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A snakes and ladders representation of stock prices and returns

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Abstract

Snakes and ladders is an ancient Indian game of chance that offers amusement as well as a metaphor for life's many ups and downs. Games offer useful and fun ways of conveying ideas as well as solution techniques and this game has considerable mathematical tractability.

This note shows how snakes and ladders can be used to represent the ups and downs of *share ownership* and solve for fair values of a *multistage project* that pays fixed dividends at uncertain completion times and has random returns.

1 The game

Traditionally a six sided die is rolled and a counter is advanced along the board, starting with square 1, toward the final goal of reaching 100. If a snake is encountered, the player must retreat to an earlier square and recover the same territory but if a ladder is reached, a short cut toward the goal is offered and the counter advanced, skipping over territory. In a competitive two player version, on reaching the final square first, the winner takes all (of whatever prize is at stake) but here different rewards are proposed.

2 Project completion

The 10×10 snakes and ladders board is taken and the 100 squares used as timeline for a *firm's progress towards a stage completion.* On reaching the last square, a fixed reward (or payoff) is achieved and the game starts again. This is not dissimilar to what companies do when managing projects to completion, when a payoff is achieved and they move onto the next project, often of similar nature.

Alternatively, the process can be thought to represent ownership of a *stock or share* over time; slow or fast company progress being represented by the snakes or ladders encountered before the arrival of the next known dividend. Although real companies offer dividends of *unknown quantity* at *fixed times*, we shall see that offering *fixed* dividends at *random times* still offers highly plausible price and return series. Good news (a ladder) is to be interpreted as the firm overcoming some difficulty faster than expected and conversely, bad news (a snake) is an outcome or resolution of some uncertainty in a worse manner than expected.

Two such boards from [1] and [2] have been examined to determine with how long such a game would last.

3 Transition matrix P

In Markov chains, a transition matrix $\mathbf{P}$ is composed of elements $p_{i,j}$ that each determine the probability of moving from state i to state j; for the snakes and ladders game in question some of the elements are shown

$$\mathbf{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & \vdots & p_{1,100} \\ p_{2,1} & p_{2,2} & \vdots & p_{2,100} \\ \vdots & \vdots & \vdots & \vdots \\ p_{100,1} & p_{100,2} & \vdots & p_{100,100} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

The example transition matrix, $\mathbf{P}$, shows the first and last three states in detail, with attendant transition probabilities of $\frac{1}{6}$ as well as one putative ladder, from state 8 to 99. In the lower right quadrant, probabilities have been included that enforce strict achievement at square 100, e.g. if any number higher than a one is rolled[1] from square 99 (with probability $\frac{5}{6}$), the counter stays where it is. This guarantees that each game circuit ends on square 100 before recommencing. These latter squares are the only entries on the leading diagonal. Thus the penultimate five squares represent a potential plateau that precedes the valuation peak at 100 where this circuit's payoff and "dividend" are achieved.

Note that a player will never terminate a round on the head of a snake or the foot of a ladder since this will involve a move to a new square, thus not all states will be attainable once a move is completed (e.g. with 10 snakes and 10 ladders, only 80 final states are accessible and 20 columns of $\mathbf{P}$ are empty).

Finally, since the stock market game here is to be *perpetual* in nature, once square 100 is achieved and the cash flow reward $F_{100} = £10$ realised, the game starts again (from 100 which can be viewed as a square 0 start point) with equal chance of ending on square 1-6 after the next roll (this is on the last row of $\mathbf{P}$). Thus the game revolves around a series of cashflows F, whose periodicity is random depending on the speed of progression across the board toward the recurrent goal.

4 Expected arrival times

Both [1] and [2] have analysed the probability of finishing one circuit as a function of exactly N throws of a die and expected times to game completion. Using different snake and ladder configurations, for one player games they come up with expected values for the die roll number of about 48 and 39, respectively. Thus, unconditionally, the expected payoff per die throw is a small fraction of the final £10.

5 Discounting, time value of money

Although here circuits are repeated and the game is of infinite length, if an *interest rate* is applied for every time period, *defined by one roll of the die*, dividend cashflows far in the future will not influence *current value* and the game will have a unique value at each stage; that is the sum in today's money terms of values from increasingly long horizons will converge to a constant that depends on the players location in the game alone.

[1] The two papers cited treat the end game differently, [1] allows for any die roll that allows passing of square 100 while [2] restricts only those that achieve 100 exactly. Here we follow the latter but adjustment to the former is not difficult.

