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Option bounds from concurrently expiring options when relative risk aversion is bounded

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Option Bounds from Concurrently Expiring Options When Relative Risk Aversion is Bounded

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Abstract

In this paper we derive option bounds from concurrently expiring option prices assuming the (pricing) representative investor's relative risk aversion is bounded. We show that given n concurrently expiring options, the option bounds are given by pricing kernels that have $(n+2)$-segmented piecewise constant elasticity. Closed form formulas are presented for the case where the distribution of the stock price is log-normal.

Keywords: Option bounds, Option pricing, Arbitrage pricing.

JEL Classification Numbers: G13.

Introduction

Various efforts have been made to derive Option pricing bounds when it is difficult to derive exact option prices, particularly when the market is incomplete. Merton (1973) gives option pricing bounds based only on a no-arbitrage argument. These bounds are improved by Perrakis and Ryan (1984), Ritchken (1985), and Levy (1985) under the assumption of risk aversion or second degree stochastic dominance. Option bounds can be further improved by imposing stronger assumptions on investors' risk preferences. For example, Ritchken and Kuo (1989) further improve the option bounds by assuming higher order stochastic dominance rules. Basso and Pianca (1997) and Mathur and Ritchken (2000) derive option bounds under the assumption of decreasing absolute (relative) risk aversion (hereafter DARA (DRRA)). Huang (2004) obtain option pricing bounds assuming the representative investor has bounded relative risk aversion.

Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000) present new approaches to option bounds. Cochrane and Saa-Requejo derive option bounds using restrictions on the volatility of the pricing kernel, while Bernardo and Ledoit derive option pricing bounds using restrictions on the deviation of the pricing kernel from a benchmark pricing kernel.

If we have observed prices of concurrently expiring options then we can further improve option bounds. Ryan (2003) improve the second order stochastic dominance option bounds by using one concurrently expiring option at a time. Huang (2004a) uses a new methodology to further improve risk aversion option bounds based on the observed prices of a number of options and discusses the second order arbitrage opportunities in the markets of concurrently expiring options. Using a similar method, Huang (2004b) improves higher order stochastic dominance option bounds based on the observed prices of options.

In this paper we derive option bounds from concurrently expiring options when the pricing representative investor's relative risk aversion is bounded.[1] Because of the close relationship between the elasticity of the pricing kernel and the representative investor's relative risk aversion, restrictions on the coefficient of risk aversion lead to restrictions on the elasticity of the pricing kernel, which enables us to derive meaningful option bounds.[2] Assuming the pricing representative investor's relative risk aversion is bounded above and below by $\overline{\gamma}$ and $\underline{\gamma}$ respectively, we show that given n observed options, the upper (lower) option bound is given by a pricing kernel that has $(n+2)$-segmented piecewise constant elasticity. More specifically, the elasticity of the pricing kernel that gives the upper (lower) option bound is equal to $\overline{\gamma}$ ($\underline{\gamma}$)on even segments and $\underline{\gamma}$ ($\overline{\gamma}$) on odd segments.

As is well known, constant relative risk aversion and log-normality lead to

[1]See, for example, Rubinstein (1976) and Brennan (1979) about a representative investor; see Benninga and Mayshar (2000) about a pricing representative investor.

[2]See, for example, Rubinstein (1976), Brennan (1979), FSS (1999), and Huang (2004) on the relationship between the elasticity of the pricing kernel and investors's coefficients of relative risk aversion.

the celebrated Black-Scholes formula. From our results, the option bounds are given by pricing representative investors who have piecewise constant relative risk aversion; and piecewise constant relative risk aversion and lognormality also give convenient formulas. An example is given in this paper.

The structure of the paper is as follows: In Section 1 we introduce the assumptions. In Section 2 we derive option bounds when no observed options are used. In Sections 3 and 4 we deal with the cases where there is one or two observed options respectively. Section 5 deals with the general case. In Section 6 we give closed form formulas when lognormaility is assume. The final section concludes the paper.

1 Assumptions

We assume that there is only one share of a stock in an economy on which option contracts are written. The price of the stock at time t is denoted by S_t. Let $u(x)$ be the pricing representative investor's utility function. As is well known, the pricing kernel for the contingent claims on the stock is equal to the discounted marginal utility of wealth of the pricing representative investor.[3] That is

$$\phi(S_t) = \frac{u'(S_t)}{Eu''^{(S_t)}} B_0$$

where $\phi(S_t)$ denotes the pricing kernel and B_0 is the time 0 value of a unit zero coupon bond.[4]

[3]See, for example, Rubinstein (1976), Brennan (1979), and Benninga and Mayshar (2000).

