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arbitrage opportunities**

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Stochastic Dominance Option Bounds and N th
Order Arbitrage Opportunities

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Abstract

In this paper we first derive N th order stochastic dominance option bounds from concurrently expiring options. We show that these bounds are given by pricing kernels that have piecewise constant $(N - 2)$ th derivatives. When these option bounds are violated there are N th order arbitrage opportunities interpreted as (weighted average) conditional expected return comparison. We then establish a way to explore these arbitrage opportunities in option markets.

Keywords: Option bounds, Option pricing, Stochastic Dominance.

JEL Classification Numbers: G13.

Introduction

It has been recognized that meaningful option bounds can be obtained under less strong assumptions than exact option prices. Perrakis and Ryan (1984), Ritchken (1985), and Levy (1985) derive option bounds under the assumption of risk aversion or second order stochastic dominance.¹ Ritchken and Kuo (1989) derive option bounds under the assumption of higher order stochastic dominance rules. Basso and Pianca (1997) and Mathur and Ritchken (2000) obtain option pricing bounds by assuming decreasing absolute (relative) risk aversion (hereafter DARA (DRRA)).

Ryan (2003) tightens the second order stochastic dominance option bounds by using the observed price of one concurrently expiring option at a time. Huang (2004b) uses a new methodology to further improve second stochastic dominance option bounds and discusses the second order arbitrage opportunities in the markets of concurrently expiring options. The methodology is presented by Huang (2004a) which takes the advantage of options's distinctive features. Using the same methodology, Huang (2004c) improves the DARA and DRRA bounds by using the observed prices of concurrently expiring options. Huang (2004d) derives option bounds from concurrently expiring options when the pricing representative investor's relative risk aversion is bounded.

¹We use SSD, TSD, and NSD to denote the second, third, and N th order stochastic dominance.

In this paper we tighten the N th ($N \geq 3$) stochastic dominance (hereafter NSD) option bounds by using the observed prices of concurrently expiring options. We show that given the prices of a unit bond, underlying stock, and n option prices, the k th order stochastic dominance option bounds are given by a pricing kernel whose $(N - 2)$ th derivative is $(n/2)$ -segmented and piecewise constant if n is even or $((n + 1)/2)$ -segmented and piecewise constant if n is odd.

When the N th stochastic dominance option bounds are violated then there are N th order arbitrage opportunities. An N th order arbitrage opportunity can be similarly understood to the second order arbitrage opportunity, which is interpreted by Ryan (2003) as (conditional) expected return comparison.² An N th order arbitrage opportunity can be interpreted as a weighted average conditional expected return comparison but the conditional expectation has to be taken average $N - 2$ times with less weights on lower stock prices. We present the N th order arbitrage portfolios which can be used to make profits when the NSD option bounds are violated.

This paper is also related to the important works by Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000). Cochrane and Saa-Requejo derive option bounds using restrictions on the volatility of the pricing kernel, while Bernardo and Ledoit derive option pricing bounds using restrictions on the deviation of the pricing kernel from a benchmark pricing kernel.

Other related works include Lo (1987), Grundy (1991), and Constantinides

²See Huang (2004b).

and Zariphopoulou (1999, 2001) who all derive option bounds under different conditions.

The structure of the remaining paper is as follows: In Section 1 we derive option bounds from concurrently expiring options when the third order stochastic dominance rule applies. In Section 2 we generalize the results in Section 1 to the case where the N th ($N \geq 3$) order stochastic dominance rule applies. In Section 3 we present the arbitrage portfolios. The final section concludes the paper.

1 TSD Option Bounds

In this section we derive option bounds from concurrently expiring options assuming third order stochastic dominance. According to Ritchken and Kuo (1989), applying third order stochastic dominance we have a pricing kernel that is decreasing and convex in the underlying stock price.

1.1 Option Bound Problem and Its Dual

Given the prices of a unit bond, a stock and some European options written on the stock with the same maturity, we want to know the bounds on the price of another option which has the same maturity when N th ($N \geq 3$) order stochastic dominance rule applies. That is,

$$\max \text{ (or min) } E(c_X(S_T)\phi(S_T))B_0$$

subject to

$$\phi(x) \geq 0; \quad \phi'(x) \leq 0; \quad \phi'(x) \text{ is increasing in } x.$$

$$E(\phi(S_T)) = 1$$

$$E(S_T \phi(S_T)) B_0 = S_0$$

$$E(c_i(S_T) \phi(S_T)) B_0 = c_{i0}, \quad i = 1, \dots, n.$$

The dual problem is³

$$\min \text{ (or max) } \alpha_1 B_0 + \alpha_2 S_0 + \sum_{i=1}^n \alpha_i c_0^i$$

subject to

$$\begin{aligned} & \frac{\int_0^s E(\alpha_1 + \alpha_2 x + \sum_{i=1}^n \alpha_i c_i(x) | x < x_0) Pr(x < x_0) dx_0}{\int_0^{S_T} Pr(x < x_0) dx_0} \\ & \geq (\leq) \frac{\int_0^{S_T} E(c_X(x) | x < x_0) Pr(x < x_0) dx_0}{\int_0^{S_T} Pr(x < x_0) dx_0}, \quad \text{for all } s. \end{aligned} \quad (1)$$

The above dual problem suggests a third order arbitrage opportunity if the option bound is violated. Suppose, for example, the upper bound is violated. Then by selling the over priced option c^X and buying the arbitrage portfolio $L = (\alpha_1, \dots, \alpha_{n+2})$, we make a profit $(c_0^X - L_0 > 0)$ at time 0. Now consider the net payoff $(L(S_t) - C_0^X(S_t))$ at the maturity of the option. The conditional expectation of the net payoff of our position

$$\frac{\int_0^s E(L(S_t) - C_0^X(S_t) | S_t < x_0) Pr(S_t < x_0) dx_0}{\int_0^{S_T} Pr(S_t < x_0) dx_0}$$

will be non-negative for every s .

³The proof of the duality can be given by using the Ritchken and Kuo's (1989) results in the discrete case. For brevity, it is omitted.

To understand third order arbitrage opportunities we first analyze the above expression. Note is the expected net payoff under the condition that the stock price will be lower than x_0 . In the case of second order arbitrage opportunity, this conditional expectation of the net payoff will be always non-negative.⁴ However, in the case of the third order arbitrage opportunity, this is not guaranteed to be non-negative; only a weighted average of this conditional expectation is guaranteed to be non-negative. Since for $E(L(S_t) - C_0^X(S_t)|S_t < x_0)$, the weight is $Pr(S_t < x_0)$, the lower x_0 , the less weight the conditional expectation carries. Hence the comparison of these weighted average conditional expectations is weaker the comparison of the conditional expectations. Thus the third order arbitrage opportunities are weaker than the second order arbitrage opportunities.

