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Abstract

Most decision making research in real options focuses on revenue uncertainty assuming discount rates remain constant. For many decisions, however, revenue or cost streams are relatively static and investment is driven by interest rate uncertainty, for example the decision to invest in durable machinery and equipment. Using interest rate models from Cox et al. (1985b), we generalize the work of Ingersoll and Ross (1992) in two ways. Firstly, we include real options on perpetuities (in addition to "zero coupon" cash flows). Secondly, we incorporate abandonment or disinvestment as well as investment options and thus model interest rate hysteresis [parallel to revenue uncertainty, Dixit (1989a)]. Under stochastic interest rates, economic hysteresis is found to be significant, even for small sunk costs.

Keywords: real options, interest rate uncertainty, perpetuities, investment hysteresis.
JEL Classification: G31; D92; D81; C61.

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1 Introduction

The capital theory of investment has typically ranged from models where investment is costlessly reversible to models where investment is completely irreversible. The traditional case of costlessly reversible investment occurs when there is no difference between the price at which the firm can purchase capital and the price at which it can sell capital. Thus, with perfect reversibility the wedge between the investment cost and the disinvestment proceeds is zero and the optimal investment policy of a firm maintains the marginal revenue product of capital equal to the Jorgenson (1963) user cost of capital. The case of costlessly reversible investment is not realistic since it is not expected that a firm can disinvest at no cost[1]. At the other opposite extreme, lies the case of complete irreversible investment when the sale price of capital is zero, i.e., the firm cannot recoup any fraction of the investment cost initially supported. For the sake of simplicity, the extreme assumption that resale of capital goods is impossible, i.e., disinvestment proceeds are zero, is initially introduced by Arrow (1968) and subsequently used by the majority of the literature on optimal investment under uncertainty. This assumption is more realistic since in many economic situations the sale of capital invested cannot be accomplished at the same price. In the limiting case of complete irreversibility, firms are not able to recoup any fraction

[1]Most investment expenditures are at least in part irreversible, i.e., are sunk costs that cannot be totally recouped should market conditions change adversely. Although some investments can be reversible, the majority of them are at least partly irreversible because firms cannot recover all the investment costs. On the contrary, in some cases additional costs of detaching and moving machinery may exist. Since most of the capital expenditures are firm or industry specific they cannot be used in a different firm or different industry. Therefore, they should be considered as largely a sunk cost. But even if the capital expenditures would not be firm or industry specific, they could not be totally recovered due to the "lemons" problem of Akerlof (1970). Hence, major investment costs are in a large part irreversible. As a result, the full cost of investment must be the sum of two terms: the cost of investment itself (a direct cost of investment) and the opportunity cost value of the lost option (an indirect cost of investment). An extensive literature has shown how this opportunity cost of the lost option can be evaluated and demonstrated that its value is extremely sensitive to uncertainty and can have a large impact on investment spending. See Pindyck (1991) and Dixit (1992) for an overview of the literature. Dixit and Pindyck (1994) provides an excellent revision of the various approaches and applications. A complementary survey may be found in Caballero (1999).

of the investment cost[2]. However, the most common and realistic case is characterized by investments with costly reversibility in which a firm can purchase capital at a given price (by paying an investment cost $\overline{I}$) and sell capital at a lower price (receive the disinvestment proceeds $\underline{I}$), i.e., there is a fraction α of the invested capital, $\alpha = \underline{I}/\overline{I}$ (with $0 < \alpha < 1$), that a firm can recoup when disinvesting[3]. Examples of an analysis for reversible investment decisions include, among others, Dixit (1989a), Abel et al. (1996), Abel and Eberly (1996) and Kandel and Pearson (2002) in which capital can be abandoned at a cost since only a fraction of the entry cost can be recovered on exit[4].

Decisions made under an uncertain environment where it is costly to reverse economic actions will lead to an intermediate range of the state variable, called hysteretic band, where inaction is the optimal policy. Several models of entry and exit decisions have shown that the range of inaction can be remarkably large [see, for example, Brennan and Schwartz (1985), Dixit (1989a,b) and Abel and Eberly (1996)]. The economic hysteresis effect is also found to be wide in the optimal consumption and portfolio choice literature [see, for example, Constantinides (1986)]. Therefore, such effect seems to be extremely relevant for many economic applications. Since interest rates are also an important determinant of investment and disinvestment decisions it is important to analyze the economic hysteresis effect provoked by interest rate uncertainty. To our knowledge, this effect has not been previously analyzed under stochastic interest rates.

Most decision making research in real options focuses on revenue uncertainty assuming discount rates remain constant. For many decisions, however, revenue or cost streams are relatively static and investment is driven by interest rate uncertainty, for example the decision to invest in durable machinery and equipment. Using interest rate models from

[2]Moreover, there may even exist cases where additional costs of closing a project may exist, such as the cases of a copper mine or a nuclear power station where environmental clean costs may have to be supported.

[3]$\alpha = 0$ represents the case where investment is completely irreversible, while $\alpha = 1$ stands for the case of costlessly reversible investment. For the cases where it is necessary to a pay a lump-sum cost to exit the disinvestment proceeds $\underline{I}$ is of negative sign.

[4]It should be noted that in Dixit (1989a) model firms can decide to suspend operations but have to pay a lump-sum exit cost l to do so. However, the case in which a part of the entry cost, k, can be recovered on exit can easily extended to the costly reversible investment case by changing the sign of l.

Cox et al. (1985b), we generalize the work of Ingersoll and Ross (1992) in two ways. Firstly, we include real options on perpetuities (in addition to "zero coupon" cash flows). Secondly, we incorporate abandonment or disinvestment as well as investment options and thus model interest rate hysteresis [parallel to revenue uncertainty, Dixit (1989a)].

Our paper is also related to a body of literature that examines the investment decision problem under stochastic interest rates. Using the insights of the influent work of Ingersoll and Ross (1992), Ross (1995) derives an approximation rule for the optimal hurdle rate at which a project should be undertaken. Lee (1997) proposes a method for computing the value of an investment-timing option on a postponable project with a finite maturity that has multiple cash flows. In addition to interest rate volatility, the effects of mean reversion are also included. His results confirms the evidence of Ingersoll and Ross (1992) and Ross (1995) that interest rate uncertainty has a significant impact on NPV, because sizable gains in NPV are obtained by waiting to invest. More recently, Alvarez and Koskela (2005) generalize the findings of Ingersoll and Ross (1992) by allowing a stochastic interest rate of a mean-reverting type. More specifically, they study the impact of interest rate uncertainty on irreversible investment decisions using the mean-reverting model of Merton (1975) as the underlying stochastic interest rate dynamics. Allowing for interest rate volatility increases both the required exercise premium of the investment opportunity and the value of waiting and, as a consequence, it decelerates investment. Thus, the sign of the relationship between interest rate volatility and investment is unambiguously negative, which is in concordance with previous results. They also extend their analysis by exploring the interaction between the stochastic term structure and stochastic revenue dynamics and conclude that increased revenue volatility strengthen the negative effect of interest rate uncertainty on irreversible investment decisions and vice versa. However, it should be noted that none of the cited papers consider abandonment options and thus the hysteresis modelling problem under stochastic interest rates is not previously addressed.

Our results allow us to conclude that when there is some level of interest rate uncertainty, the hysteresis level emerges very quickly even for very small investment costs. This means that apart from the output price uncertainty [see, for example, Dixit (1989a)], interest rate uncertainty also plays a critical role for widening the hysteretic band. When interest rates fall, firms make durable investments, that is to say that they switch from cash (an immediate asset) to longer lived assets with cash flows further ahead in time. When interest rates rise, they will stop undertaking any durable projects. Furthermore, if flexibility exists they will also try and reverse the investment process, i.e. disinvest away from projects with long lived cash flows into projects with more immediate payoffs.

An outline of this paper is as follows: Section 2 describes the interest rate process in a CIR economy and the behaviour near the natural zero interest rate boundary and details the necessary risk adjustment for risk-neutral valuation in a CIR framework. Section 3 presents the solutions for the perpetuity function in a CIR economy. Section 4 discusses the investment hysteresis problem under stochastic interest rates and solves it numerically using the single-factor pure diffusion process of Ingersoll and Ross (1992). Section 5 concludes.