[4]As there is only one share of the stock the pricing representative investor's wealth is equal to the stock price S.

Thus we have

$$S_0 = B_0 E(\phi(S_t) S_t)$$

Denote time t value of a contingent claim by $c(S_t)$, which is dependent on S_t; denote its time 0 value by c_0. Then we have $c_0 = B_0 E(\phi(S_t) c(S_t))$.

Let $R(S_t) = -u''(S_t)/u'(S_t)$, which is the pricing representative investor's absolute risk aversion. Let $\gamma(S_t) = S_t R(S_t$, which is the pricing representative investor's relative risk aversion.

Note we also have $\gamma(S_t) = -S_t \phi'(S_t)/\phi(S_t)$. That is the pricing representative investor's relative risk aversion is equal to the elasticity of the pricing kernel. Because of this, we will use the two terms interchangeably.

In this paper we are going to derive option bounds assuming $\gamma(S_t)$ is bounded. We will show in this paper that under this condition, the option bounds are given by a pricing representative investor who has piecewise constant relative risk aversion, where the number of segments of the risk aversion depends on the number of observed option prices.

In order to explain the solutions more clearly we start with the case where we have no observed options then continue with the cases where we have one or two observed options. Building on the above examples we explore the general case where we have n observed options.

2 With No Observed Options

We first present a lemma.

Lemma 1 (FSS (1999)) *Assume two pricing kernels give the same stock price. If they intersect twice, then the pricing kernel with fatter tails gives higher prices of convex-payoff contingent claims writhen on the stock.*

Proof: See Huang (2004a) or FSS (1999).

Proposition 1 *Assume the elasticity of the pricing kernel is bounded from above by $\overline{\gamma}$ and below by $\underline{\gamma}$. Assume the prices of a unit bond and the underlying stock are B_0 and S_0 respectively.*

- *The upper option bound is given by the pricing kernel ϕ_0^{**} that has two-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for $S_t < s^{**}$ and $\underline{\gamma}$ for $S_t > s^{**}$, where s^{**} is to be determined by the underlying stock price. That is,*

$$\phi_0^{**}(x) = \{ \begin{array}{cc} ax^{-\overline{\gamma}}, & \text{for } x < s^{**} \\ as^{**\underline{\gamma}-\overline{\gamma}}x^{-\underline{\gamma}}, & \text{for } x \geq s^{**} \end{array}$$

where a and s^{**} are to be determined such that

$$E(\phi_0^{**}(x)) = 1 \quad \text{and} \quad E(S_t\phi_0^{**}(S_t))B_0 = S_0.$$

- *The lower option bound is given by the pricing kernel ϕ_0^{*} that has two-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\underline{\gamma}$ for $S_t < s^{*}$ and $\overline{\gamma}$ for $S_t > s^{*}$, where s^{*} is to be determined by the underlying stock price. That is,*

$$\phi_0^{*}(x) = \{ \begin{array}{cc} ax^{-\underline{\gamma}}, & \text{for } x < s^{*} \\ as^{*\overline{\gamma}-\underline{\gamma}}x^{-\overline{\gamma}}, & \text{for } x \geq s^{*} \end{array}$$

where a and s^* are to be determined such that

$$E(\phi_0^*(x)) = 1 \quad \textit{and} \quad E(S_t \phi_0^*(S_t))B_0 = S_0.$$

Proof: From Lemma 1 we need only prove that the true pricing kernel intersects $\phi_0^{**}(x)$ or $\phi_0^*(x)$ exactly twice and then examine which one has a fatter left tail.

We first examine ϕ_0^{**}. Note it has two-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for $S_t < s^{**}$ and $\underline{\gamma}$ for $S_t > s^{**}$. Obviously the elasticity of the true pricing kernel intersects that of ϕ_1^{**} at most once; thus the true pricing kernel will intersect ϕ_1^{**} at most twice. However, because they give the same stock price, they must intersect at least twice. Thus they intersect exactly twice. It is not difficult to verify that ϕ_1^{**} has fatter tails. For ϕ_1^* the proof is similar. Q.E.D.

In Proposition 1, when $\overline{\gamma}$ becomes larger and larger and approaches infinity and $\underline{\gamma}$ becomes smaller and smaller and approaches zero, we eventually obtain the second stochastic dominance option bounds.

