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Improved Bid Prices for Choice-Based Network Revenue Management

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In many implemented network revenue management systems, a bid price control is being used. In this form of control bid, prices are attached to resources, and a product is offered if the revenue derived from it exceeds the sum of the bid prices of its consumed resources. This approach is appealing because once bid prices have been determined, it is fairly simple to derive the products that should be offered. Yet it is still unknown how well a bid price control actually performs. Recently, considerable progress has been made with network revenue management by incorporating customer purchase behavior via discrete choice models. However, the majority of authors have presented control policies for the booking process that are expressed in terms of which *combination of products* to offer at a given point in time and given resource inventories. The recommended combination of products as identified by these policies might *not* be representable through bid price control. If demand were independent from available product alternatives, an optimal choice of bid prices is to use the marginal value of capacity for each resource in the network. But under dependent demand, this is not necessarily the case. In fact, it seems that these bid prices are typically not restrictive enough and result in buy-down effects.

We propose (1) a simple and fast heuristic that iteratively improves on an initial guess for the bid price vector; this first guess could be, for example, dynamic estimates of the marginal value of capacity. Moreover, (2) we demonstrate that using these dynamic marginal capacity values directly as bid prices can lead to significant revenue loss as compared to using our heuristic. Finally, (3) we investigate numerically how much revenue performance is lost due to the confinement of product combinations that can be represented by a bid price.

Our heuristic is not restricted to a particular choice model and can be combined with any method that provides estimates of the marginal values of capacity. In our numerical experiments, we test the heuristic on some popular networks examples taken from peer literature. We use a multinomial logit choice model which allows customers from different segments to have products in common that they are considering purchasing. In most instances, our heuristic policy results in significant revenue gains over some currently available alternatives at low computational cost.

Key words: revenue management, network, bid prices, choice model

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Network revenue management (RM) refers to the activity of a firm that is endowed with limited quantities of multiple resources and that sells products, each consuming a bundle of resources, which controls product availability over time with the objective to maximize revenue. This sort of problem appears in the hotel, railway, car rental, and tour operator business, but air travel is perhaps the most well-known source for such problems. Therefore, we will stick to that terminology. A product in this setting is composed of seats on one or more flight legs, potentially some fare rules and an associated fare. The firm faces stochastic demand, and the capacity of the resources is limited. In the context of this paper, we treat capacities as fixed; in particular, we do not consider overbooking.

The incorporation of choice behavior into network RM has increasingly gained attention in recent years as the means of segmentation erode in many markets. Traditional models for revenue management have worked under what is known as *independent demand* assumption. This assumption postulates that every customer is interested only in a single product and makes a purchase or no-purchase decision independent of other offers available by the firm or competitors. This assumption is reasonable if products are perfectly fenced, but with the severe cuts of fare restrictions that traditional airlines made in response to low-cost competition, it cannot be upheld in many markets. We still assume that customers' purchase behavior is myopic so that demand at any point in time does not depend on previous or anticipated future demand.

The problem can be formulated as a dynamic program, unfortunately one with a computationally intractable state-space even for networks of moderate size. Therefore, significant research has been devoted to approximating the value function to obtain heuristic policies. Many of this recent work proposes policies under the assumption that any combination of products can in principle be made available. However, in practice, this is not the case as some product combinations might not be representable via the two dominant methodologies, i.e., virtual nesting controls and bid-prices. Adoption of either method seems to be driven by corporate history rather than an informed choice, but it is noteworthy that in both cases bid-prices have to be computed. For a detailed discussion see Chaneton and Vulcano (2009). These authors have recently proposed a stochastic gradient method based on simulated sample paths. The authors show that their algorithm converges under mild assumptions to a stationary point and improves previous methods to calculate bid prices.

In this paper, we present a simple, yet effective, way to improve bid prices that can be based on any choice-based network RM method that provides estimates of the marginal value of capacity, irrespective of the choice model (as long as it is reasonably fast to calculate purchase probabilities). The basic idea is to start with some initial bid price (based, e.g., on estimated marginal capacity values), and then to raise bid prices in a greedy fashion to exclude products that have a negative impact on overall profits because of buy-down effects. Numerical experiments confirm that this new method performs very well when compared with other available approaches.

1. Literature Review

Network Revenue Management (RM) are computationally intensive even without consideration of customer choice behavior, thus research has been primarily concentrated on finding good heuristics. A comprehensive description of both scientific and applied RM can be found in the book of Talluri and van Ryzin (2004b), and the reader interested in a general overview of research over the last decades shall be referred to the reviews McGill and van Ryzin (1999) and Chiang et al. (2007). We focus in the following on papers more closely related to our approach.

Independent demand is a valid assumption in the case that customer segments are well fenced off, and recent work includes Adelman (2007) and Topaloglu (2009). Adelman (2007) proposes a time-dependent approximation and shows that upper bounds on the optimal objective value are tightened relative to the standard so-called deterministic linear programming (DLP) approach, and that the obtained policies perform better in a simulation study. Similarly, Topaloglu (2009) improves on the DLP by using Lagrangian relaxation to obtain a time- and inventory-level-dependent approximation. Farias and Van Roy (2007) introduce a linear programming approach to approximate dynamic programming that depends on both time and inventory level. The same approximation was independently proposed by Talluri (2008), who focuses on the relationships of upper bounds on the optimal objective value of the aforementioned approaches by Topaloglu and Adelman, as well as the DLP and a randomized linear programming model.

The earliest contributions to single leg RM with choice behavior include Brumelle et al. (1990) and Belobaba and Weatherford (1996), amongst others, and for networks, the PODS simulation studies by Belobaba and Hopperstad (1999). Zhang and Cooper (2005) consider an inventory control problem of a set of parallel flights including a customer choice model yielding a stochastic optimization problem which is being solved by simulation-based methods. Another simulation-based approach is given by van Ryzin and Vulcano (2008), who compute virtual nesting controls by constructing a stochastic steepest ascent algorithm designed to find stationary points of the expected revenue function.