2 CIR's Term Structure Interest Rate Dynamics

The well-known valuation framework of asset pricing in a continuous-time competitive economy developed by Cox et al. (1985a) has been the basis for many equilibrium models of contingent claims valuation. For example, the general equilibrium approach to term structure modelling developed by Cox et al. (1985b) is an application of their more general equilibrium framework. In their single-factor model of the term structure of interest rates they assume that the interest rate dynamics can be expressed as a diffusion process known as the mean-reverting square-root process[5]:

[5]It should be noted that the CIR model is a single-factor model and may be criticized on these grounds. The criticism arises because in a single-factor Markovian model it is implicit that price changes in bonds of different maturities are perfectly correlated and the long-term interest rate is constant. It also implies that bond prices do not depend on the path followed by the spot interest rate in reaching its current level. The Vasicek (1977) model is also a very popular one-factor model for the term structure dynamics of interest rates. However, the criticism that is applied to Vasicek's arbitrage model does not apply to the CIR intertemporal general equilibrium term structure model, because the latter does not allow negative interest rates which is a desirable and more realistic feature for the term structure dynamics of interest rates [see Rogers (1995)]. Therefore, we will use the CIR framework for the valuation of perpetuities and to study the economic hysteresis effects under stochastic interest rates. It should be noted that by passing

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \ , \qquad r(0) = r_0 \tag{1}$$

where κ is the parameter that determines the speed of adjustment (reversion rate), i.e., it measures the intensity with which the interest rate is drawn back towards its long-run mean, θ is the long-run mean of the instantaneous interest rate (asymptotic interest rate), σ is the volatility of the process, r_t is the instantaneous interest rate and dW_t is a standard Gauss-Wiener process. Moreover, it is usually assumed that κ, θ and σ are strictly positive constants[6]. The drift term of the process, $\kappa(\theta - r_t)$, is a restoring force

to multi-factor models one should get an improved fit to observed prices, but there is a heavy price to pay since the resulting partial differential equation would have a higher dimension. If our objective were to calculate prices of some interest rate derivatives then other factors could be included in the analysis in order to match observed prices. However, since our focus is on the effects of interest rate uncertainty on investment decisions a single-factor model of interest rates seems suitable due to its simplicity and tractability. By choosing only one state variable (i.e., the interest rate r) we are making an effort to achieve a reasonable compromise between the richness and understandability of a useful framework for capital budgeting decisions. Adding more than one state variable yields more flexibility but at a cost of much greater complication in analysis and possibly without any commensurate improvement in insights. However, it would be possible to expand the problem to include other factors without changing the essential nature of the analysis [see, for example, the multi-factor model specification of Chen and Scott (1993) for the term structure of interest rates].

[6]One of the key issues of the square-root diffusion is the role played by the term $\kappa\theta$, which is closely related with the dimension δ of a squared Bessel process ($\delta = 4\kappa\theta/\sigma^2$), and have important implications for the boundary conditions of the problem [see, for example, Feller (1951); for a complete description of the boundary classification for one-dimensional diffusions see Karlin and Taylor (1981, chap. 15) and Borodin and Salminen (2002, chap. II)]. The values of the function both at $r = 0$ and $r = +\infty$ are of particular interest when we are dealing with interest rate problems. From these two points, only the first one deserves particular attention since no key phenomenon occurs at infinity, because the infinite point is a non-attracting natural boundary for all specifications of $\kappa\theta$. But, at $r = 0$ the specification of the $\kappa\theta$ term completely changes the behaviour of the problem. Three important properties are of particular interest: (i) if $2\kappa\theta \geq \sigma^2$, $r = 0$ is an entrance, but not exit, boundary point for the process. This means that 0 acts both as absorbing and reflecting barrier such that no homogeneous boundary conditions can be imposed there. Thus, the origin is inaccessible and the CIR process stays strictly positive; (ii) if $0 < 2\kappa\theta < \sigma^2$, $r = 0$ is a reflecting boundary (exit and entrance), i.e., 0 is chosen to be an instantaneously reflecting regular boundary; (iii) if $\kappa\theta = 0$, $r = 0$ is a trap or an absorbing point and no boundary condition can be imposed there. Thus, when the CIR diffusion process hits 0 it is extinct, i.e.,

which always pull the stochastic interest rate toward a long-term value of θ. The diffusion term of the process, $\sigma^2 r_t$, represents the variance of instantaneous changes in the interest rates.

Under this framework, the fundamental partial differential equation to price a default-free discount bond, P, promising to pay one unit of capital at time T, is equal to:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 P(r)}{\partial r^2} + \kappa(\theta - r)\frac{\partial P(r)}{\partial r} + \frac{\partial P(r)}{\partial t} - \lambda r \frac{\partial P(r)}{\partial r} - rP(r) = 0 \tag{2}$$

with the boundary condition $P(r, T, T) = 1$. Since the first three terms of equation (2), which come from Ito's formula, represent the expected price change for the bond, the expected return on the bond is $r + (\lambda r \frac{\partial P(r)}{\partial r} \times \frac{1}{P})$. The factor λr represents the covariance of changes in the interest rate with percentage changes in optimally invested wealth and λ is the "market" risk parameter or price of interest rate risk[7]. Due to the fact that $\frac{\partial P(r)}{\partial r} < 0$, positive premiums will exist if $\lambda < 0$, i.e., if the covariance is negative. The discount bond price is then equal to:

$$P(r, t_0, T) = A(t_0, T)\; e^{-B(t_0,T)\; r(t_0)} \tag{3}$$

where

$$A(t_0, T) = \left[\frac{2\omega e^{[(\kappa+\lambda+\omega)(T-t_0)]/2}}{(\omega + \kappa + \lambda)(e^{\omega(T-t_0)} - 1) + 2\omega}\right]^{2\kappa\theta/\sigma^2} \tag{4a}$$

$$B(t_0, T) = \frac{2(e^{\omega(T-t_0)} - 1)}{(\omega + \kappa + \lambda)(e^{\omega(T-t_0)} - 1) + 2\omega} \tag{4b}$$

$$\omega = \left[(\kappa + \lambda)^2 + 2\sigma^2\right]^{1/2} \tag{4c}$$

Although this solution was first introduced in finance by Cox et al. (1985b), the formula was already obtained by Pitman and Yor (1982) but in a different context [see, for example, Delbaen (1993) and Geman and Yor (1993)]. Thus, the price at time $t = t_0$ of a zero coupon bond maturing at time T is also equal to:

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it remains at 0 forever (absorbing or exit boundary).

[7]More specifically, the parameter λ is related to the market price of risk $\lambda^*(r, t) = -\lambda\sqrt{r(t)}/\sigma$. It turns out that in equilibrium the market price of risk is restricted to be of this particular functional form.

$$P(r, t_0, T) = \mathbb{E}_{t_0}^{\mathcal{Q}}\left[e^{-\int_{t_0}^{T} r(s)\, ds}\right] = A(t_0, T)\, e^{-B(t_0,T)\, r(t_0)} \tag{5}$$

where $\mathbb{E}_{t_0}^{\mathcal{Q}}$ denotes the expectation under the risk-neutral probability $\mathcal{Q}$ (or martingale measure $\mathcal{Q}$), at time $t = t_0$, with respect to the risk-adjusted process for the instantaneous interest rate that can be written as the following stochastic differential equation:

$$dr_t = [\kappa\theta - (\lambda + \kappa)\, r_t]\, dt + \sigma\sqrt{r_t}\, dW_t \tag{6}$$

and where dW_t is a standard Brownian motion under $\mathcal{Q}$. It should be noted that option pricing analysis usually resort in the so-called risk-neutral valuation which is essentially based in replication and continuous trading arguments[8]. However, the interest rate r is not the price of a traded asset, since there is no asset on the market whose price process is given by r. This means that the present framework is somewhat more complicated than a Black-Scholes setting due to the appearance of the market price of risk λ, which is not determined separately within the model but rather obtained as part of the equilibrium.