3 With One Observed Option

Lemma 2 *Assume two pricing kernels give the same prices of the underlying stock and an option with strike price K. If they intersect three times, then the pricing kernel with fatter left tail will give higher [lower] prices for all options with strike prices below [above] K than the other.*

Proof: See Huang (2004a).

Proposition 2 *Assume the elasticity of the pricing kernel is bounded from above by $\overline{\gamma}$ and below by $\underline{\gamma}$. Assume the prices of a unit bond, the underlying stock, and an option on the stock with strike price K are B_0, S_0, and c_{K0} respectively.*

- *The upper bound for an option with a strike price below K is given by the pricing kernel ϕ_1^{**} that has three-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for $S_t < s_1^{**}$ and $S_t > s_2^{**}$ and $\underline{\gamma}$ for $s_1^{**} < S_t < s_2^{**}$, where $s_1^{**} < s_2^{**}$ are to be determined. That is,*

$$\phi_1^{**}(x) = \left\{ \begin{array}{ll} ae^{-\overline{\gamma}\ln x}, & x < s_1^{**} \\ a{s_1^{**}}^{\underline{\gamma}-\overline{\gamma}} e^{-\underline{\gamma}\ln x}, & s_1^{**} \leq x < s_2^{**} \\ a{s_1^{**}}^{\underline{\gamma}-\overline{\gamma}} {s_2^{**}}^{\overline{\gamma}-\underline{\gamma}} e^{-\overline{\gamma}\ln x}, & x \geq s_2^{**} \end{array} \right.$$

  where a, s_1^{**}, and s_2^{**} are to be determined such that

$$E(\phi_1^{**}(x)) = 1, \quad E(S_t\phi_1^{**}(S_t))B_0 = S_0, \quad E(c_K(S_t)\phi_1^{**}(S_t))B_0 = c_{K0}.$$

- *The lower bound for an option with a strike price below K is given by the pricing kernel ϕ_1^{*} that has three-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\underline{\gamma}$ for $S_t < s_1^{*}$ and $S_t > s_2^{*}$ and $\overline{\gamma}$ for $s_1^{*} < S_t < s_2^{*}$, where $s_1^{*} < s_2^{*}$ are to be determined. That is,*

$$\phi_1^{*}(x) = \left\{ \begin{array}{ll} ae^{-\underline{\gamma}\ln x}, & x < s_1^{*} \\ a{s_1^{**}}^{\overline{\gamma}-\underline{\gamma}} e^{-\overline{\gamma}\ln x}, & s_1^{*} \leq x < s_2^{*} \\ a{s_1^{**}}^{\overline{\gamma}-\underline{\gamma}} {s_2^{**}}^{\underline{\gamma}-\overline{\gamma}} e^{-\underline{\gamma}\ln x}, & x \geq s_2^{*} \end{array} \right.$$

  where a, s_1^{*}, and s_2^{*} are to be determined such that

$$E(\phi_1^{*}(x)) = 1, \quad E(S_t\phi_1^{*}(S_t))B_0 = S_0, \quad E(c_K(S_t)\phi_1^{*}(S_t))B_0 = c_{K0}.$$

- *The lower (upper) bound for an option with a strike price above K is given by the pricing kernel $\phi_1^{**}(S_t)$ ($\phi_1^*(S_t)$).*

Proof: From Lemma 2 we need only prove that the true pricing kernel intersects $\phi_0^{**}(x)$ or $\phi_0^*(x)$ exactly three times and then examine which one has a fatter left tail.

We first examine ϕ_0^{**}. Note it has three-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for $S_t < s_1^{**}$ and $S_t > s_2^{**}$ and $\underline{\gamma}$ for $s_1^{**} < S_t < s_2^{**}$. Obviously the elasticity of the true pricing kernel intersects that of ϕ_1^{**} at most twice; thus the true pricing kernel will intersect ϕ_1^{**} at most three times. However, because they give the same prices of the stock and option, from Lemma 1 they must intersect at least three times. Thus they intersect exactly three times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

4 The Case with Two Observed Options

Lemma 3 *Assume two pricing kernels give the same prices of the underlying stock and two options with strike prices K_1 and K_2, where $K_1 < K_2$. If they intersect four times, then the pricing kernel with fatter left tail will give higher (lower) prices for options with strike prices outside (inside) (K_1, K_2).*

Proof: See Huang (2004a).

Proposition 3 *Assume the elasticity of the pricing kernel is bounded from above by $\overline{\gamma}$ and below by $\underline{\gamma}$. Assume the price of a unit bond is B_0, the un-*

derlying stock price is S_0, and the prices of two options with strike prices K_1 and K_2 are c_{10} and c_{20} respectively, where $K_1 < K_2$.