The incorporation of choice behavior into network RM has increasingly gained attention as the means of segmentation erode in many markets by the arrival of competitors employing a low-cost strategy. In this situation, the inclusion of choice behavior becomes a crucial element for any RM system. Among the first approaches with a general model of customer choice is Talluri and van Ryzin (2004a) for a single flight leg problem. Among the efficient techniques that have been proposed for the network context is the so-called choice-based linear program (CDLP) of Gallego and Phillips (2004). Based on this work, Liu and van Ryzin (2008) present an extension of the standard deterministic linear program approach to include choice behavior, albeit with customer segments that do not consider the same products. The result is an indication of the number of time periods out the finite time horizon that an offer set should be available. A dynamic programming decomposition approach is taken to obtain policies from the static solution of the CDLP and applied to the multinomial logit (MNL) choice model with disjoint consideration sets. Furthermore, the solution to the CDLP constitutes an upper bound on the optimal expected revenue. The notion of efficient sets introduced by Talluri and van Ryzin (2004a) for the single leg case is translated into the network context, and these authors show that CDLP only uses efficient sets in its optimal solution. Unfortunately, for the network problem the optimal policy does not necessarily only use efficient sets like the single leg case, but Liu and van Ryzin (2008) can show asymptotic optimality of the CDLP which indicates that using efficient sets might be a good choice. Kunnumkal and Topaloglu (2008) propose an alternative deterministic linear programming approach (ADLP) with very similar structure like the CDLP, but they try to address its shortcoming by calculating time dependent bid prices in contrast to the static ones produced by the CDLP. Although neither CDLP nor ADLP can be proven to be theoretically superior, numerical experiments indicate ADLP results in tighter upper bounds on the optimal expected revenue and better policies, as well. Kunnumkal and Topaloglu (2008) also apply their model to the MNL choice model with disjoint consideration sets. Similar results like for the CDLP are presented, including asymptotic optimality, the fact that ADLP provides an upper bound on the objective value and a dynamic programming decomposition approach. The extension though comes at the cost of having significantly more constraints in the arising linear program. A generalization of the CDLP that can also handle the MNL choice model with overlapping consideration sets is presented in Miranda Bront et al. (2009), who employ column generation to solve the arising large linear program. Meissner and Strauss (2009) extend the approach of Adelman (2007) to include bid prices that depends on the remaining inventory level as well as time.

Bid prices in network revenue management have been discussed in Talluri and van Ryzin (1998), who show that the resulting control is not necessarily optimal. Recently, Chaneton and Vulcano (2009) have proposed a stochastic gradient method to optimize bid prices based on simulated sample paths and show that the resulting algorithm converges under mild conditions. Prior simulations-based methods include Topaloglu (2008), which builds on van Ryzin and Vulcano (2008); however, these papers focus on optimizing virtual nesting rather than bid prices.

2. Problem Formulation

We face a network with m resources—flight legs in the airline application—and n products. A product j is a seat on one or several flight legs and has a fixed fare f_j and potentially some fare rules associated with it. The set of all products is denoted by $N = \{1, \dots, n\}$. Which resources a product requires is defined in a matrix $A \in \{0,1\}^{m,n}$ whose component a_{ij} represents whether product j requires resource i, so we assume that there are no group requests. We write A_j for the jth column of A and A^i for its ith row. The notation $i \in A_j$ ($j \in A^i$) represents resources i that are used by product j (products j that use resource i).

Customers arrive continuously over time while decisions on which products to offer are made at discrete points in time such that the time intervals are small enough to have a negligible probability that two or more arrivals occur. A customer arrives in time periods t with probability λ. For the sake of simplicity, we assume that λ is constant over all time periods. However, the extension to the time-heterogeneous case is not difficult. The decision time periods are indexed with t starting at time $t = 1$ until the end of the booking horizon $t = \tau$. All flights depart at time $t = \tau + 1$. The index t can also refer to the time interval between decisions at t and $t + 1$ and will be clear from the context. Given that we offer a set $S \subset N$ of products at time t, a customer purchases product $j \in S$ with probability $P_j(S)$ and does not purchase with probability $P_0(S)$. The choice option $j = 0$ stands for the non-purchase alternative and can be used to reflect the attractiveness of competition. The choice probabilities are derived by some choice model such that $\sum_{j \in S} P_j(S) + P_0(S) = 1$ and do not depend on time (the same comments as for λ apply concerning time-dependence). All customers show up and do not cancel so that no overbooking is required.

Each resource i initially has a capacity c_i available, and the state vector $x \in \mathbb{N}_0^m$ indicates how much inventory is still available. Although x is clearly time-dependent, we do not use a subscript t because it will be clear from the context. The remaining inventory also affects which products can be offered; since we exclude overbooking, we require that sufficient inventory must be available to provide a product. The set of all feasible products is then $N(x) = \{j \in N : A_j \le x\}$.

Let us denote the optimal expected revenue obtainable from time t until the end of the booking horizon given remaining capacity x by $v_t(x)$, usually referred to as the value function. A common assumption in recent work on this kind of network RM problem is that we can offer any combination of products at any time, subject to sufficient remaining inventory. Under this assumption, $v_t(x)$ can be written as follows:

$$v_t(x) = \max_{S \subset N(x)} \sum_{j \in S} \lambda P_j(S) \Big[f_j + v_{t+1}(x - A_j) \Big] + \Big[1 - \lambda + \lambda P_0(S) \Big] v_{t+1}(x)$$

$$= \max_{S\subset N(x)} \left\{ \sum_{j\in S} \lambda P_j(S) \Big[f_j - \big(v_{t+1}(x) - v_{t+1}(x-A_j) \big) \Big] \right\} + v_{t+1}(x), \qquad \forall\, t, x. \tag{1}$$

The boundary conditions are given by $v_{\tau+1}(x)=0$ for all inventory states x. Note that the expression $\big(v_{t+1}(x)-v_{t+1}(x-A_j)\big)$ represents the opportunity cost of selling product j. If we have a good approximation of $v_t(x)$ for all time periods t and all inventory state vectors x, then we can use this as an approximation of the opportunity cost. Various approximations have been proposed, see e.g. Miranda Bront et al. (2009), Liu and van Ryzin (2008), Meissner and Strauss (2009), or Zhang and Adelman (2007). They all construct a policy of the following kind:

$$S^* = \arg\max_{S\subset N(x)} \left\{ \sum_{j\in S} \lambda P_j(S) \Big[f_j - \big(\text{approx. opportunity cost}(t,x,j)\big) \Big] \right\}. \tag{2}$$

In words, the policy recommends to offer the product set S^* at time period t when we have inventory x still available in the network.