We see that the value at time $t = t_0$ of a zero coupon bond with maturity date T is given as the expected value of the final payoff of one dollar discounted to t_0. This expected value is stated by equation (5), but in this case the expectation is not to be taken using the objective probability measure $\mathcal{P}$. Instead, a martingale measure $\mathcal{Q}$ must be used to denote that the expectation is taken with respect to a risk-adjusted process, where the risk adjustment is determined by reducing the drift of the underlying variable by a factor risk premium λr. Therefore, the risk-adjusted drift of the interest rate square-root process is denoted by the term $[\kappa\theta - (\lambda + \kappa)\, r_t]$. It should also be emphasized that although risk premiums for interest rates may be introduced, they cannot be observed or measured separately. We know that the CIR model has four parameters in addition to the current interest rate r_t: the parameters associated with the objective probability measure $\mathcal{P}$ (e.g., κ, θ and σ) and the risk premium of the single-factor which drives the economy under the

[8]For example, it is possible to compute the Black and Scholes (1973) arbitrage free prices using such arguments and a risk-neutral valuation approach, because there is a risk-neutral probability measure $\mathcal{Q}$ equivalent to the real world probability measure $\mathcal{P}$ [see, for example, Cox and Ross (1976), Harrison and Kreps (1979) and Harrison and Pliska (1981)].

risk-neutral world (e.g., λ). It turns out that contingent claim prices depend only on the parameters of the risk-adjusted process. These are the current interest rate r_t, the interest rate volatility σ, and two parameters that are combinations of the remaining ones, i.e., $\kappa\theta$ and $\lambda + \kappa$. That is why the market price of risk is not determined separately within the model, but rather obtained as part of the equilibrium. Obviously it is possible to exogenously specify or impose a given λ, especially if we want to perform some numerical analysis. If this is the case, however, there may be no underlying equilibrium that could support the imposed premiums, but, on the other hand, there is no risk-neutral measure either in such a situation [see Rogers (1995, section 6) for an excellent exposition about this issue]. If a risk factor term is to be introduced it is determined by things such as the forms of risk aversion possessed by the various agents on the market. This means that if one makes an *ad-hoc* choice of $\lambda = 0$, then he is implicitly making an assumption concerning the aggregate risk aversion on the market[9].

3 Valuation of Perpetuities under Stochastic Interest Rates within the CIR Framework

Following Cox et al. (1985a,b), the price of any interest rate contingent claims satisfies the following partial differential equation:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 F(r)}{\partial r^2} + \kappa(\theta - r)\frac{\partial F(r)}{\partial r} + \frac{\partial F(r)}{\partial t} - \lambda r \frac{\partial F(r)}{\partial r} - rF(r) + C(r,t) = 0 \tag{7}$$

This equation is similar to equation (2). The only difference is the new term $C(r,t)$ which represents the net cash paid out to the claim[10]. For the valuation of a default-free discount bond $C(r,t) = 0$, but for a perpetuity its value is 1 since a perpetuity is a default-free financial instrument that pays a constant stream of one unit of capital[11]. In addition, for

[9] For a detailed technical exposition regarding these issues see, for example, Björk (2004).

[10] We also change the function notation to distinguish the value of a default-free discount bond, $P(r)$, from the value of a perpetuity, $F(r)$.

[11] Another common name for a perpetuity is consol.

a perpetuity the term $\frac{\partial F(r)}{\partial t}$ will vanish as t goes to infinity. Thus, equation (7) can be restated as:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 F(r)}{\partial r^2} + \kappa(\theta - r)\frac{\partial F(r)}{\partial r} - \lambda r \frac{\partial F(r)}{\partial r} - rF(r) + 1 = 0 \tag{8}$$

It is well known that the solution to this ordinary differential equation is the value of a perpetuity in a CIR framework, that can be computed as follows[12]:

$$F(r) = \mathbb{E}^{\mathcal{Q}}_{t_0}\left[\int_{t_0}^{\infty} e^{-\int_{t_0}^{t} r(s)\, ds}\, dt\right] = \int_{t_0}^{\infty} P(r, t_0, t)\, dt \tag{9}$$

As we will see later, we need to use the first derivative of the perpetuity function. Differentiation under the integral sign is allowed, even when a limit is infinite, and this gives us:

$$F'(r) = \frac{d}{dr}\int_{t_0}^{\infty} P(r, t_0, t)\, dt = \int_{t_0}^{\infty} \frac{\partial P(r, t_0, t)}{\partial r}\, dt = -\int_{t_0}^{\infty} A(t_0, t)B(t_0, t)\, e^{-B(t_0,t)\, r(t_0)}\, dt \tag{10}$$

4 The Hysteresis Problem Assuming There Is No Mean Reversion

4.1 Perpetual Investment and Disinvestment Opportunities

To concentrate on the effects of interest rates on investment decisions we use a particular model of real interest rates. To do so, we follow the single-factor pure diffusion process of Ingersoll and Ross (1992) assuming that changes in the instantaneous interest rate, r, satisfy the following Itô equation:

$$dr_t = \sigma\sqrt{r_t}\, dW_t \tag{11}$$

[12]The valuation of perpetuities using the methodology of Bessel processes under stochastic interest rates within the CIR's framework can be found in Delbaen (1993), Geman and Yor (1993) and Yor (1993). However, their analytical solutions cannot be used in our framework since we will consider that the $\kappa\theta$ term is zero and thus we have to rely on numerical methods.

where σ is constant. This is equivalent to the interest rate dynamics of the risk-adjusted stochastic process $dr_t = -\lambda r_t\, dt + \sigma\sqrt{r_t}\, dW_t$ for risk-neutral pricing in the case of a nonzero term premium λ where it is assumed that λ is constant and $\lambda < 0$ corresponds to positive risk premiums. The process followed here restricts the more general mean-reverting drift process of Cox et al. (1985b). Since we want to focus on the effects of interest rate uncertainty on the investment and disinvestment decisions, the Ingersoll and Ross (1992) process with a zero expected interest rate change allows the simplification of our analysis.

According to Cox et al. (1985a,b), the price of any interest-rate contingent claims satisfies the following partial differential equation (for the case where mean reversion is not considered):

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 F(r)}{\partial r^2} - \lambda r\frac{\partial F(r)}{\partial r} + \frac{\partial F(r)}{\partial t} - rF(r) + C(r,t) = 0 \tag{12}$$

where C is the net cash paid out to the claim and λ measures the price of interest-rate risk. Following the ideas that underlies most of the real options' framework, we assume a very long time to maturity options. This technique was firstly raised by Merton (1973) to obtain closed-form solutions for the perpetual calls and puts options. Using this technique the problem stated in equation (12) becomes time independent since the term $\frac{\partial F(r)}{\partial t}$ will vanish as t becomes very long $\left(\frac{\partial F(r)}{\partial t} \to 0\right)$. In addition, for a very long time to maturity the net cash paid out to the claim will be 1 dollar. For this perpetual case, equation (12) reduces to an ordinary differential equation of the form:

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 F(r)}{\partial r^2} - \lambda r\frac{\partial F(r)}{\partial r} - rF(r) + 1 = 0 \tag{13}$$

Looking at equation (13) it is easy to see that it does not have constant coefficients since they are dependent on r. But with a single change we can turn the problem easier. Thus, dividing both sides of the equation by r and rearranging we get:

$$\frac{1}{2}\sigma^2\frac{\partial^2 F(r)}{\partial r^2} - \lambda\frac{\partial F(r)}{\partial r} - F(r) = -\frac{1}{r} \tag{14}$$

Now, equation (14) is a linear nonhomogeneous constant coefficient equation. The general solution to this equation is the sum of the complementary solution, $y(r)$, and the particular solution, $Y(r)$. A possible and natural lower barrier for an interest rate process would be

$r = 0$, but for this single-factor pure diffusion process such control is not possible because the term $\kappa\theta$ is equal to zero. As a result, we have to determine the barriers, as well as the constants of the complementary solution, numerically since no closed-form solution is available.

Since we want to consider models of investment and disinvestment we will add a new state variable to the decision problem, a discrete variable that will indicate if the firm is active (1) or idle (0). It turns out that the value of an idle or not active firm, $F_0(r)$, is obtained by the solution of the complementary function of equation (14):

$$\frac{1}{2}\sigma^2\frac{\partial^2 F_0(r)}{\partial r^2} - \lambda\frac{\partial F_0(r)}{\partial r} - F_0(r) = 0 \tag{15}$$

and the value of an active firm, $F_1(r)$, is the solution of the entire equation (14):

$$\frac{1}{2}\sigma^2\frac{\partial^2 F_1(r)}{\partial r^2} - \lambda\frac{\partial F_1(r)}{\partial r} - F_1(r) = -\frac{1}{r} \tag{16}$$

Let us now proceed with the solution of the complementary functions together, since they are similar linear homogeneous equations with constant coefficients. Trying a solution of the form $F(r) = e^{mr}$, we find that $F'(r) = me^{mr}$ and $F''(r) = m^2e^{mr}$. Substitution yields:

$$\left(\frac{1}{2}\sigma^2m^2 - \lambda m - 1\right)e^{mr} = 0 \tag{17}$$

Hence $F(r) = e^{mr}$ is a solution of Equation (14) when m is a root of

$$\frac{1}{2}\sigma^2m^2 - \lambda m - 1 = 0 \tag{18}$$

or

$$\phi(m) = m^2 - vm - w = 0 \tag{19}$$

where we define $v = 2\lambda/\sigma^2$ and $w = 2/\sigma^2$. The convergence condition of equation (19) is $w > 1 - v$. Then, it turns out that $\phi(0) = -w < 0$ and $\phi(1) = 1 - v - w < 0$. Since $\phi''(m) = 2 > 0$ it means that the auxiliary equation has two roots, where one of them must be greater than one (we will call it a) and the other one must be less than zero (we will call it b). The discriminant of the characteristic equation is positive, $\Delta = v^2 + 4w > 0$, which means that the respective solutions are real. Therefore, the two roots can be written out as:

$$a = \frac{+v + \sqrt{v^2 + 4w}}{2} > 1 \tag{20a}$$

$$b = \frac{+v - \sqrt{v^2 + 4w}}{2} < 0 \tag{20b}$$

Thus, we can write the general solution of equation (15) as:

$$F_0(r) = C_1 e^{ar} + C_2 e^{br} \tag{21}$$

and the general solution of equation (16) as:

$$F_1(r) = C_3 e^{ar} + C_4 e^{br} + Y(r) \tag{22}$$

where C_1, C_2, C_3 and C_4 are constants to be determined from boundary conditions.