To solve the above option bound problem, we first solve a similar but more general problem in which we assume that not only the third order stochastic dominance rule applies but also the absolute value of the pricing kernel's first derivative is bounded from above.

We will show in this paper that under this condition, the option bounds are given by a pricing kernel that has piecewise constant first derivative, where the number of segments of the risk aversion depends on the number of observed option prices.

Moreover, we will see that for an even number of observed option prices the pricing kernel that gives the option bounds has a certain pattern while for an

⁴See Ryan (2003) or Huang (2004b).

odd number of observed option prices the pricing kernel that gives the option bounds has a different pattern. Thus in order to explain the solutions more clearly we start with the case where we have only one observed option price then continue with the case where we have two observed options. Building on the above two cases we explore the general case where we have n observed options.

1.2 The Case with One Observed Option

In this subsection we deal with the case where we observe the price of one concurrently expiring option. Before we proceed, we introduce two lemmas.

Lemma 1 (FSS (1999)) *Assume two pricing kernels give the same stock price. If they intersect twice, then the pricing kernel with fatter tails gives higher prices of convex-payoff contingent claims written on the stock.*

Proof: See the proof of Theorem 1 in FSS (1999).

Lemma 2 *Assume two pricing kernels give the same prices of the underlying stock and an option with strike price K . If they intersect three times, then the pricing kernel with fatter left tail will give higher [lower] prices for all options with strike prices below [above] K than the other.*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's

first derivative is bounded from above.

Lemma 3 *Assume the pricing kernel is decreasing and convex in S_t and its first derivative is bounded below by $-\delta$. Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 and the current price of an option with strike price K is c_{K0} .*

- *Then the upper bound for an option with strike price below K is given by a pricing kernel, $\phi_1^{**}(x)$, which has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$. That is,*

$$\phi_1^{**}(x) = \begin{cases} \delta^{**}(s_2^{**} - s_1^{**}) + \delta(s_1^{**} - x), & x < s_1^{**} \\ \delta^{**}(s_2^{**} - x), & x \in (s_1^{**}, s_2^{**}) \\ 0, & x \geq s_2^{**}, \end{cases}$$

where s_1^{**} , s_2^{**} , and δ^{**} are to be decided such that $E(\phi_1^{**}(S_t)) = 1$, $E(S_t \phi_1^{**}(S_t))/B_0 = S_0$, and $E(c_K(S_t) \phi_1^{**}(S_t))/B_0 = c_{K0}$.

- *The lower bound for an option with strike price below K is given by a pricing kernel, $\phi_1^*(x)$, which has a two-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta^*$ for $x < s^*$, and zero for $x > s^*$, and its value is zero at its right tail. That is,*

$$\phi_1^*(x) = \begin{cases} b + \delta^*(s^* - x), & x < s^* \\ b, & x \geq s^*, \end{cases}$$

where b , δ^* , and s^* are to be decided such that $E(\phi_1^*(S_t)) = 1$, $E(S_t \phi_1^*(S_t))/B_0 = S_0$, and $E(c_K(S_t) \phi_1^*(S_t))/B_0 = c_{K0}$.

- The upper (lower) bound for an option with strike price above K is given by pricing kernel $\phi_0^*(x)$ ($\phi_0^{**}(x)$).

Proof: From Lemma 2 we need only prove that the true pricing kernel intersects the pricing kernels that give the option bounds exactly three times and then examine which one has a fatter left tail.

We first examine ϕ_1^{**} . Note it has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is zero at its right tail. Obviously we must have $0 < \delta^{**} < \delta$. Otherwise the true pricing kernel will intersect ϕ_1^{**} at most twice. In this case, applying Lemma 1, we find that the two pricing kernels cannot give the same observed option price. From this, we can immediately conclude that the true pricing kernel intersects ϕ_1^{**} exactly three times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

Proposition 1 *Assume the pricing kernel is decreasing and convex in S_t . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 and the current price of an option with strike price K is c_{K0} .*

- Then the upper bound for an option with strike price below K is given by the pricing kernel $\varphi_1^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_1^{**}(S_t)$, where $p(S_t)$ is the true probability density function and $\delta(S_t)$ is the Dirac function and

$$f_1^{**}(S_t) = \begin{cases} \delta^{**}(s^{**} - x), & x < s^{**} \\ 0, & x \geq s^{**}, \end{cases}$$

where a_0 , s^{**} , and δ^{**} are to be decided such that $E(\varphi_1^{**}(S_t)) = 1$, $E(S_t \varphi_1^{**}(S_t))/B_0 = S_0$, and $E(c_K(S_t) \varphi_1^{**}(S_t))/B_0 = c_{K0}$.

- The lower bound for an option with strike price below K is given by the pricing kernel $\varphi_1^*(x) = \phi_1^*(x)$.
- The upper (lower) bound for an option with strike price above K is given by pricing kernel $\phi_0^*(x)$ ($\phi_0^{**}(x)$).

Proof: Let $\delta \rightarrow +\infty$; we immediately obtain the result from Lemma 8.

1.3 The Case with Two Observed Options

In this subsection we deal with the case where we have two observed concurrently expiring options. We first introduce a lemma.

Lemma 4 *Assume two pricing kernels give the same prices of the underlying stock and two options with strike prices K_1 and K_2 , where $K_1 < K_2$. If they intersect four times, then the pricing kernel with fatter left tail will give higher (lower) prices for options with strike prices outside (inside) (K_1, K_2) .*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's first derivative is bounded from above.

Lemma 5 *Assume the pricing kernel is decreasing and convex in S_t and its first derivative is bounded below by $-\delta$. Assume the current price of a unit bond*

is B_0 , the current price of the underlying stock is S_0 , and the current prices of two options with strike prices K_1 and K_2 are c_{10} and c_{20} respectively.

- Then the upper bound for an option with a strike price below K_1 or above K_2 is given by a pricing kernel, $\phi_2^{**}(x)$, which has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is a positive constant b at its right tail. That is,

$$\phi_2^{**}(x) = \begin{cases} b + \delta^{**}(s_2^{**} - s_1^{**}) + \delta(s_1^{**} - x), & x < s_1^{**} \\ b + \delta^{**}(s_2^{**} - x), & x \in (s_1^{**}, s_2^{**}) \\ b, & x \geq s_2^{**}, \end{cases}$$

where b , s_1^{**} , s_2^{**} , and δ^{**} are to be decided such that $E(\phi_1^{**}(S_t)) = 1$, $E(S_t \phi_1^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_1^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2$.