Problem (2) assumes that we are able to offer any combination of products for which we have sufficient inventory left, thus the maximization is over $S\subset N(x)$. However, this is not necessarily true if we are forced to use bid price control (e.g. owing to the restrictions imposed by global distribution systems). In this case, we need to set a bid price $b_i(t,x)\in\mathbb{R}$ at each time t for each resource i given remaining capacity x, which is subsequently used in a so-called bid price control:

Definition 1 (Talluri and van Ryzin (2004b)) *A control $u(t,x,f)$ is a* bid-price control *if there exist real-valued functions $b(t,x)=[b_1(t,x),\dots,b_m(t,x)]$, $t=1,2,\dots,\tau$, (called* bid prices*), such that*

$$u_j(t,x,f_j) = \begin{cases} 1 & \text{if } f_j > \sum_{i\in A_j} b_i(t,x),\ A_j \le x, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The control u maps into a binary m-vector that represents which products shall be offered.

Two central issues arise. First, we need a way to compute bid prices dynamically. If demand were independent from the firm's control, then an optimal choice of bid prices would be to use the marginal value of capacity $\delta(t,x,i)$ for each resource i since this ensures that each product with positive contribution $f_j - \sum_{i\in A_j}\delta(t,x,i)$ will be offered, where we approximated the opportunity cost with the sum of marginal capacity values. However, if the customers' choices do depend on the firm's control, it might be better to close some products—even though they might have a positive contribution—if this influences other purchase probabilities so as to improve the overall objective. Closing such a product could induce buy-up effects.

Second, if we are required to use a bid price control due to technical or other reasons, the formulation of the dynamic program (1) is incorrect in as far as some sets $S\subset N(x)$ might not be feasible under bid price control. Intuitively, the value function $v_t(x)$ therefore overestimates the "true" value function because we are only able to offer combinations of products that can be represented by a bid price.

In the following, we elaborate on these two research questions. Our numerical experiments are based on the example of the Choice-based Deterministic Linear Program (CDLP) with dynamic programming decomposition as described in Miranda Bront et al. (2009)—we refer the interested reader to the appendix for details concerning CDLP. We keep the discussion general since our bid price policy below can be combined with other existing methods as well. Essentially, we only require a good estimate of the marginal value of capacity of each resource and the ability to quickly evaluate choice probabilities under given bid prices.

3. Heuristic Bid Price Improvement

Let us begin with the problem of improving bid prices dynamically. We assume we used some solution approach to solve the dynamic program (1) approximately so that we have estimates of the marginal values of capacity $\delta(t,x,i)$. These we can use to approximate opportunity cost and to obtain a policy of the form (2). We seek to develop a heuristic that attempts to maximize the objective $f(b)$ over bid price vectors $b \in \mathbb{R}^m$; note that any bid price vector can be mapped into a corresponding offer set of products S_b using Definition 1. The objective function is given by

$$f(b) := \sum_{j \in S_b} \lambda P_j(S_b)\big(f_j - \sum_{i \in A_j} \delta(t,x,i)\big),$$

for given time period t and remaining inventory x.

The underlying idea of the heuristic is to start with an initial estimate for the bid price vector and then to iteratively improve it by increasing one component at a time. A good initial candidate is to use the estimate of the marginal value of capacity $\delta(t,x,i)$ for each corresponding resource. One could also start with a vector of zeros or even negative values as bid prices, but our numerical tests showed no policy improvement over using the best available marginal capacity value estimate as initial bid price; on the contrary, policy performance was worse. We ensure that $S_b \subset N(x)$ by setting the initial bid price for any resource i with $x_i = 0$ sufficiently high so that no product using this leg is offered.

The offer set S_b associated with $b(t,x,i) = \delta(t,x,i)$ is the set of all products whose contribution in terms of revenue minus estimated opportunity cost is positive, that is, $f_j - \sum_{i \in A_j} \delta(t,x,i) > 0$. Intuitively, this is probably not restrictive enough as this corresponds to the optimal bid price under independent demand assumption that ignores buy-down effects. Hence, we wonder which products should be closed in order to achieve a higher objective. Closing a product can be accomplished by increasing a bid price; if we steadily increase $b(t,x,i)$, then we must reach a threshold where one (or more) products $j \in A^i$ are closed for the first time (unless all products using this leg are closed already, in which case we do not change $b(t,x,i)$). We perform this smallest possible change of b on every resource separately, obtain accordingly m candidates, and select the one that yields the largest improvement in the objective. The heuristic is more formally given in Algorithm 1, where we suppress the dependence of the bid price vector b on t and x to improve readability:

Algorithm 1 Bid Price Heuristic under a General Choice Model for fixed t and x

for all resources i set bid price $b_i \leftarrow$ estimated marginal capacity value $\delta(t,x,i)$
repeat
    for all $i = 1:m$ **do**
        $\Delta b_i = \min_{j \in A^i \cap S_b}\{f_j - \sum_{k \in A_j} b_k\}$
        $\tilde{b}^i \leftarrow [b_1, \ldots, b_i + \Delta b_i, \ldots, b_m]$
    end for
    $\hat{b} \leftarrow \arg\max\{f(\tilde{b}^1), f(\tilde{b}^2), \ldots, f(\tilde{b}^m)\}$
    if $f(\hat{b}) > f(b)$ **then**
        $b \leftarrow \hat{b}$
    end if
until no further improvements
return bid price vector b

The heuristic can be applied to any choice-based method that provides estimates of marginal values of capacity, disregarding the choice model being used. Naturally, we have to restrict ourselves to choice models that allow quick evaluation of the purchase probabilities $P_j(S)$ since otherwise the evaluation of the objective function might be too expensive. The heuristic provides optimal bid prices under independent demand if the marginal capacity value estimates are sufficiently accurate, since, in that case, the initial bid price vector is already optimal.