A simple economic intuition tells us that for very high interest rate levels idle firms are not induced to invest. Therefore, the option of activating the firm should be nearly worthless for this level rates. As a result, we need that the constant $C_1 = 0$ (associated with the positive root a). This means that the expected net present value of making an investment in the idle state is:

$$F_0(r) = C_2 e^{br} \tag{23}$$

Since an idle firm is not operating does not have any return from the project yet. Therefore, equation (23) is just the option value of a perpetual investment opportunity, $IO(r)$. Over the range interval of interest rates $(\underline{r}, \infty)$, an idle firm will not exercise its option to invest. To simplify our analysis, we will consider that once the investment commitment has been made, the investment project return is identical to a perpetuity making a continuous payment of one unit over time. Thus, no additional resources or expenditures apart from the initial investment are required to maintain the rights over the project or to sustain the project after it has been accepted. A similar assumption is also used by Ingersoll and Ross (1992), but in their case the project returns are identical to a T-period zero-coupon bond with a real face value of one dollar since they are considering finite maturities, whereas we are considering infinite maturities. This assumption implies that operating profits never become negative in our project. Such assumption is also used by, among others, McDonald and Siegel (1986), Pindyck (1988) and Bertola (1998).

The value of an active firm is the sum of two components, the expected present value of the profits and an option value of terminating the project. We know that for very low interest rates an active firm will be induced to continue its operations and not disinvest. Since the value of the abandonment option should go to zero as r becomes very low, we must set $C_4 = 0$ (associated with the negative root b). Therefore, the value of a firm for the active state is:

$$F_1(r) = C_3 e^{ar} + F(r) \tag{24}$$

where $F(r)$ is the particular solution $Y(r)$ of the ordinary differential equation (16). It follows that a particular solution to this equation is, as it was already stated before, the value of a perpetuity making a continuous payment of one unit over time, i.e, $\int_0^\infty P(r, 0, t)\, dt$, where we are setting $t_0 = 0$. It should be noted that we use equation (9) as $F(r)$ but we have to impose a fixed number T in the upper limit of the integral because the $\kappa\theta$ parameter is zero. Since the perpetuity value represents the expected present value that can be obtained from the project if it is maintained active forever, the remaining part of equation (24) must be the value of a perpetual option to disinvest optimally, i.e., $DO(r) = C_3 e^{ar}$. Over the interest rate range $(0, \overline{r})$ an active firm will continue its operations, holding its option to abandon alive.

4.2 Option to Invest

Let us suppose that an idle firm has an option to invest in a particular investment project where interest rate uncertainty is a key factor for the decision to invest, but where, for now, the disinvestment opportunity is not considered. Thus, the firm has to decide whether to continue being idle or to enter in the market. If interest rates drop to a low level, the firm may be induced to change an option to invest paying an investment cost $\overline{I}$ by a perpetuity making a continuous payment of one unit over time. This investment cost is considered a sunk capital cost since it cannot be recouped if the firm should decide to quit at a later date. The investment strategy can be stated as follows:

$$\overline{I} + IO(r) \rightarrow F(r)$$

The optimal policy to invest is determined using one value matching condition and one smooth pasting condition (also called high contact condition)[13]. This yields a system of two non-linear equations in two variables (C_2 and $\underline{r}$):

$$\overline{I} + C_2 e^{b\underline{r}} = \int_0^{\infty} A(0,t) e^{-B(0,t)\underline{r}}\, dt \tag{25a}$$

$$bC_2 e^{b\underline{r}} = \int_0^{\infty} -B(0,t)A(0,t) e^{-B(0,t)\underline{r}}\, dt \tag{25b}$$

4.3 Option to Disinvest

Let us now suppose that an active firm is operating and its payoff is a perpetuity making a continuous payment of one unit over time. But if interest rates start rising to very high rates the firm may be induced to temporarily shut down or even abandon the project. If a project is closed temporarily it turns out that the firm will incur some fixed maintenance costs, but may be opened up again without having to pay again entry costs, i.e., $\overline{I}$. If the project is to be permanently abandoned it will incur no maintenance costs, but if the firm wants to enter again in the market has to pay a new lump-sum cost $\overline{I}$. This possibility (i.e., a reentry option) will be ignore for now. In our case we will assume that once the state variable reaches the upper trigger point it is optimal to abandon the project, and such abandonment policy will not involve any costs. The disinvestment strategy can be stated as follows:

$$\underline{I} \leftarrow F(r) + DO(r)$$

[13]Our real options problems are of American-type nature since they are time-independent and, therefore, can be exercised at any time before maturity. Thus, they are optimal stopping problems. The optimality conditions for such problems were introduced in the financial economics literature by Samuelson (1965), McKean (1965) and Merton (1973). For a general treatment of such conditions in a simpler setting see, for example, Dixit (1991b, 1993) and Dumas (1991).

where $\underline{I}$ takes a positive value since when the firm close its operations will not incur any cost to disinvest. Obviously, there may be situations where firms have to incur an extra cost when they want to close, such as the cases of a copper mine or a nuclear power station where some environmental clean costs have to be supported. In our case, we want to focus our analysis on the possibility that some fraction of the lump-sum cost $\overline{I}$ can be recouped if firms decide to abandon its operations. Therefore, we will define a new variable α that will measure the degree of reversibility, i.e., $\alpha = \underline{I}/\overline{I}$. $\alpha = 0$ corresponds to an option in which the decision taken is irreversible and can be exercised only once. The case $0 < \alpha < 1$ corresponds to partial reversibility. We will consider three cases: $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. The case where $\alpha = 1$ represents perfect reversibility, a situation that gives rise to a flow option in which two flows can be switched continuously and costlessly [see Shackleton and Wojakowski (2001)][14].

The optimal policy to disinvest is determined using one value matching and one smooth pasting conditions. This yields a system of two non-linear equations in two variables (C_3 and $\overline{r}$):

$$\int_0^\infty A(0,t)e^{-B(0,t)\overline{r}}\,dt + C_3 e^{a\overline{r}} = \underline{I} \tag{26a}$$

$$\int_0^\infty -B(0,t)A(0,t)e^{-B(0,t)\overline{r}}\,dt + aC_3 e^{a\overline{r}} = 0 \tag{26b}$$

4.4 Switching Options

The most interesting problem is the one where optimal investment and disinvestment decisions are considered together. Thus, entry and exit decisions are valued simultaneously originating a lower bound ($\underline{r}$) and an upper bound ($\overline{r}$) with $\underline{r} < \overline{r}$, and where an idle firm is induced to invest once the state variable r crosses the action trigger point $\underline{r}$ and an active firm will be induced to disinvest if the state variable crosses the threshold point $\overline{r}$. The middle band of interest rates without entry or exit actions yields what is usually

[14]This limiting situation corresponds to the case where the two threshold will collapse to one common switching level that will determine the optimal exercise strategy. The strategy change will occur when the so-called Jorgenson (1963) user costs of capital are equals.

called by economic hysteresis, since the optimal policy is to maintain the actual status quo, whether the firm is operating or not.