If the *interest rate per die throw* is, say 1% ($R = 0.01$), the value of monies at the end of the next go are worth a small amount less due to *discounting.* Alternatively, if £100 units of currency were borrowed at the beginning of a player's go, £1 unit of interest would accrue and the final value owed on the account would be £101 at the end of the turn.

The objective is to work out the value of the game at each of the 100 game squares.[2] These values $V_1, V_2 ... V_{100}$ are placed in the vector $\mathbf{V}$. Other useful matrices in the solution process are the 100×100 identity matrix $\mathbf{I}$, an interest rate[3] scaled version $\mathbf{R} = R\mathbf{I}$ and their sum $\mathbf{I} + \mathbf{R}$. These will be used in conjunction with the vector $\mathbf{F}$ which contains just one element F_{100}, the dividend reward on achieving square 100. These will be used to calculate the time value of money of future cash receipts and the current value at each game stage V_i.

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{99} \\ V_{100} \end{bmatrix} \quad \mathbf{I} + \mathbf{R} = \begin{bmatrix} 1.01 & 0 & \vdots & 0 & 0 \\ 0 & 1.01 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & 1.01 & 0 \\ 0 & 0 & \vdots & 0 & 1.01 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F_{100} \end{bmatrix} \tag{2}$$

6 Valuation

If the game is to be *fair*[4], how much should a player pay to enter, V_{100}? Is it the same at each stage V_i? If not by how much does it increase with proximity to 100 and does it always increase from V_1 to V_{100}?

Were a player on square 100 to *borrow* an amount V_{100} in order to speculate on arriving on (the best of) squares 1-6, he would have to repay $(1 + R)V_{100}$ after having rolled the die and used one turn. This must be compared with the expected benefit of the initial game round, a sum which

[2]Since it repeats, the value at each square does not depend on the number of rounds completed or past dividends received.

[3]If the interest rate depended on the current state j, the entries on the diagonal of $\mathbf{R}$ could be varied. However it would be harder to adjust interest rates or probabilities over calendar time as opposed to state.

[4]By this we mean that the amount charged to enter the game at any stage V_i could, on average net of periodic loan repayments F_{100}, be financed at a rate per throw of R. The balance on such a hypothetical account, would be expected to remain constant whatever time horizon were considered.

is given by the *expectation* across the six possible die throws

$$(1+R)\,V_{100} = \frac{1}{6}\left(V_1+V_2+V_3+V_4+V_5+V_6\right) \qquad \text{and}$$
$$(1+R)\,V_1 = \frac{1}{6}\left(V_2+V_3+V_4+V_5+V_6+V_7\right) \qquad \text{etc.}$$

If the game is *fairly valued*, the expected benefit just matches the cost with borrowing, i.e. the increased repayment amount $(1+R)\,V_i$.

Without any snakes or ladders on the first six or seven squares this is an average of a sequence (above) but once diversions are encountered, the valuation of states becomes more convoluted. For example, the hypothetical ladder from square 8 to 99 shown above yields the following expectation for V_2

$$(1+R)\,V_2 = \frac{1}{6}\left(V_3+V_4+V_5+V_6+V_7+V_{99}\right).$$

More generally for each of the 100 starting squares (indexed i) the expected value after one die throw is given by

$$\text{Expectation}\left[V_j|\,V_i\right] = \sum_j p_{i,j} \times V_j.$$

In order for the game to have value at *any* stage, a cash reward must be gained in *at least one* stage of the game.[5] This comes on reaching square 100 so the valuation of any square that has a chance of reaching 100 has to be modified to include the dividend flow F_{100}, e.g.

$$(1+R)\,V_{98} = \frac{4}{6}V_{98} + \frac{1}{6}V_{99} + \frac{1}{6}\left(V_{100}+F_{100}\right) \qquad (3)$$
$$(1+R)\,V_{99} = \frac{5}{6}V_{99} + \frac{1}{6}\left(V_{100}+F_{100}\right) \text{ etc.}$$

Since squares that can achieve the reward at 100 can also yield no progress, these valuation formulae are self referential, having identical terms on both sides (e.g. V_{99}). More generally, all 100 state values depend on each other through 100 simultaneous equations, which fortunately can be solved by the demonstration and use of *matrix inversion*.[6]

[5] Cash flows other than the one F_{100}, could easily be included in $\mathbf{F}$ to reflect additional cash penalties or rewards on encountering snakes and/or ladders.