4. Approximating the Value Function under Bid Price Control

In order to approximate the intractable dynamic program (1), many choice-based network approaches require to solutions to the maximization problems over the set of all possible product combinations. However, since we assume that we are confined to those sets that can be represented via bid prices, we could modify such approaches in that we only allow offer sets that have this property, represented by $S \in \mathcal{B}$, where

$$S \in \mathcal{B} \quad \Leftrightarrow \quad \exists\, b \in \mathbb{R}^m : \begin{cases} f_j > \sum_{i \in A_j} a_{ij} b_i & \forall\, j \in S, \\ f_j \le \sum_{i \in A_j} a_{ij} b_i & \forall\, j \notin S. \end{cases}$$

In words, a subset of fares $S \subset N$ can be represented by a bid price if and only if there exists a real-valued vector b such that the bid price control accepts every product in S and rejects all others.

This formulation, however, entails several problems. First, in order to implement the strict inequality one needs to approximate it with a "$\geq$" constraint: $f_j \geq \sum_{i \in A_j} a_{ij} b_i + \epsilon$ for some small $\epsilon > 0$ (along with appropriate binary variables to represent the offer set S, of course). Numerical problems result when the right-hand side is about equal to the revenue. Furthermore, we need to frequently solve problems of this kind like in the policy (2) constrained to subsets of $\mathcal{B}$. Depending on the choice model that underlies the probabilities $P_j(S)$, this problem can be very difficult. For instance, Miranda Bront et al. (2009) show that it becomes NP-hard for the multinomial logit choice model when customer segments can consider common products for purchase.

We are thus led to the conclusion that a heuristic approach is required. A modification of our Heuristic 1 to solve the subproblems arising when solving the CDLP with dynamic programming decomposition provided no satisfactory results, possibly because the errors made in each subproblem of the dynamic programming method amplified themselves. However, we were surprised to find that our heuristic bid price policy could produce quite large improvements based on marginal capacity value estimates that were computed *without* restriction to bid price control and, in a number of test instances, even reached simulated expected revenue levels similar to those achieved without bid price limitations.

Therefore, we conclude that an improvement of the bid price control alone can produce significant revenue gains even though the value function approximation does not take this limitation into account, and so we focus on improving the policy only.

5. Numerical Results

We test various policies on three groups of test problems that are frequently being used in peer publications, see Liu and van Ryzin (2008), Miranda Bront et al. (2009), Chaneton and Vulcano (2009). These policies are:

1. *BP-MCV*: We solve the CDLP and use the optimal dual solution for the dynamic programming decomposition as it was proposed by Liu and van Ryzin (2008) and Miranda Bront et al.

(2009) *without* further adjustments for bid price control. We obtain leg-level value function approximations v^i for all legs i as described in the appendix. Bid prices can be obtained by setting each bid price equal to the estimated leg-level marginal capacity value (MCV) unless there is no inventory left of resource i; in this case, we set the bid price sufficiently high so that the product cannot be offered:

$$b(t,x,i) = \begin{cases} v^i_{t+1}(x_i) - v^i_{t+1}(x_i - 1) & \text{for } x_i \geq 1, \\ \max_j f_j & \text{for } x_i = 0. \end{cases}$$

2. *BP-SG*: Stochastic gradient method with 5 re-optimizations as proposed by Chaneton and Vulcano (2009). This method is of interest particularly because it accounts for customer choice behavior, uses bid price control and is applicable to problems where the considered product sets of customer segments overlap.

3. *BP-Heu*: Our proposed method; bid prices are computed by Heuristic 1 using the dynamic marginal capacity value estimates from CDLP-based dynamic programming decomposition *without* further adjustments for bid price control.

4. *GOS*: This "General Offer Set" policy is also based on approximating the opportunity cost in the general policy (2) using CDLP-based dynamic programming decomposition without further adjustments for bid price control; however, we assume here that *any* combination of products can be offered. This is the type of policy that was frequently used in recent work and serves us as an upper bound on policy performance. We obtain a solution in terms of a combination of products by solving the following problem, using the heuristic from Miranda Bront et al. (2009):

$$\max_{S \subset N(x)} \sum_{j \in S} \lambda P_j(S) \left[f_j - \sum_{i \in A_j} \left(v^i_{t+1}(x_i) - v^i_{t+1}(x_i - 1) \right) \right]. \tag{4}$$

We test the different policies on three example networks that were used by Miranda Bront et al. (2009) and Chaneton and Vulcano (2009). All examples use the multinomial logit choice model with overlapping consideration sets which we specify in the appendix. For each test network, we compare the revenue performance of the four policies by varying leg capacities and customers' no-purchase preferences.

5.1. Parallel Flights Example

The first network example consists of three parallel flight legs as depicted in Figure 1 with initial leg capacity 30, 50 and 40, respectively. On each flight there is a low and a high fare class L and H, respectively, with fares as specified in Table 1. We define four customer segments in Table 2; note that we do not give the preference values for the no-purchase option at this point. This is

Figure 1 Parallel Flights Example.

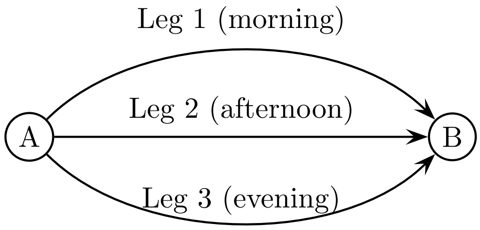

Table 1 Parallel Flights

Product	Leg	Class	Fare
1	1	L	400
2	1	H	800
3	2	L	500
4	2	H	1000
5	3	L	300
6	3	H	600

Product definitions.

Table 2 Parallel Flights

Segment	Consideration set	Pref. vector	λ_l	Description
1	{2,4,6}	[5,10,1]	0.1	Price insensitive, afternoon preference
2	{1,3,5}	[5,1,10]	0.15	Price sensitive, evening preference
3	{1,2,3,4,5,6}	[10,8,6,4,3,1]	0.2	Early preference, price sensitive
4	{1,2,3,4,5,6}	[8,10,4,6,1,3]	0.05	Price insensitive, early preference

Segment definitions.

because we consider various scenarios of this network by varying both the vector of no-purchase preferences and the network capacity. The sales horizon consists of 300 time periods.