The corresponding strategy for the entry and exit case can be stated as follows:

$$\overline{I} + IO(r) \rightarrow F(r) + DO(r)$$

$$\underline{I} + IO(r) \leftarrow F(r) + DO(r)$$

In this case, the optimal policy is determined using two value matching and two smooth pasting conditions resulting in a two-sided $(\underline{r}, \overline{r})$ policy [other examples of two-sided policies include Dumas and Luciano (1991), Shackleton and Wojakowski (2001), among others]. It is important to note that the investment and disinvestment opportunities at the lower threshold are, respectively, $IO(r = \underline{r}) = C_2 e^{b\underline{r}}$ and $DO(r = \underline{r}) = C_3 e^{b\underline{r}}$. Similarly, the investment and disinvestment opportunities at the upper threshold are, respectively, $IO(r = \overline{r}) = C_2 e^{b\overline{r}}$ and $DO(r = \overline{r}) = C_3 e^{b\overline{r}}$. This yields a system of four non-linear equations in four variables (C_2, C_3, $\overline{r}$ and $\underline{r}$):

$$\overline{I} + C_2 e^{b\underline{r}} = \int_0^\infty A(0,t) e^{-B(0,t)\underline{r}} \, dt + C_3 e^{a\underline{r}} \tag{28a}$$

$$bC_2 e^{b\underline{r}} = \int_0^\infty -B(0,t) A(0,t) e^{-B(0,t)\underline{r}} \, dt + aC_3 e^{a\underline{r}} \tag{28b}$$

$$\int_0^\infty A(0,t) e^{-B(0,t)\overline{r}} \, dt + C_3 e^{a\overline{r}} = \underline{I} + C_2 e^{b\overline{r}} \tag{28c}$$

$$\int_0^\infty -B(0,t) A(0,t) e^{-B(0,t)\overline{r}} \, dt + aC_3 e^{a\overline{r}} = bC_2 e^{b\overline{r}} \tag{28d}$$

The above equations are highly non-linear and, as a result, a closed-form solution is not available. Although we have to rely on numerical methods to get the solution for the two thresholds and the two constants, such numerical solutions are quite easy to obtain using the numerical routines for solving simultaneous non-linear equations that are available in many scientific computing software, such as *Mathematica*. However, some important economic properties of the solution can also be obtained by analytical methods similar to the ones employed by Dixit (1989a). We will use such technique to get a better understanding of the hysteresis effect.

4.5 Economic Hysteresis Effect

Decisions made under an uncertain environment where it is costly to reverse economic actions will lead to an intermediate range of the state variable, called hysteretic band, where inaction is the optimal policy. Several models of entry and exit decisions have shown that the range of inaction can be remarkably large [see, for example, Brennan and Schwartz (1985), Dixit (1989a,b) and Abel and Eberly (1996)]. The economic hysteresis effect is also found to be wide in the optimal consumption and portfolio choice literature [see, for example, Constantinides (1986)]. Therefore, such effect seems to be extremely relevant for many economic applications. Since interest rates are also an important determinant of investment and disinvestment decisions it is interesting to analyze the economic hysteresis effect provoked by interest rate uncertainty. To our knowledge, this effect has not been previously analyzed under stochastic interest rates.

The economic hysteresis effect produces a range of values for the state variable that is usually defined by highly non-linear equations that need numerical solutions. In some cases, it is possible to use analytic approximations that allow the use of explicit solutions to help understanding the importance of the hysteresis effect [see, for example, Dixit (1991a)]. In our case, we have highly non-linear equations that use functions with integrals. As a result, such analytic approximations are very difficult to obtain and we do not attempt to use them. However, we can established a general property of the solution that yields economic hysteresis. A procedure like this was previously used by Dixit (1989a) to examined the nature of hysteresis when the source of uncertainty arises from the output market price, whereas in our case the uncertainty comes from the stochastic nature of the interest rate term structure. To do so, let us define the following function:

$$V(r) = F_1(r) - F_0(r) \tag{29}$$

Using the solutions stated by equations (23) and (24) we have:

$$V(r) = C_3 e^{ar} - C_2 e^{br} + F(r) \tag{30}$$

where $F(r)$ represents the perpetuity value. For small values of r the term with the negative root b dominates. The term is negative, increasing and concave. For very high interest rate levels the dominant term is the one associated with the positive root a. The term is positive, increasing and convex. For the intermediate range, it is the perpetuity value that plays a critical role.

Now, the two value matching and the two smooth pasting conditions can be defined in terms of V as:

$$V(\underline{r}) = \overline{I}\,,\;\; V'(\underline{r}) = 0\,,\;\; V(\overline{r}) = \underline{I}\,,\;\; V'(\overline{r}) = 0 \tag{31}$$

To get some analytical results it is important to note that:

$$V''(\underline{r}) < 0\,,\;\; V''(\overline{r}) > 0 \tag{32}$$

since $V(r)$ is concave at $\underline{r}$ and convex at $\overline{r}$[15]. Subtracting equation (15) from equation (16) we see that the function $V(r)$ satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V(r)}{\partial r^2} - \lambda \frac{\partial V(r)}{\partial r} - V(r) = -\frac{1}{r} \tag{33}$$

Now evaluating this differential equation at $\underline{r}$ and using the conditions (31) and (32) we get:

$$-\frac{1}{\underline{r}} = \frac{1}{2}\sigma^2 \frac{\partial^2 V(\underline{r})}{\partial \underline{r}^2} - \lambda \frac{\partial V(\underline{r})}{\partial \underline{r}} - V(\underline{r}) < -\overline{I} \tag{34}$$

Using the same approach at the $\overline{r}$ we obtain:

$$-\frac{1}{\overline{r}} = \frac{1}{2}\sigma^2 \frac{\partial^2 V(\overline{r})}{\partial \overline{r}^2} - \lambda \frac{\partial V(\overline{r})}{\partial \overline{r}} - V(\overline{r}) > -\underline{I} \tag{35}$$

Rearranging we get, respectively:

$$\underline{r} < 1/\overline{I} \equiv M_{\underline{r}} \tag{36}$$

and

[15] Using all these conditions it is easy to show that there exists an optimal policy similar in spirit to the diagrams presented by Constantinides and Richard (1978), Harrison et al. (1983) and Dixit (1989a, 1991b).

$$\overline{r} > 1/\underline{I} \equiv M_{\overline{r}} \tag{37}$$

where $M_{\underline{r}}$ and $M_{\overline{r}}$ can be viewed, respectively, as the Marshallian trigger interest rates for investment and disinvestment. Taking decisions using this traditional concept can lead to myopic actions since we are implicitly assuming a static expectation for the interest rate dynamics. Thus, the differences between our thresholds and the Marshallian ones comes from the uncertainty effect. Conditions (36) and (37) highlights that uncertainty widens the Marshallian range of inaction.

Using these theoretical insights we can consider some limiting cases. The case where both $\overline{I}$ and $\underline{I}$ tend to zero is not very interesting, since both $\underline{r}$ and $\overline{r}$ would diverge. When they tend to extremely large values, both $\underline{r}$ and $\overline{r}$ tend to the common limit 0. But these two issues indicate that sunk costs and disinvestment proceeds are essential for the hysteresis effect. If we fix α at a level of 0.50 for example, and we impose a higher investment cost both the lower and the upper thresholds diminish. Using equations (A.6) and (A.10) that we present in appendix A, it is straightforward to show that $d\underline{r} < 0$ when $d\overline{I} > 0$ and $d\overline{r} < 0$ when $d\underline{I} > 0$, respectively. But the upper threshold falls at a higher rate. Thus, there is a tendency for narrowing the hysteretic band in this case. Let us now fix the investment cost and analyze the impact of different levels of investment recoup. Maintaining the level of investment cost fixed and rising the α parameter, originates a very small rise in the lower threshold (using equation (A.11) we can show that $d\underline{r} > 0$ when $d\underline{I} > 0$) and a sharply decrease in the upper trigger point (using equation (A.10) we can show that $d\overline{r} < 0$ when $d\underline{I} > 0$). Thus, the band of inaction will be narrower.

If we set $\alpha = 0$ the upper threshold diverges to ∞. Thus, an active firm would never abandon the project. But there is still a finite interest rate level that will induce an idle firm to invest in a project that afterwards will not be abandoned. This issue derives from the fact that the profits from the project never become negative. As a result, the firm will continue its operations. But if there were some variable costs these could lead to negative profits and then there would be an upper trigger point that would induce the firm to shut down optimally. Similarly, if the investment costs to reenter again (if a firm decides to shut down its operations previously) goes to ∞, the option to reinvest becomes worthless. An idle firm never invests due to high entrance costs and an operating firm may be induced to optimally abandon its operations when the interest rate crosses the upper trigger point. This point indicates how bad things must be, i.e., how high interest rates should rise before an active firm abandon its operations, since it knows that due to high entrance costs it can never reinvest later again.