- *Then the upper bound for options with strike prices below K_1 or above K_2 is given by the pricing kernel $\phi_2^{**}(x)$ that has four-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for $S_t < s_1^{**}$ and $S_t \in (s_2^{**}, s_3^{**})$, and $\underline{\gamma}$, for $S_t \in (s_1^{**}, s_2^{**})$ and $S_t > s_3^{**}$, where $s_1^{**} < s_2^{**} < s_3^{**}$ are to be determined. That is,*

$$\phi_2^{**}(x) = \{ \begin{array}{cc} ax^{-\overline{\gamma}}, & x < s_1^{**} \\ a{s_1^{**}}^{\underline{\gamma}-\overline{\gamma}} e^{-\underline{\gamma}\ln x}, & x \in (s_1^{**}, s_2^{**}) \\ a{s_1^{**}}^{\underline{\gamma}-\overline{\gamma}} {s_2^{**}}^{\overline{\gamma}-\underline{\gamma}} e^{-\overline{\gamma}\ln x}, & x \in (s_2^{**}, s_3^{**}) \\ a{s_1^{**}}^{\underline{\gamma}-\overline{\gamma}} {s_2^{**}}^{\overline{\gamma}-\underline{\gamma}} {s_3^{**}}^{\underline{\gamma}-\overline{\gamma}} e^{-\underline{\gamma}\ln x}, & x > s_3^{**} \end{array}$$

  where a, s_1^{**}, s_2^{**}, and s_3^{**} are to be determined such that

$$E(\phi_2^{**}(x)) = 1,\ E(S_t\phi_2^{**}(S_t))B_0 = S_0,\ E(c_i(S_t)\phi_2^{**}(S_t))B_0 = c_{i0},\ i = 1, 2.$$

- *The lower bound for options with strike prices below K_1 or above K_2 is given by the pricing kernel $\phi_2^{*}(x)$ that has four-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\underline{\gamma}$ for $S_t < s_1^{*}$ and $S_t \in (s_2^{*}, s_3^{*})$, and $\overline{\gamma}$, for $S_t \in (s_1^{*}, s_2^{*})$ and $S_t > s_3^{*}$, where $s_1^{*} < s_2^{*} < s_3^{*}$ are to be determined. That is,*

$$\phi_2^{*}(x) = \{ \begin{array}{cc} ax^{-\underline{\gamma}}, & x < s_1^{*} \\ a{s_1^{*}}^{\overline{\gamma}-\underline{\gamma}} e^{-\overline{\gamma}\ln x}, & x \in (s_1^{*}, s_2^{*}) \\ a{s_1^{*}}^{\overline{\gamma}-\underline{\gamma}} {s_2^{*}}^{\underline{\gamma}-\overline{\gamma}} e^{-\underline{\gamma}\ln x}, & x \in (s_2^{*}, s_3^{*}) \\ a{s_1^{*}}^{\overline{\gamma}-\underline{\gamma}} {s_2^{*}}^{\underline{\gamma}-\overline{\gamma}} {s_3^{*}}^{\overline{\gamma}-\underline{\gamma}} e^{-\overline{\gamma}\ln x}, & x > s_3^{*} \end{array}$$

where a, s_1^*, s_2^*, and s_3^* are to be determined such that

$$E(\phi_2^*(x)) = 1,\ E(S_t\phi_2^*(S_t))B_0 = S_0,\ E(c_i(S_t)\phi_2^*(S_t))B_0 = c_{i0},\ i = 1, 2.$$

- *The upper (lower) bound for options with strike prices between K_1 and K_2 is given by the pricing kernel $\phi_2^*(S_t)$ $(\phi_2^{**}(S_t))$.*

Proof: From Lemma 3 we need only prove that the true pricing kernel intersects $\phi_0^{**}(x)$ or $\phi_0^*(x)$ exactly four times and then examine which one has a fatter left tail.

We first examine ϕ_0^{**}. Note it has four-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for $S_t < s_1^{**}$ and $S_t \in (s_2^{**}, s_3^{**})$, and $\underline{\gamma}$, for $S_t \in (s_1^{**}, s_2^{**})$ and $S_t > s_3^{**}$. Obviously the elasticity of the true pricing kernel intersects that of ϕ_1^{**} at most three times; thus the true pricing kernel will intersect ϕ_1^{**} at most four times. However, because they give the same prices of the stock and two options, from Lemma 2 they must intersect at least four times. Thus they intersect exactly four times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

5 The General Case

Lemma 4 *Assume two pricing kernels give the same prices of the underlying stock and options with strike prices K_1, K_2, ..., K_n, where $0 = K_0 < K_1 < K_2 < ... < K_n < K_{n+1} = +\infty$. If the two pricing kernels intersect $n + 2$ times then the one with fatter left tail will give higher (lower) prices for all options with strike prices between (K_{2i-2}, K_{2i-1}) $((K_{2i-1}, K_{2i}))$, $i = 1, 2, ...$*

Proof: See Huang (2004a).