We have run 2000 simulations with each bid price policy and report the average revenue results in Table 3. The relative error was less than 0.5% with 95% confidence. BP-Heu yields an average improvement of 3.16% over BP-MCV and 0.33% over BP-SG. In fact, BP-Heu reaches in almost all cases the benchmark of the GOS. This implies that bid price control does not necessarily deteriorate revenues in this example, as long as the bid prices are being optimized. Apparently, this network has no products that use more than one resource. Therefore, the only combinations of products that cannot be offered under bid price control would be sets where we offer class L but close class H on some flight leg. This clearly would not be optimal, so it is not surprising to see that bid price controls can reach the revenue levels of GOS. On the other hand, choosing poor bid prices (for

Table 3 Policy results for Parallel Flights Example assuming Bid Price Control

α	v_0	GOS	LF	BP-MCV	LF	BP-SG	LF	BP-Heu	LF	$\Delta\frac{\text{BP-Heu}}{\text{BP-MCV}}$	$\Delta\frac{\text{BP-Heu}}{\text{BP-SG}}$	$\Delta\frac{\text{BP-Heu}}{\text{GOS}}$
	[1,5,5,1]	38,963	0.99	39,158	1.00	39,115	1.00	38,974	0.99	(0.47)	(0.36)	0.03
0.4	[1,10,5,1]	38,958	0.99	39,157	1.00	39,167	1.00	38,973	0.99	(0.47)	(0.5)	0.04
	[5,20,10,5]	36,673	0.97	36,534	0.99	37,027	0.99	36,957	0.99	1.16	(0.19)	0.77
	[1,5,5,1]	55,547	0.99	53,957	0.99	55,964	0.99	55,965	0.99	3.72	0.00	0.75
0.6	[1,10,5,1]	55,376	0.99	53,932	0.99	55,683	0.98	55,886	0.98	3.62	0.36	0.92
	[5,20,10,5]	51,409	0.95	52,395	0.99	52,029	0.98	51,402	0.95	(1.89)	(1.21)	(0.01)
	[1,5,5,1]	69,573	0.96	69,804	0.99	69,024	0.99	69,673	0.96	(0.19)	0.94	0.14
0.8	[1,10,5,1]	69,124	0.95	69,563	0.99	69,337	0.98	69,210	0.96	(0.51)	(0.18)	0.12
	[5,20,10,5]	60,056	0.90	59,167	0.92	59,132	0.92	60,056	0.90	1.50	1.56	0.00
	[1,5,5,1]	76,979	0.95	71,268	0.97	75,073	0.96	76,746	0.96	7.69	2.23	(0.30)
1.0	[1,10,5,1]	75,695	0.90	70,549	0.94	74,064	0.92	75,605	0.92	7.17	2.08	(0.12)
	[5,20,10,5]	62,599	0.77	59,850	0.79	62,105	0.76	62,603	0.77	4.60	0.80	0.01

LF: load factor. $\Delta\frac{a}{b} \equiv 100 * a/b - 100$: percentage gap. Results for BP-SG taken from Chaneton and Vulcano (2009).

example, according to BP-MCV) has a potentially dramatic effect of up to almost 8% revenue loss as compared to BP-Heu.

5.2. Small Network Example

Next, we test the policies on a network with seven flight legs as depicted in Figure 2. In total, 22 products are defined in Table 4 and the network capacity is $c = [100, 150, 150, 150, 150, 80, 80]$, where c_i is the initial seat capacity of flight leg i. In Table 5, we summarize the segment definitions according to desired origin-destination (O-D), price sensitivity and preference for earlier flights. The booking horizon has $\tau = 1000$ time periods.

Table 6 contains the policy results of BP-MCV, BP-SG, BP-Heu and GOS with relative error less than 0.5% with 95% confidence. We observe that BP-Heu again achieves in most cases the revenue levels of GOS. That means bid prices are also in this example mostly able to capture the best product combinations if they are properly optimized. However, the importance of further bid price optimization over simply choosing the marginal capacity values as in BP-MCV becomes apparent. The policies deliver similar results when the network is highly congested ($\alpha = 0.4$) since the optimal policy becomes simple then; we usually just offer the products with highest revenue. If capacity is less tight, the gains of BP-Heu over BP-MCV become very large, particularly for the cases of non-purchase preferences $[1, 5]$ and α being 0.8 and 1. In these cases, the decision problem of the policy is particularly difficult, and thus better decisions have a large impact. The last test instance, on the contrary, has high non-purchase preferences and large network capacity so

Figure 2 Small Network example.

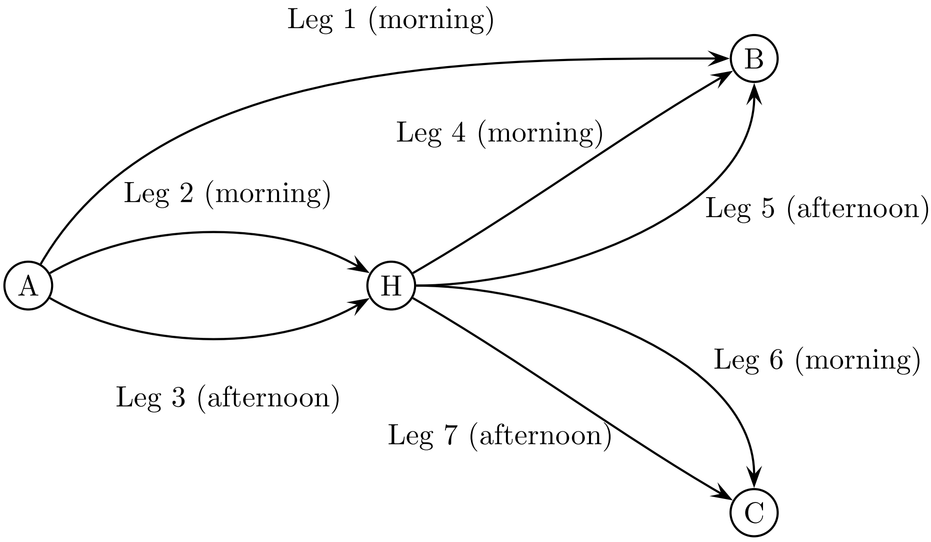

Table 4 Small Network example

Product	Legs	Class	Fare	Product	Legs	Class	Fare
1	1	H	1000	12	1	L	500
2	2	H	400	13	2	L	200
3	3	H	400	14	3	L	200
4	4	H	300	15	4	L	150
5	5	H	300	16	5	L	150
6	6	H	500	17	6	L	250
7	7	H	500	18	7	L	250
8	2,4	H	600	19	2,4	L	300
9	3,5	H	600	20	3,5	L	300
10	2,6	H	700	21	2,6	L	350
11	3,7	H	700	22	3,7	L	350

Product definitions.

that the capacity constraints are often not binding; accordingly, offering the unconstrained revenue maximizing set of fare is here usually optimal, and the policies do not differ much.