If interest rate uncertainty goes to zero, then $\underline{r} \to M_{\underline{r}}$ and $\overline{r} \to M_{\overline{r}}$. This imply that without uncertainty the range of inaction is determined by the Marshallian trigger points. As the interest rate volatility starts rising, the lower threshold will fall and the upper threshold will rise, which will lead to a wider hysteretic band. Now, maintaining the volatility fixed at a positive level and letting $\overline{I} \to 0$, we have $d\underline{r}/d\overline{I} \to -\infty$ and $d\overline{r}/d\overline{I} \to \infty$. This can be confirmed using, respectively, equations (A.6) and (A.7) presented in appendix A. This means that when there is some level of interest rate uncertainty, the hysteresis level emerges very quickly even for very small investment costs. This also means that apart from the output price uncertainty [see, for example, Dixit (1989a)], the interest rate uncertainty also plays a critical role for widening the hysteretic band. All these theoretical insights can be confirmed with the numerical results that we present next.

4.6 Numerical Analysis

After providing some theoretical insights about the economic hysteresis effect under stochastic interest rates, let us now proceed with some numerical results that will confirm our analytical results. For simplicity we assume that $\kappa = \theta = \lambda = 0$. We will consider the base case volatility level $\sigma = 0.0854$ [taken from Chan et al. (1992)] and a smaller and a higher volatility ($\sigma = 0.03$ and $\sigma = 0.3$, respectively) for comparative purposes. In addition, we use two different levels of investment cost, $\overline{I} = 10$ and $\overline{I} = 7.5$. We also establish three degrees of reversibility, $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. We also present the case of perfect reversibility, $\alpha = 1.00$, to illustrate the limiting case. The case where $\alpha = 0$ is also illustrated since it falls in the single investment strategy situation.

We have to use equations (9) and (10) to compute the perpetuity and the derivative of the perpetuity functions, respectively. However, we have to set a fixed upper limit for the integrals in both functions, otherwise their values would be $+\infty$ and $-\infty$, respectively. A question now arises. What value T should we use at the upper limits of the integrals? We have tried several time values such as $T = 100$, $T = 500$, $T = 1000$, $T = 3000$, etc. It turns out that from a practical point of view any of these values can be considered as a sufficient large maturity, which gives a time-independent resemblance for the problem[16]. For the single investment strategy the choice of T is not sensible for the lower threshold point. For the disinvestment single strategy, however, this is not the case. Indeed, the upper threshold is rising with T, although not too much strongly marked, especially for low and moderate volatility levels. But for the combined entry and exit strategy, which is the most interesting one, the use of any of those T values does not produce any significative change on both lower and upper trigger points. Yet, we present two different time values, $T = 500$ and $T = 1000$, which confirms what we have described.

Table 1 presents the lower trigger points for the single investment strategy considering different investment cost levels and different interest rate volatilities. Under this strategy, an active firm never shuts its project, since the profits never become negative. Thus, these thresholds indicate which is the interest rate level that will induce an idle firm to enter in a project and continue its operations forever since the option to shut down is worthless (i.e., it corresponds to the case where $\alpha = 0$). For example, considering the base case volatility level and $\overline{I} = 10$ it would be necessary that the interest rate falls to 1.94% to induce an idle firm to invest. At this rate level, the firm will change the full cost of investment (i.e., option to invest plus investment cost) by a project paying a perpetuity. From the table we can conclude that: (i) for a given level of volatility the lower threshold falls as the investment cost rises. This confirms our theoretical results stated by equation (A.6); and (ii) for a given investment cost level the lower trigger point falls as the volatility level rises. In addition, all investment thresholds when uncertainty is considered are lower than the Marshallian investment trigger point (0.10 and 0.1333 for $\overline{I} = 10$ and $\overline{I} = 7.5$, respectively). It should be noted that for high levels of uncertainty firms would never invest.

[16]For example, Schaefer and Schwartz (1984) calculate the price of a consol by actually computing the price of a 200-year annuity.

[Insert Table 1 Here]

Table 2 presents the upper trigger points for the single disinvestment strategy considering different investment cost levels and interest rate volatilities and different fractions of disinvestment proceeds to investment costs. This case corresponds to the one where the option to reinvest becomes worthless since the investment costs to reenter again are extremely high. Thus, idle firms do not invest and active firms will only disinvest if interest rates goes to sufficiently high values, because they know they cannot reinvest again. Considering the volatility level of $\sigma = 0.0854$, an investment cost of $\overline{I} = 10$ and $\alpha = 0.50$, it would be necessary that interest rates would be around 40% to induce an active firm to abandon its operations, knowing that its action could not be reversed later. From the table we can take the following conclusions: (i) for a given level of volatility and investment cost the upper threshold falls when the parameter α rises (i.e., the disinvestment proceeds rises). This confirms our theoretical results stated by equation (A.10); (ii) for a given volatility level and α parameter the upper trigger point falls as the investment cost rises (at the end it originates a rise on the disinvestment proceeds). Once again, this confirms equation (A.10); and (iii) for a given investment cost and α parameter the optimal disinvestment threshold rises as volatility rises. All disinvestment thresholds when uncertainty is considered are higher than the Marshallian disinvestment trigger points. Thus, higher uncertainty produces higher upper trigger points. It should be noted that for high levels of uncertainty firms would never disinvest almost for sure. Even for moderate and low levels of uncertainty it would be necessary that firms could recoup a very high fraction of the investment costs to induce firms to abandon permanently their operations. This issue arises from the fact that operating profits can never become negative and there is no option to reenter again. As we will see below, when the option to invest again is considered these thresholds will fall significantly due to the reentry option effect.

[Insert Table 2 Here]

Tables 3 and 4 present both the lower and upper trigger points for the entry and exit combined strategy considering different investment cost levels, $\overline{I} = 10$ for Table 3 and $\overline{I} = 7.5$ for Table 4, and different interest rate volatilities and fractions of disinvestment proceeds to investment costs. This case corresponds to the one where idle firms are induced to invest if the interest rate value falls to a sufficient low level, but they own an option to abandon later if interest rates rise to very high values. Once the project is abandoned, firms own an option to reinvest again if interest rates reverse to very low levels again. It turns out that this combined strategy originates a range where inaction is the optimal policy, i.e., idle firms do not invest and active firms do not abandon their operations. Considering a volatility level of $\sigma = 0.0854$, an investment cost of $\overline{I} = 10$ and $\alpha = 0.50$, the lower and upper trigger points are, respectively, $\underline{r} = 0.0199$ and $\overline{r} = 0.2641$, which originates a range of inaction of 0.2442 ($\overline{r} - \underline{r}$). Figure 1 depicts this numerical example. It is possible to see that an unique optimal solution exists. In this case, the Marshallian trigger points would be $M_{\underline{r}} = 1/\overline{I} = 0.10$ and $M_{\overline{r}} = 1/\underline{I} = 0.20$, originating a band of inaction of 0.10 ($M_{\overline{r}} - M_{\underline{r}}$). Our lower trigger point is approximately 80 percent below the Marshallian investment threshold and our upper trigger point is approximately 32 percent above the corresponding Marshallian exit point. Therefore, uncertainty and the embedded option values are responsible for the hysteretic band widening.

[Insert Table 3 Here]

[Insert Table 4 Here]

[Insert Figure 1 Here]

Considering these parameters, it is possible to see that the lower trigger point for the single investment strategy (i.e., without the exit option) is $\underline{r} = 0.0194$ (see Table 1). For the combined strategy, this lower trigger point is now $\underline{r} = 0.0199$ (see Table 3). The small difference in value arises from the firm's possibility to shut down later if interest rates start rising for very high levels. Similarly, the upper trigger point for the single disinvestment strategy (i.e., without the reentry option) is $\overline{r} = 0.3818$ (see Table 2). For the combined strategy, the upper trigger point is now $\overline{r} = 0.2641$ (see Table 3). This difference comes from the value of the reentry option that is owned by the firm. But now the difference is much more pronounced. There is an economic explanation for this fact. Thus, when the firm invests in a project and has an option to shut down in the future, the disinvestment proceeds present value is very small because the prospect of close its operations is sufficiently far in the future. As a result, the impact on the lower trigger point is very diminutive. However, when a firm is operating and decides to shut down its activities it will receive almost immediately the disinvestment proceeds value, which originates a bigger present value and justifies the greater differences between the two upper trigger points.

We can take the following conclusions from the tables: (i) for a given level of volatility and investment cost the lower threshold rises and the upper threshold falls when the parameter α rises (i.e., the disinvestment proceeds rises). Thus, the range of inaction will be narrower. This confirms our theoretical results stated by equations (A.11) and (A.10), respectively; (ii) for a given volatility level and α parameter both the lower and upper trigger points falls as the investment cost rises (at the end it originates a rise on the disinvestment proceeds). This confirms the theoretical insights of equations (A.6) and (A.10), respectively; and (iii) for a given investment cost and α parameter the optimal lower trigger point falls and the upper threshold rises as volatility rises, widening the hysteretic range[17].

