
	1.4	1.4	1.5	1		1.4	1.4	14.0	1

p refers to the number of prices points from which at most four may be chosen, and #GC denotes the number of generated columns.