Proposition 4 *Assume the elasticity of the pricing kernel is bounded from above by $\overline{\gamma}$ and below by $\underline{\gamma}$. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of n options with strike prices K_1, K_2, ..., K_n are c_{10}, c_{20}, ..., and c_{n0} respectively, where $0 = K_0 < K_1 < K_2 < ... < K_n < K_{n+1} = +\infty$.*

- *Then the upper bound for options with strike prices between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...$, is given by the pricing kernel ϕ_n^{**} that has $(n+2)$-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ at odd segments and $\underline{\gamma}$ at even segments. That is, $\phi_n^{**}(x) = \phi_{ni}^{**}(x)$, for $x \in (s_{i-1}^{**}, s_i^{**})$, $i = 1, 2, ..., n+2$, where $s_0^{**} = 0$, $s_{n+2} = +\infty$,*

$$\phi_{n1}^{**}(x) = ae^{-\overline{\gamma}\ln x}, \quad \phi_{n2}^{**}(x) = as_1^{**\underline{\gamma}-\overline{\gamma}}e^{-\underline{\gamma}\ln x},$$

$$\phi_{ni}^{**}(x) = \phi_{n(i-1)}^{**}(s_{i-1}^{**})s_{i-1}^{**\overline{\gamma}}e^{-\overline{\gamma}\ln x},\ \text{if } i \in (2, n+2] \text{ is odd},$$

$$\phi_{ni}^{**}(x) = \phi_{n(i-1)}^{**}(s_{i-1}^{**})s_{i-1}^{**\underline{\gamma}}e^{-\underline{\gamma}\ln x},\ \text{if } i \in (2, n+2] \text{ is even},$$

  where a, s_1^{**}, ..., s_{n+1}^{**} are to be determined such that $E(\phi_n^{**}(x)) = 1$, $E(S_t\phi_n^{**}(S_t))B_0 = S_0$, and $E(c_i(S_t)\phi_n^{**}(S_t))B_0 = c_{i0}$, $i = 1, 2, ..., n$.

- *The lower bound for options with strike prices between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...$, is given by the pricing kernel ϕ_n^* that has $(n+2)$-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\underline{\gamma}$ at odd segments and $\overline{\gamma}$ at even segments. That is, $\phi_n^*(x) = \phi_{ni}^*(x)$, for $x \in (s_{i-1}^*, s_i^*)$, $i = 1, 2, ..., n+2$, where $s_0^* = 0$, $s_{n+2} = +\infty$,*

$$\phi_{n1}^*(x) = ae^{-\underline{\gamma}\ln x}, \quad \phi_{n2}^*(x) = as_1^{*\overline{\gamma}-\underline{\gamma}}e^{-\overline{\gamma}\ln x},$$

$$\phi^*_{ni}(x) = \phi^*_{n(i-1)}(s^*_{i-1}) s^{*\underline{\gamma}}_{i-1} e^{-\underline{\gamma} \ln x}, \text{ if } i \in (2, n+2] \text{ is odd,}$$

$$\phi^*_{ni}(x) = \phi^*_{n(i-1)}(s^*_{i-1}) s^{*\overline{\gamma}}_{i-1} e^{-\overline{\gamma} \ln x}, \text{ if } i \in (2, n+2] \text{ is even,}$$

where a, s^*_1, ..., s^*_{n+1} *are to be determined such that* $E(\phi^*_n(x)) = 1$, $E(S_t \phi^*_n(S_t)) B_0 = S_0$, *and* $E(c_i(S_t) \phi^*_n(S_t)) B_0 = c_{i0}$, $i = 1, 2, ..., n$.

- *The lower (upper) bound for options with strike prices between* $(K_{2i-1}$, $K_{2i})$, $i = 1, 2, ...$, *is given by the pricing kernel* $\phi^{**}_n(S_t)$ $(\phi^*_n(S_t))$.

Proof: From Lemma 4 we need only prove that the true pricing kernel intersects $\phi^{**}_0(x)$ or $\phi^*_0(x)$ exactly $n+2$ times and then examine which one has a fatter left tail.