[6] A word of caution regarding risk is necessary here. Game players here are treated as *risk neutral* in the sense that they care only for expected outcomes and not, for example, their variances. Thus the probabilities are to be interpreted as risk neutral or as already adjusted for risk preferences. Were this not the case and players require risk premia to be present, probabilities could be adjusted state by state to reflect this (see [3]).

7 Solution

Stacking all 100 such equations into a vector, the most general expression is one where the time value of money for one die roll is exactly compensated by expected gains in value **PV** and cashflow **PF**

$$\begin{aligned} (\mathbf{I}+\mathbf{R})\,\mathbf{V} &= \mathbf{P}\,(\mathbf{V}+\mathbf{F}) \\ \mathbf{V} &= (\mathbf{I}+\mathbf{R}-\mathbf{P})^{-1}\,\mathbf{PF}. \end{aligned} \tag{4}$$

Thus assuming the matrix to be inverted is of *full rank*, the unique vector of values at stages 1-100 is derived from the inverted, time–value–less transition matrix,[7] applied to the one roll ahead expected cashflow vector **PF**.

In conjunction with a dividend reward of $F_{100} = £10$ and a cost of capital R per roll[8] of 1%, embedding the set of 20 snakes and ladders from [1][9] within **P**, yields the following values **V** from equation 4 for the eighty states on which a counter can come to rest (twenty are ruled out since they are at the foot of a ladder or head of a tail).

The base value V_{100} of just over £16.32 at the outset (and a borrowing cost of 1% per roll) is supported by a stream of £10 dividends coming at expected time intervals of about 48 interest rate periods. This is because over a complete cycle (expected to last 48 rolls) the compounded cost rate R_{48} generated by borrowing V_{100} to enter the game, while rolling up all interest until the dividend is reached, will be compensated by the periodic payment of the £10 dividend.

The **P** matrix here has a different and cyclical end structure to [1] which will give it different expected times even if it has the same snake and ladder features. However R_{48} is very close to the periodic 1% rate compounded by the 47.98 in that article

$$\begin{aligned} R_{48} \times V_{100} &= £10 : R_{48} = \frac{10}{16.32} = 61.27\% \\ R_{48} &\approx (1+R)^{47.98} - 1 = 61.19\%. \end{aligned}$$

[7] This has an interpretation as a *discount* matrix.

[8] As we shall see, over the many rolls taken to complete a round this rate compounds considerably!

[9] Note that this does not have the illustrative ladder from 8 to 99 used earlier.

Snakes

27	55	61	69	79	81	87	91	95	97
10	16	14	50	5	44	31	25	49	59

Ladders

6	8	13	20	33	37	41	57	66	77
23	30	47	39	70	75	62	83	89	96

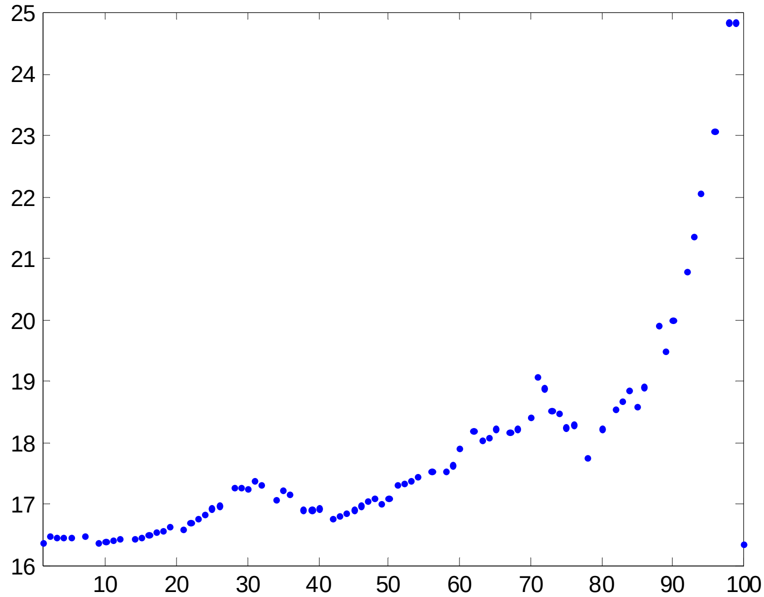

Figure 1: Valuation V_i of the 80 attainable states on the 10×10 snakes and ladders board (i.e excluding snake heads and ladder foots). Note that V_{100} does not include the payoff of 10 gained on achieving the last square.

This explains why the game values V are so low compared to F for a low borrowing cost per period R ([2] is a faster game and therefore more valuable).