5.3. Hub & Spoke Network Example

Consider the Hub & Spoke network in Figure 3. It has eight flight legs, one hub and four spokes. Each flight i has initial capacity $c_i = 200$ and the booking horizon is divided into $\tau = 2000$ time periods. There are 80 products in total which we define in Table 7 in the following way: product 1 corresponds to the trip ATL-BOS using leg 3 in class Y, product 4 is ATL-BOS in class Q, product 5

Table 5 Small Network example

Segment	O-D	Consideration set	Pref. vector	λ_l	Description
1	A→B	{1,8,9,12,19,20}	(10,8,8,6,4,4)	0.08	less price sensitive, early pref.
2	A→B	{1,8,9,12,19,20}	(1,2,2,8,10,10)	0.2	price sensitive
3	A→H	{2,3,13,14}	(10,10,5,5)	0.05	less price sensitive
4	A→H	{2,3,13,14}	(2,2,10,10)	0.2	price sensitive
5	H→B	{4,5,15,16}	(10,10,5,5)	0.1	less price sensitive
6	H→B	{4,5,15,16}	(2,2,10,8)	0.15	price sensitive, slight early pref.
7	H→C	{6,7,17,18}	(10,8,5,5)	0.02	less price sensitive, slight early pref.
8	H→C	{6,7,17,18}	(2,2,10,8)	0.05	price sensitive
9	A→C	{10,11,21,22}	(10,8,5,5)	0.02	less price sensitive, slight early pref.
10	A→C	{10,11,21,22}	(2,2,10,10)	0.04	price sensitive

Segment definitions.

Table 6 Policy results for Small Network Example assuming Bid Price Control

α	v_0	GOS	LF	BP-MCV	LF	BP-SG	LF	BP-Heu	LF	$\Delta\frac{\text{BP-Heu}}{\text{BP-MCV}}$	$\Delta\frac{\text{BP-Heu}}{\text{BP-SG}}$	$\Delta\frac{\text{BP-Heu}}{\text{GOS}}$
	[1,5]	149,300	0.98	149,287	0.99	149,693	0.99	149,287	0.98	0.00	(0.27)	(0.00)
0.4	[5,10]	144,193	0.98	142,572	0.99	145,010	0.98	144,218	0.98	1.15	(0.55)	0.02
	[10,20]	134,370	0.96	134,001	0.98	134,345	0.96	134,450	0.97	0.34	0.08	0.06
	[1,5]	213,237	0.95	211,041	0.98	212,459	0.97	213,071	0.95	0.96	0.29	0.00
0.6	[5,10]	193,402	0.94	179,728	0.97	195,037	0.95	193,260	0.95	7.53	(0.91)	(0.07)
	[10,20]	167,909	0.94	158,007	0.95	165,638	0.94	167,712	0.94	6.14	1.25	(0.12)
	[1,5]	262,421	0.90	217,577	0.96	238,041	0.94	262,408	0.90	20.60	10.24	(0.01)
0.8	[5,10]	220,631	0.93	198,690	0.94	214,847	0.92	220,112	0.93	10.78	2.45	(0.24)
	[10,20]	185,943	0.88	178,364	0.90	185,150	0.88	184,118	0.90	3.23	(0.56)	(0.98)
	[1,5]	278,927	0.85	225,061	0.92	272,569	0.86	278,930	0.92	23.94	2.33	0.00
1.0	[5,10]	233,700	0.87	218,324	0.90	231,094	0.88	229,178	0.90	4.97	(0.83)	(1.93)
	[10,20]	191,421	0.79	188,778	0.82	189,349	0.82	190,254	0.82	0.78	0.48	(0.61)

LF: load factor. $\Delta\frac{a}{b} \equiv 100 * a/b - 100$: percentage gap. Results of BP-SG taken from Chaneton and Vulcano (2009). The non-purchase preference vectors are abbreviated: e.g., [1,5] stands for [1,5,1,5,1,5,1,5,1,5].

is BOS-ATL using leg 4 in class Y and so on. Definitions of the 20 customer segments for this example can be found in Table 8.

All simulation results in Table 9 have a relative error of less than 0.5% with 95% confidence. The policy results show stable improvements of BP-Heu versus both BP-MCV and BP-SG in all instances; however, BP-Heu does not achieve the performance of GOS in most cases. The fact that the Hub and Spoke Example has more multi-resource products than single-resource products might be a reason for these results, as it makes the calibration of the best bid price vector more difficult and might cause often situations where there is no bid price vector that would yield the

Figure 3 Hub & Spoke Network example.

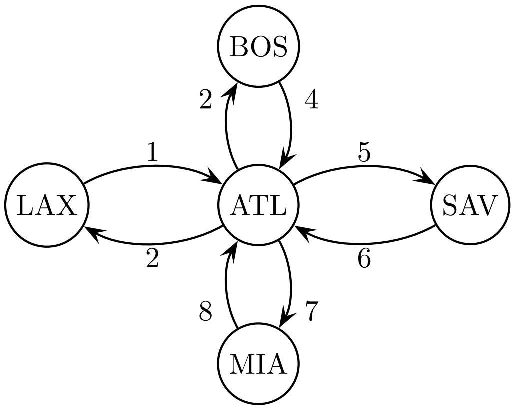

Table 7 Hub & Spoke Network Example

O-D Market	Legs	Revenue			
		Y	M	B	Q
ATLBOS/BOSATL	3/4	310	290	95	69
ATLLAX/LAXATL	2/1	455	391	142	122
ATLMIA/MIAATL	7/8	280	209	94	59
ATLSAV/SAVATL	5/6	159	140	64	49
BOSLAX/LAXBOS	4,2/1,3	575	380	159	139
BOSMIA/MIABOS	4,7/8,3	403	314	124	89
BOSSAV/SAVBOS	4,5/6,3	319	250	109	69
LAXMIA/MIALAX	1,7/8,2	477	239	139	119
LAXSAV/SAVLAX	1,5/6,2	502	450	154	134
MIASAV/SAVMIA	8,5/6,7	226	168	84	59

Product definitions.

offer set that GOS would recommend. However, it is not clear whether the under-performance of BP-Heu with respect to GOR, particularly for the difficult instances corresponding to $[1,5]$, can be attributed to faults inherent to the heuristic itself or to the fact that there exists no bid price that can represent the offer set recommended by GOR. In the last problem instance in Table 9 there is a slightly positive gap between BP-Heu and GOR. This is because we solved the dynamic policy problem (4) with the heuristic proposed in Miranda Bront et al. (2009), so that sometimes BP-Heu might yield better results. The difference is not statistically significant.