We first examine ϕ^{**}_0. Note it has $(n+2)$-segmented piecewise constant elasticity. More precisely, its elasticity is equal to $\overline{\gamma}$ for odd segments and $\underline{\gamma}$ for even segments. Obviously the elasticity of the true pricing kernel intersects that of ϕ^{**}_1 at most $n+1$ times; thus the true pricing kernel will intersect ϕ^{**}_1 at most $n+2$ times. However, because they give the same prices of the stock and n options, from Lemma 4 they must intersect at least $n+2$ times. Thus they intersect exactly $n+2$ times. It is not difficult to verify that ϕ^{**}_1 has fatter left tail. For ϕ^*_1 the proof is similar. Q.E.D.

6 Option Bounds with Lognormality

In this section we give a formula of option bounds when the underlying stock price follows a lognormal distribution. Assume the bond price B_0 and the stock price S_0 are known. Let R be the continuous interest rate implied by B_0. Let $n(x, \mu, \sigma)$ denote the normal p.d.f with mean μ and standard deviation σ.

Proposition 5 *Assume the risk neutral p.d.f can be written as $f^Q(x) = g(x)$ $n(\ln x, 0, \sigma)/x$. Assume the elasticity of $g(x)$ is bounded below by $\underline{\gamma}$ and above by $\overline{\gamma}$. Let x^* be the solution to*

$$\begin{aligned}
&e^{\frac{1}{2}(\overline{\gamma}-1)^2\sigma^2} N\left(\frac{\ln x^* + (\overline{\gamma}-1)\sigma^2}{\sigma}\right) + x^{*\underline{\gamma}-\overline{\gamma}} e^{\frac{1}{2}(\underline{\gamma}-1)^2\sigma^2} N\left(\frac{-(\underline{\gamma}-1)\sigma^2 - \ln x^*}{\sigma}\right) \\
&= e^R\left[e^{\frac{1}{2}\overline{\gamma}^2\sigma^2} N\left(\frac{\ln x^* + \overline{\gamma}\sigma^2}{\sigma}\right) + x^{*\underline{\gamma}-\overline{\gamma}} e^{\frac{1}{2}\underline{\gamma}^2\sigma^2} N\left(\frac{-\underline{\gamma}\sigma^2 - \ln x^*}{\sigma}\right)\right]. \qquad (1)
\end{aligned}$$

For a call option with strike price K, if $K < S_0 x^*$, the upper bound of its price is

$$\begin{aligned}
&a e^{-R} x^{*\underline{\gamma}-\overline{\gamma}}\left[S_0 e^{\frac{1}{2}(\underline{\gamma}-1)^2\sigma^2} N\left(\frac{-(\underline{\gamma}-1)\sigma^2 - \ln x^*}{\sigma}\right) - K e^{\frac{1}{2}\underline{\gamma}^2\sigma^2} N\left(\frac{-\underline{\gamma}\sigma^2 - \ln x^*}{\sigma}\right)\right] \\
&- a e^{-R}\left[S_0 e^{\frac{1}{2}(\overline{\gamma}-1)^2\sigma^2} N\left(\frac{\ln \frac{K}{S_0} + (\overline{\gamma}-1)\sigma^2}{\sigma}\right)) - K e^{\frac{1}{2}\overline{\gamma}^2\sigma^2} N\left(\frac{\ln \frac{K}{S_0} + \overline{\gamma}\sigma^2}{\sigma}\right))\right] \\
&+ a e^{-R}\left[S_0 e^{\frac{1}{2}(\overline{\gamma}-1)^2\sigma^2} N\left(\frac{\ln x^* + (\overline{\gamma}-1)\sigma^2}{\sigma}\right) - K e^{\frac{1}{2}\overline{\gamma}^2\sigma^2} N\left(\frac{\ln x^* + \overline{\gamma}\sigma^2}{\sigma}\right)\right], \qquad (2)
\end{aligned}$$

and if $K \geq S_0 x^*$ the upper bound of its price is

$$a e^{-R} x^{*\underline{\gamma}-\overline{\gamma}}\left[S_0 e^{\frac{1}{2}(\underline{\gamma}-1)^2\sigma^2} N\left(\frac{-(\underline{\gamma}-1)\sigma^2 - \ln \frac{K}{S_0}}{\sigma}\right) - K e^{\frac{1}{2}\underline{\gamma}^2\sigma^2} N\left(\frac{-\underline{\gamma}\sigma^2 - \ln \frac{K}{S_0}}{\sigma}\right)\right], \qquad (3)$$

where

$$a = 1/\left[e^{\frac{1}{2}\overline{\gamma}^2\sigma^2} N\left(\frac{\ln x^* + \overline{\gamma}\sigma^2}{\sigma}\right) + x^{*\underline{\gamma}-\overline{\gamma}} e^{\frac{1}{2}\underline{\gamma}^2\sigma^2} N\left(\frac{-\underline{\gamma}\sigma^2 - \ln x^*}{\sigma}\right)\right]. \qquad (4)$$