Firstly, when judged in terms of value, game progress is neither monotonic in time or location, as can be seen from Figure 1 (where inaccessible states have been omitted). Due to the intricate interplay between snake and ladder paths, some subsequent states are worth less than their predecessors. If these could be avoided, a shorter expected and more valuable path to the next dividend at 100 could be achieved.

Secondly, on successfully passing from square 99 to 100, the value falls; this is not because of the adverse effect of a snake's bad news on the project or firm, but because when the dividend is realised it is separated from the remaining value (of future dividends). This is equivalent to the stock changing from *cum-dividend* to *ex-dividend* status, an event which occurs regularly in the stock market every time ownership of a share and its most imminent dividend are legally separated. The *return* at such a time needs to take this value transfer into account (add back on the £10 that was separated).

As a seasoned player of almost any age knows, the figure also shows that the stakes in this game increase with proximity to the payoff. Potential returns over the final quarter of the squares exceed those from earlier in the game, but then again so do the pitfalls!

8 Expected returns

Using the previous value for $V_{100} = £16.32$ and also $V_{99} = £24.83$, the second part of equation 3 can be confirmed as offering a 1% return if the dividend payment of £10 is also included; there is a $\frac{5}{6}$ chance of no gain and a $\frac{1}{6}$ chance of income of £10 and a capital loss (on ex-dividend) of $V_{100} - V_{99} = -£8.510$, a net gain of £1.489, or a probabilistic gain of £0.248. This just compensates for the cost of capital $R \times V_{99}$ on the initial state. This can be see by rearranging the second part of equation 3 so

$$RV_{99} = \frac{1}{6}\left(V_{100} - V_{99} + F_{100}\right).$$

Since $V_{98} = V_{99}$ (they are both on the final valuation plateau) the first part of equation 3 yields identical results. Although V_{96}, V_{94} are different from V_{99}, V_{98} their return equations also yield the correct expected return of 1% on their initial value if cash received is taken into account. All other value states (except those like V_{97}, V_{95} that are snake heads or ladder foots) also yield a 1% return but without a dividend payment and commensurate drop in capital value.

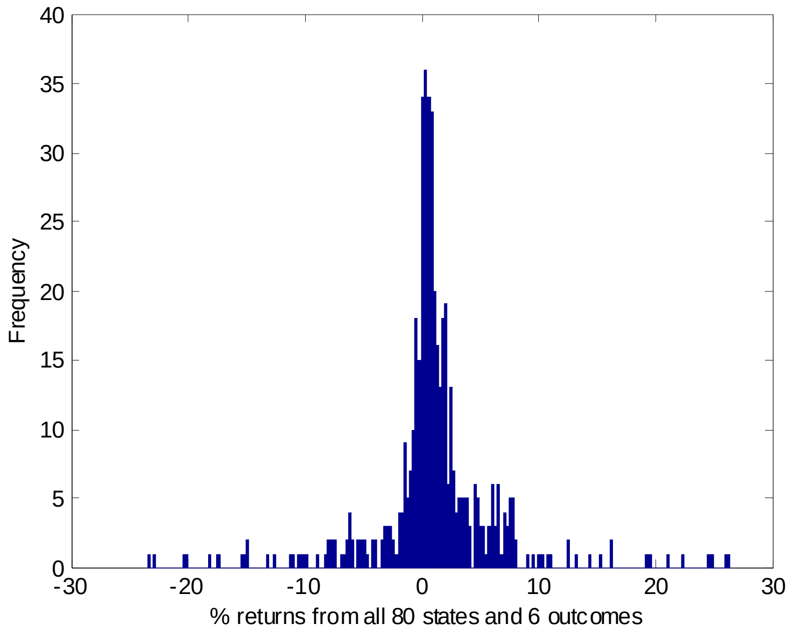

Figure 2: A histogram of return frequency for all 480 possible moves.

If the perpetual game were well under way (far from its start time), due to the end of round rules and plateau, square 99 also has the distinction of being the most likely square on which a counter would be found. Its unconditional probability is 5.77% compared to an average of 1.25% across the 80 accessible states and 2.68% for square 16, the most likely to be occupied in the non cyclical version in [1].

9 Returns from each of the die's six outcomes

Breaking **P** down into component parts that represent each individual die throw from 1-6, allows the 480 possible returns (6 options from each of the 80 states) to be evaluated. Figure 2 shows the results; the minimum return is -23.55% (throwing a 1 from 96 and landing on the 97 snake head) and maximum +26.37% (throwing a 2 from 75 and landing on the 77 ladder foot), the standard deviation is 5.5% (the mean is 1.0% by construction but the median is 0.7%).