Table 8 Hub & Spoke Network Example.

Segment	C_l	v_l	λ_l	Segment	C_l	v_l	λ_l
ATL/BOS H	{1,2,3,4}	{6,7,9,10}	0.015	BOS/MIA H	{41,42,43,44}	{6,7,10,10}	0.008
ATL/BOS L	{3,4}	{8,10}	0.035	BOS/MIA L	{43,44}	{8,10}	0.03
BOS/AT H	{5,6,7,8}	{6,7,9,10}	0.015	MIA/BOS H	{45,46,47,48}	{6,7,10,10}	0.008
BOS/ATL L	{7,8}	{8,10}	0.035	MIA/BOS L	{47,48}	{8,10}	0.03
ATL/LAX H	{9,10,11,12}	{5,6,9,10}	0.01	BOS/SAV H	{49,50,51,52}	{5,6,9,10}	0.01
ATL/LAX L	{11,12}	{10,10}	0.04	BOS/SAV L	{51,52}	{8,10}	0.035
LAX/ATL H	{13,14,15,16}	{5,6,9,10}	0.01	SAV/BOS H	{53,54,55,56}	{5,6,9,10}	0.01
LAX/ATL L	{15,16}	{10,10}	0.04	SAV/BOS L	{55,56}	{8,10}	0.035
ATL/MIA H	{17,18,19,20}	{5,5,10,10}	0.012	LAX/MIA H	{57,58,59,60}	{5,6,10,10}	0.012
ATL/MIA L	{19,20}	{8,10}	0.035	LAX/MIA L	{59,60}	{9,10}	0.028
MIA/ATL H	{21,22,23,24}	{5,5,10,10}	0.012	MIA/LAX H	{61,62,63,64}	{5,6,10,10}	0.012
MIA/ATL L	{23,24}	{8,10}	0.035	MIA/LAX L	{63,64}	{9,10}	0.028
ATL/SAV H	{25,26,27,28}	{4,5,8,9}	0.01	LAX/SAV H	{65,66,67,68}	{6,7,10,10}	0.016
ATL/SAV L	{27,28}	{7,10}	0.03	LAX/SAV L	{67,68}	{10,10}	0.03
SAV/ATL H	{29,30,31,32}	{4,5,8,9}	0.01	SAV/LAX H	{69,70,71,72}	{6,7,10,10}	0.016
SAV/ATL L	{31,32}	{7,10}	0.03	SAV/LAX L	{71,72}	{10,10}	0.03
BOS/LAX H	{33,34,35,36}	{5,5,7,10}	0.01	MIA/SAV H	{73,74,75,76}	{6,7,8,10}	0.01
BOS/LAX L	{35,36}	{9,10}	0.032	MIA/SAV L	{75,76}	{9,10}	0.025
LAX/BOS H	{37,38,39,40}	{5,5,7,10}	0.01	MIA/SAV H	{77,78,79,80}	{6,7,8,10}	0.01
LAX/BOS L	{39,40}	{9,10}	0.032	MIA/SAV L	{79,80}	{9,10}	0.025

Segment definitions for Hub and Spoke Example.

Table 9 Policy results for Hub & Spoke Example assuming Bid Price Control

α	v_0	GOS	LF	BP-MCV	LF	BP-SG	LF	BP-Heu	LF	$\Delta\frac{\text{BP-Heu}}{\text{BP-MCV}}$	$\Delta\frac{\text{BP-Heu}}{\text{BP-SG}}$	$\Delta\frac{\text{BP-Heu}}{\text{GOS}}$
	[1,5]	139,453	0.97	130,613	0.98	135,452	0.95	136,180	0.98	4.26	0.54	(2.34)
0.4	[5,10]	112,730	0.97	109,532	0.98	110,287	0.93	112,322	0.97	2.55	1.85	(0.37)
	[10,20]	94,869	0.97	91,780	0.98	93,274	0.94	94,506	0.97	2.97	1.32	(0.38)
	[1,5]	160,613	0.96	147,114	0.98	152,212	0.93	155,794	0.98	5.90	2.35	(3.00)
0.6	[5,10]	130,483	0.97	123,394	0.98	126,816	0.95	128,726	0.97	4.32	1.51	(1.35)
	[10,20]	110,167	0.97	105,206	0.98	107,408	0.96	108,763	0.97	3.38	1.26	(1.27)
	[1,5]	174,469	0.96	156,306	0.98	164,418	0.95	166,640	0.98	6.61	1.35	(4.49)
0.8	[5,10]	144,039	0.97	135,892	0.98	139,038	0.96	140,972	0.97	3.74	1.39	(2.13)
	[10,20]	120,699	0.96	117,624	0.97	118,618	0.97	120,275	0.96	2.25	1.40	(0.35)
	[1,5]	183,682	0.95	167,345	0.98	170,349	0.89	176,414	0.97	5.42	3.56	(3.96)
1.0	[5,10]	153,932	0.94	147,014	0.97	150,021	0.97	152,049	0.96	3.42	1.35	(1.22)
	[10,20]	126,782	0.90	126,383	0.91	125,795	0.92	126,865	0.91	0.38	0.85	0.07

LF: load factor. $\Delta\frac{a}{b} \equiv 100 * a/b - 100$: percentage gap. Results for BP-SG taken from Chaneton and Vulcano (2009). The no-purchase preference vector are abbreviated; e.g., [1,5] represents the vector $[1,5,1,5,\dots,1,5] \in \mathbb{R}^{40}$.