- The lower bound for an option with a strike price below K_1 or above K_2 is given by a pricing kernel, $\phi_2^*(x)$, which has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta_1^*$ for $x < s_1^*$, $-\delta_2^*$ for $x \in (s_1^*, s_2^*)$, and zero for $x > s_2^*$. That is,

$$\phi_2^*(x) = \begin{cases} \delta_1^*(s_2^* - s_1^*) + \delta_2^*(s_1^* - x), & x < s_1^* \\ \delta_1^*(s_2^* - x), & x \in (s_1^*, s_2^*) \\ 0, & x \geq s_2^*, \end{cases}$$

where s_1^* , s_2^* , δ_1^* , and δ_2^* are to be decided such that $E(\phi_1^*(S_t)) = 1$, $E(S_t \phi_1^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_1^*(S_t))/B_0 = c_{i0}$, $i = 1, 2$.

Proof: From Lemma 4 we need only prove that the true pricing kernel intersects the pricing kernels that give the option bounds exactly four times and then

examine which one has a fatter left tail.

We first examine ϕ_2^{**} . Note it has a three-segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is a positive constant b at its right tail.

Obviously we must have $0 < \delta^{**} < \delta$ and $b > \inf_x \phi(x)$, where $\phi(x)$ is the true pricing kernel. Otherwise the true pricing kernel will intersect ϕ_2^{**} at most three times. In this case, applying Lemma 2, we find that the two pricing kernels cannot give the same two observed option prices. From this, we can immediately conclude that the true pricing kernel intersects ϕ_2^{**} exactly four times. It is not difficult to verify that ϕ_2^{**} has fatter left tail. For ϕ_2^* the proof is similar. Q.E.D.

Proposition 2 *Assume the pricing kernel is decreasing and convex in S_t . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of two options with strike prices K_1 and K_2 are c_{10} and c_{20} respectively.*

- *Then the upper bound for an option with a strike price below K_1 or above K_2 is given by the pricing kernel $\varphi_2^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_2^{**}(S_t)$, where $p(S_t)$ is the true probability density function and $\delta(S_t)$ is the Dirac function and*

$$f_2^{**}(S_t) = \begin{cases} b + \delta^{**}(s^{**} - x), & x < s^{**} \\ b, & x \geq s^{**}, \end{cases}$$

*where a_0 , b , s^{**} , and δ^{**} are to be decided such that $E(\varphi_2^{**}(S_t)) = 1$, $E(S_t \varphi_2^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \varphi_2^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2$.*

- The lower bound for an option with a strike price below K_1 or above K_2 is given by the pricing kernel $\varphi_2^*(x) = \phi_2^*(x)$.

Proof: Let $\delta \rightarrow +\infty$; we immediately obtain the result from Lemma 9.

1.4 The General Case

In this subsection we deal with the case where we have n observed concurrently expiring options. We first introduce a lemma.

Lemma 6 *Assume two pricing kernels give the same prices of the underlying stock and options with strike prices K_1, K_2, \dots, K_n , where $K_1 < K_2 < \dots < K_n$. Let $K_0 = 0$ and $K_{n+1} = +\infty$. If the two pricing kernels intersect $n + 2$ times then the one with fatter left tail will give higher (lower) prices for all options with strike prices between (K_{2i-2}, K_{2i-1}) ((K_{2i-1}, K_{2i})), $i = 1, 2, \dots$*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's first derivative is bounded from above.

Lemma 7 *Assume the pricing kernel is decreasing and convex in S_t and its first derivative is bounded below by $-\delta$. Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of n options with strike prices K_1, \dots, K_n are c_{10}, \dots, c_{n0} respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

- Assume n is odd. Let $m = (n + 1)/2$.

– Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel, $\phi_n^{**}(x)$, which has a $(m+2)$ -segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta_1^{**}$ for $x \in (s_1^{**}, s_2^{**})$, ..., $-\delta_m^{**}$ for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$. That is, $\phi_n^{**}(x) =$

$$\begin{aligned} & \delta_m^{**}(s_{m+1}^{**} - s_m^{**}) + \dots + \delta_1^{**}(s_2^{**} - s_1^{**}) + \delta(s_1^{**} - x), & x < s_1^{**} \\ & \dots\dots \\ & \delta_m^{**}(s_{m+1}^{**} - x), & x \in (s_m^{**}, s_{m+1}^{**}) \\ & 0, & x \geq s_{m+1}^{**}, \end{aligned}$$

where s_1^{**} , ..., s_{m+1}^{**} , δ_1^{**} , ..., and δ_m^{**} are to be decided such that $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

– The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel $\phi_n^*(x)$, which has a $(m+1)$ -segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta_1^*$ for $x < s_1^*$, ..., $-\delta_m^*$ for $x \in (s_{m-1}^*, s_m^*)$, and zero for $x > s_m^*$. That is, $\phi_n^*(x) =$

$$\begin{aligned} & b + \delta_m^*(s_m^* - s_{m-1}^*) + \dots + \delta_2^*(s_2^* - s_1^*) + \delta_1^*(s_1^* - x), & x < s_1^* \\ & \dots\dots \\ & b + \delta_m^*(s_m^* - x), & x \in (s_{m-1}^*, s_m^*) \\ & b, & x \geq s_m^*, \end{aligned}$$

where b , s_1^* , ..., s_m^* , δ_1^* , ..., and δ_m^* are to be decided such that

$E(\phi_n^*(S_t)) = 1$, $E(S_t \phi_n^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^*(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

- Assume n is even. Let $m = n/2$.

- Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel, $\phi_n^{**}(x)$, which has a $(m+2)$ -segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta_1^{**}$ for $x \in (s_1^{**}, s_2^{**})$, ..., $-\delta_m^{**}$ for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is a positive constant b . That is, $\phi_n^{**}(x) =$

$$b + \delta_m^{**}(s_{m+1}^{**} - s_m^{**}) + \dots + \delta_1^{**}(s_2^{**} - s_1^{**}) + \delta(s_1^{**} - x), \quad x < s_1^{**}$$

.....

$$b + \delta_m^{**}(s_{m+1}^{**} - x), \quad x \in (s_m^{**}, s_{m+1}^{**})$$

$$b, \quad x \geq s_{m+1}^{**},$$

where b , s_1^{**} , ..., s_{m+1}^{**} , δ_1^{**} , ..., and δ_m^{**} are to be decided such that $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

- The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel, $\phi_n^*(x)$, which has a $(m+2)$ -segmented and piecewise constant first derivative. More precisely its first derivative is equal to $-\delta_1^*$ for $x < s_1^*$, ..., $-\delta_{m+1}^*$ for $x \in$

(s_m^*, s_{m+1}^*) , and zero for $x > s_{m+1}^*$. That is, $\phi_n^*(x) =$

$$\begin{aligned} & \delta_{m+1}^*(s_{m+1}^* - s_m^*) + \dots + \delta_2^*(s_2^* - s_1^*) + \delta_1^*(s_1^* - x), & x < s_1^* \\ & \dots\dots \\ & \delta_{m+1}^*(s_{m+1}^* - x), & x \in (s_{m-1}^*, s_m^*) \\ & 0, & x \geq s_{m+1}^*, \end{aligned}$$

where $s_1^*, \dots, s_{m+1}^*, \delta_1^*, \dots$, and δ_{m+1}^* are to be decided such that $E(\phi_n^*(S_t)) = 1$, $E(S_t \phi_n^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^*(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

Proof: From Lemma 6 we need only prove that the true pricing kernel intersects the pricing kernels that give the option bounds exactly $(n + 2)$ times and then examine which one has a fatter left tail.

We first examine ϕ_n^{**} . Assume n is odd. Note it has a $(m+2)$ -segmented and piecewise constant first derivative, where $m = (n + 1)/2$. More precisely, its first derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta_1^{**}$ for $x \in (s_1^{**}, s_2^{**})$, ..., $-\delta_m^{**}$ for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is 0.

Obviously we must have $\delta > \delta_1^{**} > \dots \delta_m^{**} > 0$ and $b > \inf_x \phi(x)$, where $\phi(x)$ is the true pricing kernel. Otherwise the true pricing kernel will intersect ϕ_n^{**} at most $n + 1$ times. In this case, applying Lemma 6, we find that the two pricing kernels cannot give the same n observed option prices. From this, we can immediately conclude that the true pricing kernel intersects ϕ_n^{**} exactly $n + 2$ times. It is not difficult to verify that ϕ_n^{**} has fatter left tail.

Assume n is even. Note it has a $(m+2)$ -segmented and piecewise constant first derivative, where $m = n/2$. More precisely, its first derivative is equal to

$-\delta$ for $x < s_1^{**}$, $-\delta_1^{**}$ for $x \in (s_1^{**}, s_2^{**})$, ..., $-\delta_m^{**}$ for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is a positive constant b .

Obviously we must have $\delta > \delta_1^{**} > \dots \delta_m^{**} > 0$ and $b > \inf_x \phi(x)$, where $\phi(x)$ is the true pricing kernel. Otherwise the true pricing kernel will intersect ϕ_n^{**} at most $n + 1$ times. In this case, applying Lemma 6, we find that the two pricing kernels cannot give the same n observed option prices. From this, we can immediately conclude that the true pricing kernel intersects ϕ_n^{**} exactly $n + 2$ times. It is not difficult to verify that ϕ_n^{**} has fatter left tail.

For ϕ_n^* the proof is similar. Q.E.D.

Proposition 3 *Assume the pricing kernel is decreasing and convex in S_t . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of n options with strike prices $K_1, \dots,$ and K_n are $c_{10}, \dots,$ and c_{n0} respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

- Assume n is odd. Let $m = (n + 1)/2$.

– Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots,$ is given by the pricing kernel $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_n^{**}(S_t)$, where $p(S_t)$ is the true probability density function, $\delta(S_t)$ is the Dirac function, and $f_n^{**}(x) =$

$$\delta_m^{**}(s_{m+1}^{**} - s_m^{**}) + \dots + \delta_2^{**}(s_3^{**} - s_2^{**}) + \delta_1^{**}(s_2^{**} - x), \quad x < s_2^{**}$$

.....

$$\delta_m^{**}(s_{m+1}^{**} - x), \quad x \in (s_m^{**}, s_{m+1}^{**})$$

$$0, \quad x \geq s_{m+1}^{**},$$

where $a_0, s_2^*, \dots, s_{m+1}^*, \delta_1^{**}, \dots,$ and δ_m^{**} are to be decided such that $E(\phi_n^{**}(S_t)) = 1, E(S_t \phi_n^{**}(S_t))/B_0 = S_0,$ and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}, i = 1, 2, \dots, n.$

- The lower bound for options with strike prices between $(K_{2i-2}, K_{2i-1}),$ $i = 1, 2, \dots,$ is given by the pricing kernel $\varphi_n^*(x) =$

$$\begin{aligned}
 & b + \delta_m^*(s_m^* - s_{m-1}^*) + \dots + \delta_2^*(s_2^* - s_1^*) + \delta_1^*(s_1^* - x), & x < s_1^* \\
 & \dots\dots \\
 & b + \delta_m^*(s_m^* - x), & x \in (s_{m-1}^*, s_m^*) \\
 & b, & x \geq s_m^*,
 \end{aligned}$$

where $b, s_1^*, \dots, s_m^*, \delta_1^*, \dots,$ and δ_m^* are to be decided such that $E(\phi_n^*(S_t)) = 1, E(S_t \phi_n^*(S_t))/B_0 = S_0,$ and $E(c_i(S_t) \phi_n^*(S_t))/B_0 = c_{i0}, i = 1, 2, \dots, n.$

- Assume n is even. Let $m = n/2.$

- Then the upper bound for options with strike prices between $(K_{2i-2}, K_{2i-1}), i = 1, 2, \dots,$ is given by a pricing kernel $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + b + f_n^{**}(S_t),$ where $p(S_t)$ is the true probability density function, $\delta(S_t)$ is the Dirac function, and $f_n^{**}(x) =$

$$\begin{aligned}
 & \delta_m^{**}(s_{m+1}^{**} - s_m^{**}) + \dots + \delta_2^{**}(s_3^{**} - s_2^{**}) + \delta_1^{**}(s_2^{**} - x), & x < s_2^{**} \\
 & \dots\dots \\
 & \delta_m^{**}(s_{m+1}^{**} - x), & s_m^{**} < x < s_{m+1}^{**} \\
 & 0, & x \geq s_{m+1}^{**},
 \end{aligned}$$

where $a_0, b, s_2^{**}, \dots, s_{m+1}^{**}, \delta_1^{**}, \dots,$ and δ_m^{**} are to be decided such that

$E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

– The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel $\varphi_n^*(x) =$

$$\begin{aligned} & \delta_{m+1}^*(s_{m+1}^* - s_m^*) + \dots + \delta_2^*(s_2^* - s_1^*) + \delta_1^*(s_1^* - x), & x < s_1^* \\ & \dots\dots\dots \\ & \delta_{m+1}^*(s_{m+1}^* - x), & x \in (s_{m-1}^*, s_m^*) \\ & 0, & x \geq s_{m+1}^*, \end{aligned}$$

where s_1^* , ..., s_{m+1}^* , δ_1^* , ..., and δ_{m+1}^* are to be decided such that $E(\phi_n^*(S_t)) = 1$, $E(S_t \phi_n^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^*(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

Proof: Let $\delta \rightarrow +\infty$; we immediately obtain the result from Lemma 9.