We know that it is extremely complicated to analyze analytically the impact of the σ parameter on the optimal trigger points due to the effects it produces on the value function $V(r)$, because it enters in the quadratic equation with roots a and b. However, we can resort some numerical simulations that can highlight its effects. For greater visual appeal we show the corresponding picture as a continuous curve and not as a step function. Figure 2 presents the impact on the entry and exit thresholds as the volatility rises, considering $\overline{I} = 10$, $\alpha = 0.50$ and $\lambda = 0$. Clearly, there is a tendency to a wider range of inaction as the volatility rises, as we have already mentioned before.

[Insert Figure 2 Here]

[17] It should be noted that this is not always true for the upper trigger point. Thus, for very high values of α and investment costs, which produces high disinvestment proceeds, the upper threshold starts rising as the volatility rises, but for very high volatility levels the upper threshold turns its behaviour and starts falling. A possible explanation for this issue may come from the "bird-in-the-hand" argument. Thus, since uncertainty is so high and the value of the disinvestment proceeds is significant, firms may be induced to shut down at lower rates in order to get a safe present value of the disinvestment proceeds.

5 Conclusions

We model the investment hysteresis problem under stochastic interest rates explicitly using the most tractable form of interest rate uncertainty and describe the sensitivity of the interest rate band to the model parameters. We do this to analyze the beneficial effects of waiting to invest in the presence of uncertainty as well as to shed light on the macro influence of interest rate changes on investment policies.

Our results allow us to conclude that when there is some level of interest rate uncertainty, the hysteresis level emerges very quickly even for very small investment costs. This means that apart from the output price uncertainty [see, for example, Dixit (1989a)], the interest rate uncertainty also plays a critical role for widening the hysteretic band. When interest rates fall, firms make durable investments, that is to say that they switch from cash (an immediate asset) to longer lived assets with cash flows further ahead in time. When interest rates rise, they will stop undertaking any durable projects. Furthermore, if flexibility exists they will also try and reverse the investment process, i.e. disinvest away from projects with long lived cash flows into projects with more immediate payoffs.

In this paper we use the single-factor pure diffusion process of Ingersoll and Ross (1992), thus assuming that there is no mean reversion. However, the empirical evidence on interest rate behaviour seems to indicate that interest rates are pulled back to some long-run mean value over time, which implies that for optimal investment decisions or capital budgeting problems in a competitive environment under interest rate uncertainty the assumption of no mean reversion of interest rates may not be adequate. Thus, a natural extension of this work is to consider the mean reversion feature using the mean-reverting square-root process of Cox et al. (1985b). In addition, it is important to point out that the aggregate annual rate of replacement investment typically exceeds expansion or new investments by a wide margin. Thus, it is also interesting to tackle the replacement decision problem within a CIR economy. These two problems are left for future research.

Appendix A: Analytical Comparative Statics Expressions

We know that it is not possible to achieve closed-form solutions due to the high non-linearity of the equations that define the thresholds. However, it is possible to obtain some comparative statics for some parameters since the total differentials corresponding to small changes in the parameters are linear. Thus, the purpose of this appendix is to present such analytical comparative static expressions. It would be interesting to get qualitative comparative statics for all parameters. However, some of them, such as σ and λ (or even κ and θ for the mean-reverting case), produce effects on the value function $V(r)$ that are extremely complicated to analyze analytically since they enter in the quadratic equation with roots a and b. As a result, we will restrict our comparative statics to $\overline{I}$ and $\underline{I}$.

We know that the general solution of the value function $V(r)$ is as follows:

$$V(r) = C_3 e^{ar} - C_2 e^{br} + F(r) \tag{A.1}$$

where $F(r)$ represents the perpetuity value. The function is dependent of the state variable r and the options coefficients C_3 and C_2. Therefore, throughout this appendix we will write the value function as $V(r, C_3, C_2)$. Now, the value matching conditions and the smooth pasting conditions can be stated as follows:

$$V(\underline{r}, C_3, C_2) = \overline{I}\,, \qquad V(\overline{r}, C_3, C_2) = \underline{I} \tag{A.2a}$$

$$V_r(\underline{r}, C_3, C_2) = 0\,, \qquad V_r(\overline{r}, C_3, C_2) = 0 \tag{A.2b}$$

To simplify the notation we will denote the partial derivatives of V by subscripts. In addition, we will consider that $V_i(L) = V_i(\underline{r}, C_3, C_2)$ and $V_i(U) = V_i(\overline{r}, C_3, C_2)$ for $i = r, C_3, C_2$ and where L and U stands for lower and upper, respectively.

Let us now suppose that the investment cost $\overline{I}$ changes by $d\overline{I}$ and let see how the two thresholds and the two constants behave. In order to simplify the analytical expressions we will not consider the relationship between $\overline{I}$ and $\underline{I}$ explicitly. We will start by differentiating the value matching conditions (A.2a) totally:

$$V_r(L)\,d\underline{r} + V_{C_3}(L)\,dC_3 + V_{C_2}(L)\,dC_2 = d\overline{I} \tag{A.3a}$$

$$V_r(U)\,d\overline{r} + V_{C_3}(U)\,dC_3 + V_{C_2}(U)\,dC_2 = 0 \tag{A.3b}$$

From the smooth pasting conditions (A.2b) we know that $V_r(L)\,d\underline{r}$ and $V_r(U)\,d\overline{r}$ will disappear from the above system since they are zero. In addition, it is quite easy to show that $V_{C_3}(L) = e^{a\underline{r}}$, $V_{C_2}(L) = -e^{b\underline{r}}$, $V_{C_3}(U) = e^{a\overline{r}}$ and $V_{C_2}(U) = -e^{b\overline{r}}$. Thus we can solve explicitly dC_2 and dC_3 as:

$$dC_2 = -e^{a\overline{r}}\,d\overline{I}/\Delta \tag{A.4a}$$

$$dC_3 = -e^{b\overline{r}}\,d\overline{I}/\Delta \tag{A.4b}$$

where $\Delta = e^{a\overline{r}+b\underline{r}} - e^{a\underline{r}+b\overline{r}} > 0$, since $a > 1$, $b < 0$ (i.e., $a > 0 > b$) and $\overline{r} > \underline{r}$.

Now differentiating the smooth pasting condition (A.2b) at $\underline{r}$ we get:

$$V_{rr}(L)\,d\underline{r} + V_{rC_3}(L)\,dC_3 + V_{rC_2}(L)\,dC_2 = 0 \tag{A.5}$$

Noting that $V_{rC_3}(L) = ae^{a\underline{r}}$, $V_{rC_2}(L) = -be^{b\underline{r}}$ and making the appropriate substitutions yields:

$$V_{rr}(L)\,d\underline{r} = \left[ae^{a\underline{r}+b\overline{r}} - be^{a\overline{r}+b\underline{r}}\right] d\overline{I}/\Delta \tag{A.6}$$

We know that $V_{rr}(L) < 0$ because $V(r)$ is concave at $\underline{r}$ and the term in brackets is positive. Now it is easy to prove that $d\underline{r} < 0$ when $d\overline{I} > 0$. Thus, the lower trigger point falls as the investment cost rises as we expected.

Similarly, differentiating the smooth pasting condition (A.2b) at $\overline{r}$ and following the same steps as before we get:

$$V_{rr}(U)\,d\overline{r} = \left[(a-b)e^{(a+b)\overline{r}}\right] d\overline{I}/\Delta \tag{A.7}$$

Since $V(r)$ is convex at $\overline{r}$, $V_{rr}(U) > 0$. The term in brackets is also positive. As a result, $d\overline{r} > 0$ when $d\overline{I} > 0$. This means that the upper trigger point rises with the investment cost. This is economically intuitive if we consider that when the firm shuts down it will not recoup any part of the investment cost made initially (i.e., $\alpha = 0$) or if it has to pay a new lump-sum cost to abandon. In our case, we are interested in analyzing the problem when the firm has the possibility to recoup a fraction of the investment cost. Thus, we expect that as the disinvestment proceeds rises the abandonment threshold falls. It is also possible to prove analytically our intuition as we will show below.