6. Conclusion

We have presented a simple and fast heuristic policy that can be used to iteratively improve bid prices based on an initial estimate of marginal values of capacity that need to be supplied by any of the available solution techniques. It can in principle be used with any choice model that allows for fast evaluation of the objective function of the dynamic policy problem and yields promising results for the considered test scenarios using the choice-based linear program (CDLP) with dynamic programming decomposition with the multinomial logit choice model.

The simplicity and flexibility of the approach makes it attractive for practical application. It exploits that estimates of the marginal value of capacity are usually not sufficiently restrictive when demand is dependent on availability of alternative products, and accordingly tries to increase bid prices to avoid buy-down effects.

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Appendix

CDLP with Dynamic Programming Decomposition

Denote the *expected total revenue* from offering S by

$$R(S) = \sum_{j \in S} P_j(S) f_j,$$

and the *expected total consumption of resource i* from offering S by

$$Q_i(S) = \sum_{j \in S} P_j(S) a_{ij}, \qquad \forall i \in \{1, \ldots, m\}.$$

Then the choice-based deterministic linear program (CDLP) is given by

$$\begin{aligned} z_{\text{CDLP}} = \max_h & \sum_{S \subset N} \lambda R(S) h(S) \\ & \sum_{S \subset N} \lambda Q_i(S) h(S) \le c_i, && \forall i \in \{1, \ldots, m\}, \\ & \sum_{S \subset N} h(S) = \tau, \\ & h(S) \ge 0, && \forall S \in N. \end{aligned}$$

CDLP has $2^n - 1$ decision variables corresponding to the all possible offer sets; however, only $m+1$ constraints. Therefore, column generation can be used to solve (CDLP). We start from a pool of columns that allows a feasible solution (e.g., using $h(\emptyset) = \tau$ and $h(S) = 0$ for all other $S \subset N$) and solve that reduced master problem. The dual solution can be used to compute the reduced profit of any other column, and we solve a small maximization problem to identify the column with highest "reduced profit" that we subsequently add to the master problem. This process is repeated until no further column can be found with positive reduced profit, in which case an optimal solution has been identified. Let us denote the dual solution corresponding to the capacity and time constraints by the vector $\pi \in \mathbb{R}_+^m$ and $\sigma \in \mathbb{R}_+$, respectively. The reduced profit maximization is then

$$\lambda \max_{S \subset N} \{R(S) - \pi^T Q(S)\} - \sigma.$$

Dynamic Programming Decomposition

The optimal dual variables of the capacity constraints in the CDLP can be used to estimate the marginal value of capacity on each resource, however, they suffer from the static nature of the model, namely, that there is no dependency on the time or inventory. Liu and van Ryzin (2008) proposed to introduce time- and inventory dependence by using a techniques called dynamic programming decomposition. After solving CDLP and obtaining an optimal dual solution (π^*, σ^*), we decompose the network by the resource and approximate the value function $v_t(x) \approx v_t^i(x_i) + \sum_{k\neq i} \pi_k^* x_k$ for each resource i. When we substitute this approximation into the optimal dynamic programming formulation we obtain one-dimensional resource-level dynamic programs:

$$v_t^i(x_i) = \max_{S\subset N} \left\{ \sum_{j\in S} \lambda P_j(S) \left(f_j - \sum_{k\in A_j, k\neq i} \pi_k^* - \left(v_{t+1}^i(x_i) - v_{t+1}^i(x_i - 1)\right) \right) \right\} + v_{t+1}^i(x_i), \tag{5}$$

with boundary condition $v_{\tau+1}^i(x_i) = 0$ for all x_i and $v_t^i(0) = 0$ for all t. Having computed $v^i(\cdot)$ for all resources i, we can approximate the network value function by $v_t(x) \approx \sum_i v_t^i(x_i)$.

Multinomial Logit Choice Model

Multinomial Logit (MNL) is a random utility model that essentially generalises Binary Logit to a finite number of alternatives. It is based on the assumption that the random variables ξ_j are independent identically distributed random variables with a Gumbel distribution with zero mean and variance $(\mu\pi)^2/6$ for some scaling parameter μ. Under this assumption, the purchase probability for product j is given by

$$P_j(S) = \frac{\exp(\nu_j/\mu)}{\sum_{k\in S} \exp(\nu_k/\mu) + \exp(\nu_0/\mu)} = \frac{v_j}{\sum_{k\in S} v_k + v_0},$$

where $v_j := \exp(\nu_j/\mu)$ represents the preference of the customer for product j and $j = 0$ stands for the non-purchase option. We remark that the quantity v_0 can also be used to include the influence of competition on the decision in that it may reflect the attractiveness of competitive products.

We divide customers into L segments, where customers within a given segment $l \in \{1, \dots, L\}$ are assumed to be homogenous in that they all consider the same set of products $C_l \subset N$ for purchase—the so-called consideration set—and product preferences v_{lj} for all products $j \in C_l$ in their consideration set. The means of segmentation are left unspecified; they could be based, for example, on itinerary and departure time (early morning, midday etc). The probability that a customer in segment l purchases product $j \in S$ when we offer the fare set S is given by $P_{lj}(S) = v_{lj}/(\sum_{k\in C_l \cap S} v_{lk} + v_{l0})$ for $S \subseteq N$, where v_{l0} is the preference for not buying anything. An arriving customer belongs to segment l with probability p_l such that $\sum_l p_l = 1$, hence we can define arrival probabilities $\lambda_l := p_l \lambda$ for every segment where λ is the probability that a customer arrives in a given time period. Taken together we have $\lambda = \sum_l \lambda_l$. For a given segment l, let the vector u_l describe the product availability such that $u_{lj} = 1$ if product $j \in C_l$ is available and $u_{lj} = 0$ otherwise. Accordingly, the probability that a customer from segment l purchases product j can be rewritten in the following form:

$$P_{lj}(u_l) = \frac{u_{lj} v_{lj}}{\sum_{k\in C_l} u_{lk} v_{lk} + v_{l0}}.$$

If the consideration sets are allowed to *overlap*, then the firm cannot distinguish with certainty between different segments, and the purchase probability for product j given the offer set S and the arrival of a customer is defined by

$$P_j(S) = \sum_{l=1}^{L} p_l P_{lj}(u_l(S)).$$