2 NSD Option Bounds

In this section we apply the N th order stochastic dominance rule to derive option bounds from concurrently expiring options. According to Ritchken and Kuo (1989), applying N th order stochastic dominance we have such a pricing kernel that its derivatives alternate to be negative and positive up to the $(N - 1)$ th order.

2.1 Option Bound Problem and Its Dual

Given the prices of a unit bond, a stock and some European options written on the stock with the same maturity, we want to know the bounds on the price of another option which has the same maturity when N th ($N \geq 3$) order stochastic dominance rule applies. That is,

$$\max \text{ (or min) } E(c_X(S_T)\phi(S_T))B_0$$

subject to

$$\phi^{(i)}(x) \geq 0, \text{ for even } i \leq N; \quad \phi^{(i)}(x) \leq 0, \text{ for odd } i \leq N$$

$$E(\phi(S_T)) = 1$$

$$E(S_T\phi(S_T))B_0 = S_0$$

$$E(c_i(S_T)\phi(S_T))B_0 = c_{i0}, \quad i = 1, \dots, n.$$

The dual problem⁵

$$\min \text{ (or max) } \alpha_1 B_0 + \alpha_2 S_0 + \sum_{i=1}^n \alpha_{n+2} c_{i0}$$

subject to

$$\frac{\int_0^{S_T} \int_0^{x_k} \dots \int_0^{x_1} E(\alpha_1 + \alpha_2 x + \sum_{i=1}^n \alpha_{n+2} c_i(x) | x < x_0) Pr(x < x_0) dx_0 \dots dx_k}{\int_0^{S_T} \int_0^{x_k} \dots \int_0^{x_1} Pr(x < x_0) dx_0 \dots dx_k} \\ \geq (\leq) \frac{\int_0^{S_T} \int_0^{x_k} \dots \int_0^{x_1} E(c_X(x) | x < x_0) Pr(x < x_0) dx_0 \dots dx_k}{\int_0^{S_T} \int_0^{x_k} \dots \int_0^{x_1} Pr(x < x_0) dx_0 \dots dx_k}, \quad \text{for all } S_T,$$

where $k = N - 3$.

⁵The proof of the duality is similar to the case of TSD. For brevity, it is omitted.

The above dual problem suggests an N th ($N > 3$) order arbitrage opportunity if the option bound is violated. An N th order arbitrage opportunity can be understood similar to the third order arbitrage opportunity. Obviously, an N th order arbitrage opportunity is weaker than an $(N - 1)$ th order arbitrage opportunity, because the weighted average conditional expectations in the case of $(N - 1)$ th order are taken weighted average again with the same weights. This means in the case of N th order the conditional expected returns in the states of lower stock prices carry even less weights than in the case of $(N - 1)$ th order.

As in the case of TSD, to solve the above two problems we first solve a similar but more general problem in which we assume that not only the N th order stochastic dominance rule applies but also the absolute value of the pricing kernel's N th derivative is bounded above.

We will show in this paper that under this condition, the option bounds are given by pricing kernels which have piecewise constant $(N - 2)$ th derivative, where the number of segments of the $(N - 2)$ th derivative depends on the number of observed options. Moreover, we will see that for an even number of observed options the pricing kernel which gives the option bounds has a certain pattern while for an odd number of observed options the pricing kernel which gives option bounds has a different pattern. Thus in order to explain the solutions more clearly we start with the case where we have only one observed option price then continue with the case where we have two observed options. In the end we solve the general case where we have n observed options.

2.2 The Case with One Observed Option

In this subsection we deal with the case where we observe the price of one concurrently expiring option.

We first derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's $(N - 2)$ th derivative is bounded from above.

Lemma 8 *Assume the pricing kernel $\phi(x)$ satisfies the N th order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x . Assume $|\phi^{(N-2)}(x)|$ is bounded above by δ . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 and the current price of an option with strike price K is c_{K0} .*

- *Then the upper option bound is given by a pricing kernel $(\phi_1^{**}(x))$ that has a three-segmented and piecewise constant $(N - 2)$ th derivative, where its $(N - 2)$ th derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is zero at its right tail. More precisely*

$$\phi_1^{**}(x) = \begin{cases} \delta \frac{(s_1^{**} - x)^{N-2}}{(N-2)!} + \delta^{**} \sum_1^{N-2} \frac{(s_2^{**} - s_1^{**})^i}{i!} \frac{(s_1^{**} - x)^{(N-2-i)}}{(N-2-i)!}, & x < s_1^{**} \\ \frac{\delta^{**}}{(N-2)!} (s_2^{**} - x)^{N-2}, & x \in (s_1^{**}, s_2^{**}) \\ 0, & x \geq s_2^{**}, \end{cases}$$

where s_1^{**} , s_2^{**} , and δ^{**} are to be decided such that $E(\phi_1^{**}(S_t)) = 1$, $E(S_t \phi_1^{**}(S_t))/B_0 = S_0$, and $E(c_K(S_t) \phi_1^{**}(S_t))/B_0 = c_{K0}$.

- *The lower option bound is given by a pricing kernel $(\phi_0^*(x))$ that has a*

two-segmented and piecewise constant $(N-2)$ th derivative. More precisely its $(N-2)$ th derivative is equal to $-\delta^*$ for $x < s^*$, and zero for $x > s^*$.

That is,

$$\phi_1^*(x) = \begin{cases} b + \frac{\delta^*}{(N-2)!}(s^* - x)^{N-2}, & x < s^* \\ b, & x \geq s^*, \end{cases}$$

where b , δ^* , and s^* are to be decided such that $E(\phi_1^*(S_t)) = 1$, $E(S_t \phi_0^*(S_t))/B_0 = S_0$, and $E(c_K(S_t) \phi_1^*(S_t))/B_0 = c_{K0}$.