For a call option with strike price K, the lower bound of its price is obtained using the same expressions given above while replaying $\overline{\gamma}$ by $\underline{\gamma}$ and $\underline{\gamma}$ by $\overline{\gamma}$.

Proof: See appendix Appendix 1.

7 Conclusions

In this paper we have derived option bounds given the bounds of the pricing representative investor's relative risk aversion. The option bounds are given by two pricing representative investors who have piecewise constant relative risk aversion, where the number of segments is equal to the number of observed options plus two. Moreover, assuming the pricing representative investor's relative risk aversion is bounded above and below by $\overline{\gamma}$ and $\underline{\gamma}$ respectively, then the relative risk aversion of one of the two pricing representatives is equal to $\overline{\gamma}$ at even segments and $\underline{\gamma}$ at odd segments while the other is just the opposite. This implies that the two has the most polarized relative risk aversion, which is quite intuitive.

Bounding the pricing representative investor's relative risk aversion to derive option bounds has two advantages. First, the pricing representative investor's relative risk aversion can be empirically estimated.[5] Second, if log-normality is assumed, then closed form formulas can be derived as in Section 6. Thus the results given in this paper have practical use.

Since this paper reveals the relationship between option pricing bounds and the bounds of investors' coefficients of relative risk aversion, it also presents a potential method for backing out some characteristics of investors' risk aversion.

[5]See, for example, Jackwerth (2000) and Rosenberg and Engle (2002).

Appendix 1 Proof of Proposition 5

We first calculate an integral which we will use repeatedly in the proof. Let

$$g(x;\mu,\sigma,\alpha,\beta,x^*) = \begin{cases} ax^{-\alpha}n(\ln x;\mu,\sigma)/x, & \text{for } x < x^* \\ ax^{*\beta-\alpha}x^{-\beta}n(\ln x;\mu,\sigma)/x, & \text{for } x \geq x^* \end{cases}$$

where $n(x;\mu,\sigma)$ is the normal density with mean μ and standard deviation σ and the factor $x^{*\beta-\alpha}$ is used to make the function continuous at $x = x^*$.

For any $0 < A < x^*$ we have

$$\begin{aligned}
&\int_A^\infty g(x;\mu,\sigma,\alpha,\beta,x^*)dx \\
&= \int_A^{x^*} ax^{-\alpha}n(\ln x;\mu,\sigma)d\ln x + x^{*\beta-\alpha}\int_{x^*}^\infty ax^{-\beta}n(\ln x;\mu,\sigma)d\ln x \\
&= \int_{\ln A}^{\ln x^*} ae^{\omega(\alpha)\sigma^2}n(x;\mu-\alpha\sigma^2,\sigma)dx + x^{*\beta-\alpha} \\
&\quad \int_{\ln x^*}^\infty ae^{\omega(\beta)\sigma^2}n(x;\mu-\beta\sigma^2,\sigma)dx \\
&= ae^{\omega(\alpha)\sigma^2}(N(\frac{\ln x^* - (\mu-\alpha\sigma^2)}{\sigma}) - N(\frac{\ln A - (\mu-\alpha\sigma^2)}{\sigma})) \\
&\quad + ax^{*\beta-\alpha}e^{\omega(\beta)\sigma^2}(1 - N(\frac{\ln x^* - (\mu-\beta\sigma^2)}{\sigma})) \\
&= ae^{\omega(\alpha)\sigma^2}(N(\frac{\ln x^* - (\mu-\alpha\sigma^2)}{\sigma}) - N(\frac{\ln A - (\mu-\alpha\sigma^2)}{\sigma})) \\
&\quad + ax^{*\beta-\alpha}e^{\omega(\beta)\sigma^2}N(\frac{\mu-\beta\sigma^2-\ln x^*}{\sigma})
\end{aligned} \tag{5}$$

where

$$\omega(x) = -\frac{\mu}{\sigma^2}x + \frac{1}{2}x^2,$$

and $N(x)$ is the cumulative probability function of the standardized normal distribution.