Whilst we would expect that higher die throws would yield faster game progress and higher returns, interestingly the average returns from all possi-

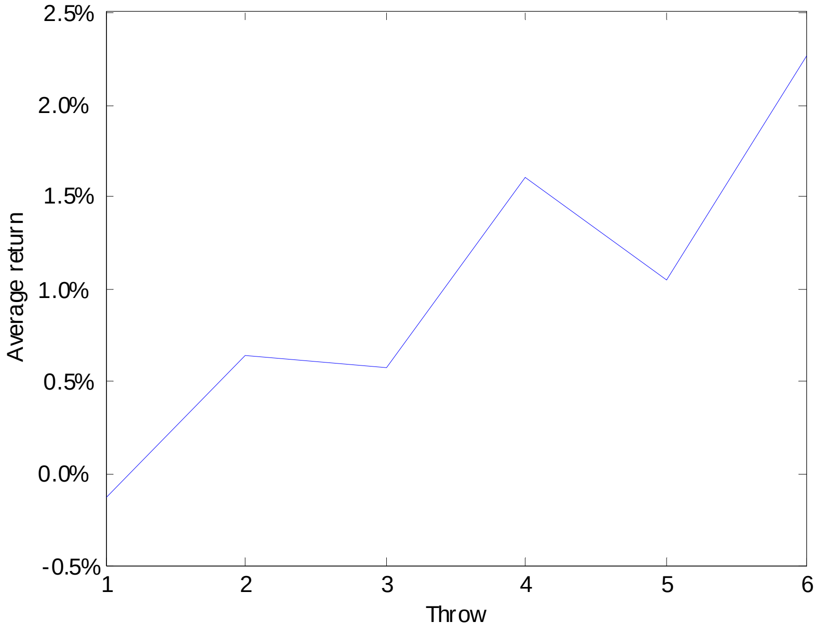

Figure 3: Average return to each throw of the die across all states (these sum to 1%). Note that 1 has a negative expected return, 2 and 3 similar gains (but less than 1%) and 5 a return very close to the cost of capital. Only 4 and in particular 6 generate significant excess returns.

ble 80 start points[10] to receiving each of 1, 2, 3, 4, 5 or 6 is *not monotonic*. Figure 3 shows that, absent information on current location, 6 indeed is highly preferred (it generates a return of 2.26%) while a 1 causes a slight loss in value (-0.13%). However, as can be also seen from Figure 3, a 4 (1.60%) is preferable to a 5 (1.05%) and a 2 just preferable to a 3 (0.64% v. 0.57%). A seemingly innocent die loaded toward even numbers could give a substantial advantage (up to 1.50% v. 0.50% per throw!).

10 Limitations and summary

Simulations of the game would generate plausible stock price series. The ups and downs of a firm's fortune would indeed be captured by the occurrence of ladders and snakes and the cum-ex dividend behaviour on project completion would also be faithfully represented.

[10]However, not weighted by their unconditional probability.

However some self criticality is due. The returns shown in Figure 2 are not independent, whilst repeated occurrence of snakes can *delay* achieving a dividend; in the long run it is likely that a similar number of ladders will have been encountered as snakes. Furthermore, if undue delay has occurred, on reaching the goal, the game is reset, whilst for a real world firm it is possible that partial or even permanent loss of value (bankruptcy) could result from a severe negative shock. At the outset (100) the game offers little downside and it would be more realistic to charge amounts of V_{25}, V_{75} (but not V_{99}) to start a game in progress when the risk profile includes some downside.

Snakes and ladders in the form represented here cannot capture all effects that are present in a real market and will present a form of weak reversion in returns. In order to capture possibly permanent features like bankruptcy or change in market shares, geometric decline (or growth) could be modelled with future dividends being a multiple or fraction of the last outcome.

Finally the transition probabilities are considered constant over time, whilst over the course of 2008 extreme market volatility across the globe led many firm's returns to be highly variable, most notably when some international banks failed and fell into bankruptcy. Although beyond the scope of the simple model here, this does alert the reader to key differences between randomness in exogenous physical and endogenous social systems.

However, the game presented here is easy to explain and would be both fun and instructive[11] to play. For those able to embrace the solution techniques, it affords insight into both the mathematics of Markovian valuation as well as the manner in which expectations are applied in the stock market to the arrival of unpredictable news and events.

11 Acknowledgements

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