Proof: From Lemma 2 we need only prove that the true pricing kernel intersects the pricing kernels that give the option bounds exactly three times and then examine which one has a fatter left tail.

We first examine ϕ_1^{**} . Note it has a three-segmented and piecewise constant $(N-2)$ th derivative. More precisely its $(N-2)$ th derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is zero at its right tail. Obviously we must have $0 < \delta^{**} < \delta$. Otherwise the true pricing kernel will intersect ϕ_1^{**} at most twice. In this case, applying Lemma 1, we find that the two pricing kernels cannot give the same observed option price. From this, we can immediately conclude that the true pricing kernel intersects ϕ_1^{**} exactly three times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

Proposition 4 *Assume the pricing kernel $\phi(x)$ satisfies the N th order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 and the current price of an*

option with strike price K is c_{K0} .

- Then the upper option bound is given by the pricing kernel $\varphi_1^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_1^{**}(S_t)$, where $p(S_t)$ is the true probability density function and $\delta(S_t)$ is the Dirac function and

$$f_1^{**}(S_t) = \begin{cases} \frac{\delta^{**}}{(N-2)!} (s^{**} - x)^{N-2}, & x < s^{**} \\ 0, & x \geq s^{**}, \end{cases}$$

where a_0 , s^{**} , and δ^{**} are to be decided such that $E(\varphi_1^{**}(S_t)) = 1$, $E(S_t \varphi_1^{**}(S_t))/B_0 = S_0$, and $E(c_K(S_t) \varphi_1^{**}(S_t))/B_0 = c_{K0}$.

- The lower option bound is given by the pricing kernel $\varphi_1^*(x) = \phi_1^*(x)$.

2.3 The Case with Two Observed Options

In this subsection we deal with the case where we observe the prices of two concurrently expiring options.

We first derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's $(N - 2)$ th derivative is bounded from above.

Lemma 9 *Assume the pricing kernel $\phi(x)$ satisfies the N th order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x . Assume $|\phi^{(N-2)}(x)|$ is bounded above by δ . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of two options with strike prices K_1 and K_2 are c_{10} and c_{20} respectively.*

- Then the upper bound for an option with a strike price below K_1 or above K_2 is given by a pricing kernel $(\phi_2^{**}(x))$ that has a three-segmented and piecewise constant $(N-2)$ th derivative, where its $(N-2)$ th derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is a positive constant b at its right tail. More precisely, $\phi_2^{**}(x) = b +$

$$\begin{cases} \frac{\delta}{(N-2)!}(s_1^{**} - x)^{N-2} + \delta^{**} \sum_1^{N-2} \frac{(s_2^{**} - s_1^{**})^i}{i!} \frac{(s_1^{**} - x)^{(N-2-i)}}{(N-2-i)!}, & x < s_1^{**} \\ \frac{\delta^{**}}{(N-2)!}(s_2^{**} - x)^{N-2}, & s_1^{**} < x < s_2^{**} \\ 0, & x \geq s_2^{**}, \end{cases}$$

where b , s_1^{**} , s_2^{**} , and δ^{**} are to be decided such that $E(\phi_1^{**}(S_t)) = 1$, $E(S_t \phi_1^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_1^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2$.

- The lower bound for an option with a strike price below K_1 or above K_2 is given by a pricing kernel $(\phi_2^*(x))$ that has a three-segmented and piecewise constant $(N-2)$ th derivative. More precisely its $(N-2)$ th derivative is equal to $-\delta_1^*$ for $x < s_1^*$, $-\delta_2^*$ for $x \in (s_1^*, s_2^*)$, and zero for $x > s_2^*$. That is,

$$\phi_2^*(x) = \begin{cases} \frac{\delta_1^*}{(N-2)!}(s_1^* - x)^{N-2} + \delta_2^* \sum_1^{N-2} \frac{(s_2^* - s_1^*)^i}{i!} \frac{(s_1^* - x)^{(N-2-i)}}{(N-2-i)!}, & x < s_1^* \\ \frac{\delta_2^*}{(N-2)!}(s_2^* - x)^{N-2}, & x \in (s_1^*, s_2^*) \\ 0, & x \geq s_2^*, \end{cases}$$

where s_1^* , s_2^* , δ_1^* , and δ_2^* are to be decided such that $E(\phi_1^*(S_t)) = 1$, $E(S_t \phi_1^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_1^*(S_t))/B_0 = c_{i0}$, $i = 1, 2$.

Proof: From Lemma 4 we need only prove that the true pricing kernel intersects the pricing kernels that give the option bounds exactly four times and then examine which one has a fatter left tail.

We first examine ϕ_2^{**} . Note it has a three-segmented and piecewise constant $(N - 2)$ th derivative. More precisely its $(N - 2)$ th derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta^{**}$ for $x \in (s_1^{**}, s_2^{**})$, and zero for $x > s_2^{**}$, and its value is a positive constant b at its right tail.

Obviously we must have $0 < \delta^{**} < \delta$ and $b > \inf_x \phi(x)$, where $\phi(x)$ is the true pricing kernel. Otherwise the true pricing kernel will intersect ϕ_2^{**} at most three times. In this case, applying Lemma 2, we find that the two pricing kernels cannot give the same two observed option prices. From this, we can immediately conclude that the true pricing kernel intersects ϕ_2^{**} exactly four times. It is not difficult to verify that ϕ_2^{**} has fatter left tail. For ϕ_2^* the proof is similar. Q.E.D.

Proposition 5 *Assume the pricing kernel $\phi(x)$ satisfies the N th order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of two options with strike prices K_1 and K_2 are c_{10} and c_{20} respectively.*

- *Then the upper bound for an option with a strike price below K_1 or above K_2 is given by the pricing kernel $\varphi_2^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_2^{**}(S_t)$, where $p(S_t)$ is the true probability density function and $\delta(S_t)$ is the Dirac function and*

$$f_2^{**}(S_t) = \begin{cases} b + \frac{\delta^{**}}{(N-2)!}(s^{**} - x)^{N-2}, & x < s^{**} \\ b, & x \geq s^{**}, \end{cases}$$

*where a_0 , b , s^{**} , and δ^{**} are to be decided such that $E(\varphi_2^{**}(S_t)) = 1$, $E(S_t \varphi_2^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \varphi_2^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2$.*

- The lower bound for an option with a strike price below K_1 or above K_2 is given by the pricing kernel $\varphi_2^*(x) = \phi_2^*(x)$.