Similarly to (5) for $A \geq x^*$ we have

$$\int_A^\infty g(x;\mu,\sigma,\alpha,\beta,x^*)dx = ax^{*\beta-\alpha}e^{\omega(\beta)\sigma^2}N(\frac{\mu-\beta\sigma^2-\ln A}{\sigma}). \tag{6}$$

Before we start to derive the formula for the option bounds, we have to determine the values of a and x^* in the pricing kernel which gives the option bounds. For convenience we normalize all prices by the current underlying stock price S_0. Let $f(x) = n(\ln x, \mu, \sigma)/x$. Let $g_+(x)$ be such that $g_+(x)f(x)$ as a risk neutral p.d.f gives the upper option bound. Since it has two-segment piecewise constant elasticity, we have

$$g_+(x) = \{ \begin{array}{ll} ae^{-\alpha \ln x}, & \text{for } x < x^* \\ a{x^*}^{\beta-\alpha} e^{-\beta \ln x}, & \text{for } x \geq x^* \end{array} \tag{7}$$

Since $\int_0^\infty g_+(x)f(x)dx = 1$, from (5) we have

$$ae^{\omega(\alpha)\sigma^2} N(\frac{\ln x^* - (\mu - \alpha\sigma^2)}{\sigma}) + a{x^*}^{\beta-\alpha} e^{\omega(\beta)\sigma^2} N(\frac{\mu - \beta\sigma^2 - \ln x^*}{\sigma}) = 1.$$

From this we immediately obtain (4).

Moreover, since $e^{-R}\int_0^\infty xg_+(x)f(x)dx = 1$, from (5) we have

$$\begin{aligned} &ae^{\omega(\alpha-1)\sigma^2} N(\frac{\ln x^* - (\mu - (\alpha-1)\sigma^2)}{\sigma}) + a{x^*}^{\beta-\alpha} e^{\omega(\beta-1)\sigma^2} \\ &N(\frac{\mu - (\beta-1)\sigma^2 - \ln x^*}{\sigma}) = e^R. \end{aligned}$$

Substituting (4) into the above equation we obtain (1). Solving this equation, we can obtain x^*.

We now start to derive the formula for the option bounds. Given a call option c with strike price $K > 0$, if $\hat{K} = K/S_0 < x^*$, its normalized time T price is given by

$$e^{-R}\int_{\hat{K}}^\infty (x - \hat{K}) g_+(x)f(x)dx = ae^{-R}\int_{\hat{K}}^\infty xg_+(x)f(x)dx - \hat{K}e^{-R}\int_{\hat{K}}^\infty g_+(x)f(x)dx.$$

Applying (5) we obtain the normalized call price

$$
\begin{aligned}
& ae^{\omega((\alpha-1))\sigma^2-R}(N(\frac{\ln x^* - (\mu-(\alpha-1)\sigma^2)}{\sigma}) - N(\frac{\ln \hat{K} - (\mu-(\alpha-1)\sigma^2)}{\sigma})) \\
& + ax^{*\beta-\alpha}e^{\omega(\beta-1)\sigma^2-R}N(\frac{\mu-(\beta-1)\sigma^2-\ln x^*}{\sigma}) \\
& -\hat{K}[ae^{\omega(\alpha)\sigma^2-R}(N(\frac{\ln x^* - (\mu-\alpha\sigma^2)}{\sigma}) - N(\frac{\ln \hat{K} - (\mu-\alpha\sigma^2)}{\sigma})) \\
& + ax^{*\beta-\alpha}e^{\omega(\beta)\sigma^2-R}N(\frac{\mu-\beta\sigma^2-\ln x^*}{\sigma})].
\end{aligned}
$$

This implies (2).

If $\hat{K} = K/S_0 \geq x^*$ from (6) we obtain the normalized call price

$$
ae^{-R}x^{*\beta-\alpha}[e^{\omega(\beta-1)\sigma^2}N(\frac{\mu-(\beta-1)\sigma^2-\ln \hat{K}}{\sigma}) - \hat{K}e^{\omega(\beta)\sigma^2}N(\frac{\mu-\beta\sigma^2-\ln \hat{K}}{\sigma})].
$$

This implies (3). Thus the upper bound of the call price is obtained.

Note that the pricing kernel which gives the lower bound of the call price has the same expression as the one gives the upper bound of the call price with α replaced by β and β replaced by α. Thus the lower bound of the call price is obtained using the same expressions as the upper bound while replaying α by β and β by α. Q.E.D.

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