To do so, let us suppose that the disinvestment proceeds $\underline{I}$ changes by $d\underline{I}$ and let see how the two thresholds and the two constants behave. Once again, we will differentiate the value matching conditions (A.2a) totally:

$$V_r(L)\, d\underline{r} + V_{C_3}(L)\, dC_3 + V_{C_2}(L)\, dC_2 = 0 \tag{A.8a}$$

$$V_r(U)\, d\overline{r} + V_{C_3}(U)\, dC_3 + V_{C_2}(U)\, dC_2 = d\underline{I} \tag{A.8b}$$

The explicit solutions for dC_2 and dC_3 are:

$$dC_2 = e^{a\underline{r}}\, d\underline{I}/\Delta \tag{A.9a}$$

$$dC_3 = e^{b\underline{r}}\, d\underline{I}/\Delta \tag{A.9b}$$

Following the same procedure used before and differentiating the smooth pasting condition (A.2b) at $\overline{r}$ we get:

$$V_{rr}(U)\, d\overline{r} = \left[be^{a\underline{r}+b\overline{r}} - ae^{a\overline{r}+b\underline{r}}\right] d\underline{I}/\Delta \tag{A.10}$$

We know that $V_{rr}(U) > 0$ and the term in brackets is negative. Therefore, the upper trigger point falls as the disinvestment proceeds rises as we intuitively expected, i.e., $d\overline{r} < 0$ when $d\underline{I} > 0$. It should be noted that the disinvestment proceeds can rise due to a higher recouped fraction α or to a higher investment cost but maintaining a positive α fixed. We are not considering the relationship between $\overline{I}$ and $\underline{I}$ explicitly in order to simplify the comparative statics expressions of this appendix. However, since we know that $\underline{I} = \alpha\,\overline{I}$ it is extremely easy to show the impact of rising α on the upper threshold. To do so, we just need to use $d\alpha\,\overline{I}$ instead of $d\underline{I}$ in equation (A.8b) and proceed as before. It turns out that $d\overline{r} < 0$ when $d\alpha > 0$.

Similarly, differentiating the smooth pasting condition (A.2b) at $\underline{r}$ and following the same steps we get:

$$V_{rr}(L)\, d\underline{r} = \left[(b-a)e^{(a+b)\underline{r}}\right] d\underline{I}/\Delta \tag{A.11}$$

$V_{rr}(L) < 0$ and the term in brackets is negative. Therefore, $d\underline{r} > 0$ when $d\underline{I} > 0$. In this case it is also easy to prove that $d\underline{r} > 0$ when $d\alpha > 0$.

Table 1: Lower thresholds for the investment option case under different levels of investment costs and interest rate volatility. CIR parameters: $\kappa = \theta = \lambda = 0$.

	$\overline{I} = 10$			$\overline{I} = 7.5$		
T	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$
500	0.0809	0.0194	−0.4470	0.1145	0.0642	−0.3450
1000	0.0809	0.0194	−0.4470	0.1145	0.0642	−0.3450

Table 2: Upper thresholds for the disinvestment option case under different levels of disinvestment proceeds and interest rate volatility and different ratios of the disinvestment proceeds to the investment costs. CIR parameters: $\kappa = \theta = \lambda = 0$.

		$\overline{I} = 10$			$\overline{I} = 7.5$		
T	α	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$
500	0.25	0.4225	0.5091	1.3648	0.5555	0.6138	1.4557
1000		0.4225	0.5286	1.4998	0.5555	0.6222	1.5871
500	0.50	0.2251	0.3818	1.1746	0.2900	0.4229	1.2499
1000		0.2260	0.4145	1.3148	0.2901	0.4520	1.3884
500	0.75	0.1659	0.3365	1.0739	0.2044	0.3675	1.1447
1000		0.1707	0.3723	1.2160	0.2062	0.4013	1.2856
500	1.00	0.1402	0.3095	1.0053	0.1659	0.3365	1.0739
1000		0.1479	0.3467	1.1484	0.1707	0.3723	1.2160

Table 3: Upper and lower thresholds for the switching option case under an investment cost of 10 for different ratios of the disinvestment proceeds to the investment costs and different interest rate volatilities. CIR parameters: $\kappa = \theta = \lambda = 0$.

		$\overline{I} = 10$					
		$\sigma = 0.03$		$\sigma = 0.0854$		$\sigma = 0.3$	
T	α	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$
500	0.25	0.0809	0.4225	0.0194	0.4725	−0.4467	0.5922
1000		0.0809	0.4225	0.0194	0.4725	−0.4467	0.5922
500	0.50	0.0810	0.2239	0.0199	0.2641	−0.4427	0.2217
1000		0.0810	0.2239	0.0199	0.2641	−0.4427	0.2217
500	0.75	0.0815	0.1556	0.0233	0.1717	−0.4273	0.0061
1000		0.0815	0.1556	0.0233	0.1717	−0.4273	0.0061
500	1.00	0.0991	0.0991	0.0671	0.0671	−0.2872	−0.2872
1000		0.0991	0.0991	0.0671	0.0671	−0.2872	−0.2872

Table 4: Upper and lower thresholds for the switching option case under an investment cost of 7.5 for different ratios of the disinvestment proceeds to the investment costs and different interest rate volatilities. CIR parameters: $\kappa = \theta = \lambda = 0$.

		$\overline{I} = 7.5$					
		$\sigma = 0.03$		$\sigma = 0.0854$		$\sigma = 0.3$	
T	α	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$
500	0.25	0.1145	0.5555	0.0642	0.6029	−0.3447	0.7614
1000		0.1145	0.5555	0.0642	0.6029	−0.3447	0.7614
500	0.50	0.1145	0.2899	0.0645	0.3355	−0.3412	0.3497
1000		0.1145	0.2899	0.0645	0.3355	−0.3412	0.3497
500	0.75	0.1148	0.2005	0.0676	0.2271	−0.3262	0.1200
1000		0.1148	0.2005	0.0676	0.2271	−0.3262	0.1200
500	1.00	0.1331	0.1331	0.1127	0.1127	−0.1838	−0.1838
1000		0.1331	0.1331	0.1127	0.1127	−0.1838	−0.1838

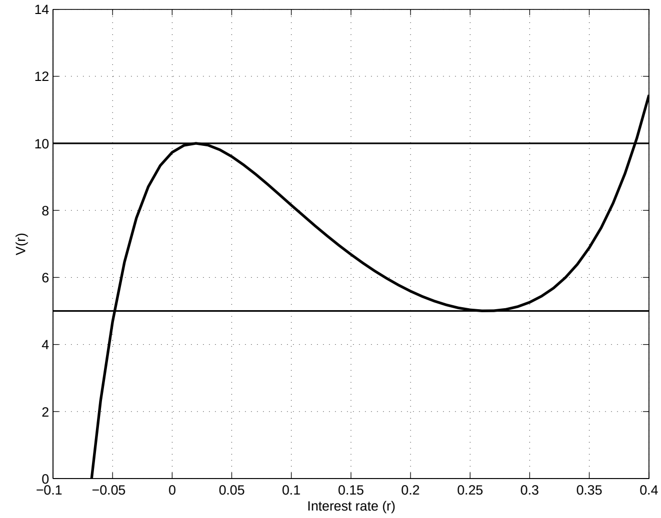

Figure 1: Determination of the numerical upper and lower thresholds. CIR parameters: $\kappa = \theta = \lambda = 0$ and $\sigma = 0.0854$. $\overline{I} = 10$ and $\alpha = 0.50$.

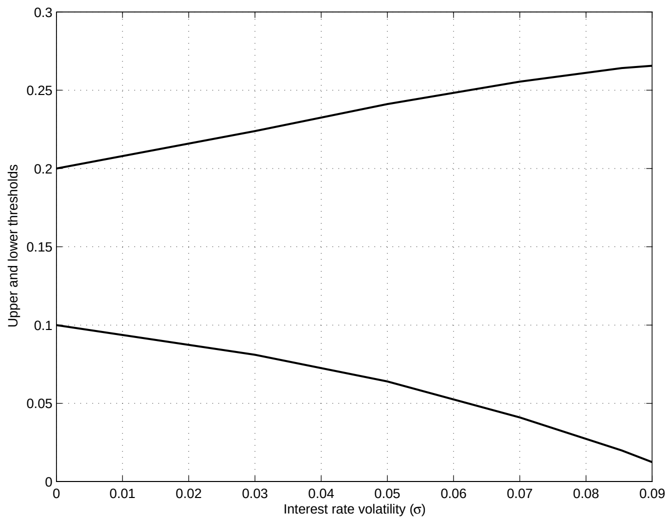

Figure 2: Entry and exit thresholds as functions of interest rate volatility. CIR parameters: $\kappa = \theta = \lambda = 0$. $\overline{I} = 10$ and $\alpha = 0.50$.

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