2.4 The General Case

In this subsection we deal with the case where we observe the prices of n currently expiring options.

We first derive the option bounds under the assumption that the third order stochastic dominance rule applies and the absolute value of the pricing kernel's $(N - 2)$ th derivative is bounded from above.

Lemma 10 *Assume the pricing kernel $\phi(x)$ satisfies the N th order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x . Assume $|\phi^{(N-2)}(x)|$ is bounded above by δ . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of n options with strike prices K_1, \dots , and K_n are c_{10}, \dots , and c_{n0} respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

- Assume n is odd. Let $m = (n + 1)/2$.
 - Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel $(\phi_n^{**}(x))$ that has a $(m+2)$ -segmented and piecewise constant $(N-2)$ th derivative, where the absolute of value of its $(N-2)$ th derivative is equal to δ for $x < s_1^{**}$, δ_1^{**} for $x \in (s_1^{**}, s_2^{**})$, ..., δ_m^{**} for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is 0. More precisely,

$\phi_n^{**}(x) = \phi_{ni}^{**}(x)$, for $x \in (s_{i-1}^{**}, s_i^{**})$, $i = 1, \dots, m+2$, where $s_0^{**} = 0$,
 $s_{m+2}^{**} = +\infty$, $\phi_{n(m+2)}^{**}(x) = 0$, $\phi_{n(m+1)}^{**}(x) = \frac{\delta_m^{**}}{(N-2)!}(s_{m+1}^{**} - x)^{N-2}$,
and for $i = 1, \dots, m$,

$$\begin{aligned} \phi_{ni}^{**}(x) &= \phi_{n(i+2)}^{**}(s_{i+1}^{**}) + \delta_{i-1}^{**} \frac{(s_i^{**} - x)^{N-2}}{(N-2)!} \\ &\quad + \delta_i^{**} \sum_1^{N-2} \frac{(s_{i+1}^{**} - s_i^{**})^j}{j!} \frac{(s_i^{**} - x)^{(N-2-j)}}{(N-2-j)!}, \end{aligned}$$

where $\delta_0^{**} = \delta$ while $s_1^{**}, \dots, s_{m+1}^{**}, \delta_1^{**}, \dots$, and δ_m^{**} are to be decided
such that $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

– The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) ,
 $i = 1, 2, \dots$, is given by a pricing kernel $(\phi_{on}^*(x))$ that has a $(m+1)$ -
segmented and piecewise constant $(N-2)$ th derivative, where the ab-
solute of value of its $(N-2)$ th derivative is equal to δ_1^* for $x < s_1^*$,
 \dots, δ_m^* for $x \in (s_{m-1}^*, s_m^*)$, and zero for $x > s_m^*$. More precisely,
 $\phi_{on}^*(x) = \phi_{ni}^*(x)$, for $x \in (s_{i-1}^*, s_i^*)$, $i = 1, \dots, m+1$, where $s_0^* = 0$,
 $s_{m+1}^* = +\infty$, $\phi_{n(m+1)}^*(x) = b$, $\phi_{nm}^*(x) = b + \frac{\delta_m^*}{(N-2)!}(s_m^* - x)^{N-2}$, and
for $i = 1, \dots, m-1$,

$$\begin{aligned} \phi_{ni}^*(x) &= \phi_{n(i+2)}^*(s_{i+1}^*) + \delta_i^* \frac{(s_i^* - x)^{N-2}}{(N-2)!} \\ &\quad + \delta_{i+1}^* \sum_1^{N-2} \frac{(s_{i+1}^* - s_i^*)^j}{j!} \frac{(s_i^* - x)^{(N-2-j)}}{(N-2-j)!}, \end{aligned}$$

where $b, s_1^*, \dots, s_m^*, \delta_1^*, \dots$, and δ_m^* are to be decided such that
 $E(\phi_{on}^*(S_t)) = 1$, $E(S_t \phi_{on}^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_{on}^*(S_t))/B_0 =$
 c_{i0} , $i = 1, 2, \dots, n$.

- Assume n is even. Let $m = n/2$.

- Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel $(\phi_n^{**}(x))$ that has a $(m+2)$ -segmented and piecewise constant $(N-2)$ th derivative, where the absolute value of its $(N-2)$ th derivative is equal to δ for $x < s_1^{**}$, δ_1^{**} for $x \in (s_1^{**}, s_2^{**})$, ..., δ_m^{**} for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$. More precisely, $\phi_n^{**}(x) = \phi_{ni}^{**}(x)$, for $x \in (s_{i-1}^{**}, s_i^{**})$, $i = 1, \dots, m+2$, where $s_0^{**} = 0$, $s_{m+2}^{**} = +\infty$, $\phi_{n(m+2)}^{**}(x) = b$, $\phi_{n(m+1)}^{**}(x) = b + \frac{\delta_m^{**}}{(N-2)!}(s_{m+1}^{**} - x)^{N-2}$, and for $i = 1, \dots, m$,

$$\begin{aligned} \phi_{ni}^{**}(x) &= \phi_{n(i+2)}^{**}(s_{i+1}^{**}) + \delta_{i-1}^{**} \frac{(s_i^{**} - x)^{N-2}}{(N-2)!} \\ &\quad + \delta_i^{**} \sum_1^{N-2} \frac{(s_{i+1}^{**} - s_i^{**})^j}{j!} \frac{(s_i^{**} - x)^{(N-2-j)}}{(N-2-j)!}, \end{aligned}$$

where $\delta_0^{**} = \delta$ while $b, s_1^{**}, \dots, s_{m+1}^{**}, \delta_1^{**}, \dots$, and δ_m^{**} are to be decided such that $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

- The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel $(\phi_{en}^*(x))$ that has a $(m+2)$ -segmented and piecewise constant $(N-2)$ th derivative, where the absolute of value of its $(N-2)$ th derivative is equal to δ_1^* for $x < s_1^*$, ..., δ_{m+1}^* for $x \in (s_m^*, s_{m+1}^*)$, and zero for $x > s_{m+1}^*$. More precisely, $\phi_{en}^*(x) = \phi_{ni}^*(x)$, for $x \in (s_{i-1}^*, s_i^*)$, $i = 1, \dots, m+2$, where $s_0^* = 0$, $s_{m+2}^* = +\infty$, $\phi_{n(m+2)}^*(x) = 0$, $\phi_{n(m+1)}^*(x) = \frac{\delta_{m+1}^*}{(N-2)!}(s_{m+1}^* - x)^{N-2}$,

and for $i = 1, \dots, m$,

$$\begin{aligned}\phi_{ni}^*(x) &= \phi_{n(i+2)}^*(s_{i+1}^*) + \delta_i^* \frac{(s_i^* - x)^{N-2}}{(N-2)!} \\ &\quad + \delta_{i+1}^* \sum_1^{N-2} \frac{(s_{i+1}^* - s_i^*)^j}{j!} \frac{(s_i^* - x)^{(N-2-j)}}{(N-2-j)!},\end{aligned}$$

where $s_1^*, \dots, s_{m+1}^*, \delta_1^*, \dots$, and δ_{m+1}^* are to be decided such that $E(\phi_{en}^*(S_t)) = 1$, $E(S_t \phi_{en}^*(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_{en}^*(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

Proof: From Lemma 6 we need only prove that the true pricing kernel intersects the pricing kernels that give the option bounds exactly $(n + 2)$ times and then examine which one has a fatter left tail.

We first examine ϕ_n^{**} . Assume n is odd. Note it has a $(m+2)$ -segmented and piecewise constant $(N - 2)$ th derivative, where $m = (n + 1)/2$. More precisely its $(N - 2)$ th derivative is equal to $-\delta$ for $x < s_1^{**}$, $-\delta_1^{**}$ for $x \in (s_1^{**}, s_2^{**})$, ..., $-\delta_m^{**}$ for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is 0.

Obviously we must have $\delta > \delta_1^{**} > \dots \delta_m^{**} > 0$. Otherwise the true pricing kernel will intersect ϕ_n^{**} at most $n + 1$ times. In this case, applying Lemma 6, we find that the two pricing kernels cannot give the same n observed option prices. From this, we can immediately conclude that the true pricing kernel intersects ϕ_n^{**} exactly $n + 2$ times. It is not difficult to verify that ϕ_n^{**} has fatter left tail.

Assume n is even. Note it has a $(m+2)$ -segmented and piecewise constant $(N - 2)$ th derivative, where $m = n/2$. More precisely, the absolute value of its $(N - 2)$ th derivative is equal to δ for $x < s_1^{**}$, δ_1^{**} for $x \in (s_1^{**}, s_2^{**})$, ..., δ_m^{**}

for $x \in (s_m^{**}, s_{m+1}^{**})$, and zero for $x > s_{m+1}^{**}$, and its value at its right tail is a positive constant b .

Obviously we must have $\delta > \delta_1^{**} > \dots \delta_m^{**} > 0$ and $b > \inf_x \phi(x)$, where $\phi(x)$ is the true pricing kernel. Otherwise the true pricing kernel will intersect ϕ_n^{**} at most $n + 1$ times. In this case, applying Lemma 6, we find that the two pricing kernels cannot give the same n observed option prices. From this, we can immediately conclude that the true pricing kernel intersects ϕ_n^{**} exactly $n + 2$ times. It is not difficult to verify that ϕ_n^{**} has fatter left tail.

For ϕ_n^* the proof is similar. Q.E.D.

Proposition 6 *Assume the pricing kernel $\phi(x)$ satisfies the N th order stochastic dominance rule, i.e., $\phi^{(i)}(x) \geq 0$, for even $i < N$, $\phi^{(i)}(x) \leq 0$, for odd $i < N$, and $|\phi^{(N-1)}(x)|$ is increasing in x . Assume $|\phi^{(N-2)}(x)|$ is bounded above by δ . Assume the current price of a unit bond is B_0 , the current price of the underlying stock is S_0 , and the current prices of n options with strike prices K_1, \dots , and K_n are c_{10}, \dots , and c_{n0} respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

1. *Assume n is odd. Let $m = (n + 1)/2$.*

(a) *Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by the pricing kernel $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_n^{**}(S_t)$, where $p(S_t)$ is the true probability density function, $\delta(S_t)$ is the Dirac function, and $f_n^{**}(x)$ has a $(m+1)$ -segmented and piecewise constant $(N-2)$ th derivative, where the absolute value of its $(N-2)$ th derivative is equal to δ_1^{**} for $x < s_1^{**}$, \dots , δ_m^{**} for $x \in (s_{m-1}^{**}, s_m^{**})$, and*

zero for $x > s_m^{**}$. More precisely, $f_n^{**}(x) = f_{ni}^{**}(x)$, for $x \in (s_{i-1}^{**}, s_i^{**})$,
 $i = 1, \dots, m+1$, where $s_0^{**} = 0$, $s_{m+1}^{**} = +\infty$, $f_{n(m+1)}^{**}(x) = 0$,
 $f_{nm}^{**}(x) = \frac{\delta_m^{**}}{(N-2)!} (s_m^{**} - x)^{N-2}$, and for $i = 1, \dots, m-1$,

$$f_{ni}^{**}(x) = f_{n(i+2)}^{**}(s_{i+1}^{**}) + \delta_i^{**} \frac{(s_i^{**} - x)^{N-2}}{(N-2)!} \\ + \delta_{i+1}^{**} \sum_1^{N-2} \frac{(s_{i+1}^{**} - s_i^{**})^j (s_i^{**} - x)^{(N-2-j)}}{j! (N-2-j)!},$$

where $a_0, s_1^{**}, \dots, s_m^{**}, \delta_1^{**}, \dots$, and δ_m^{**} are to be decided such that
 $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$,
 $i = 1, 2, \dots, n$.

(b) The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) ,
 $i = 1, 2, \dots$, is given by the pricing kernel $\varphi_n^*(x) = \phi_{\delta n}^*(x)$, which is
derived in the above proposition.

2. Assume n is even. Let $m = n/2$.

(a) Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}) ,
 $i = 1, 2, \dots$, is given by a pricing kernel $\varphi_n^{**}(S_t) = a_0 \frac{\delta(S_t)}{p(S_t)} + f_n^{**}(S_t)$,
where $p(S_t)$ is the true probability density function, $\delta(S_t)$ is the Dirac function,
and $f_n^{**}(x)$ has a $(m+1)$ -segmented and piecewise constant $(N-2)$ th derivative,
where the absolute of value of its $(N-2)$ th derivative is equal to δ_1^{**} for $x < s_1^{**}$,
 \dots , δ_m^{**} for $x \in (s_{m-1}^{**}, s_m^{**})$, and zero for $x > s_m^{**}$.
More precisely, $f_n^{**}(x) = f_{ni}^{**}(x)$, for $x \in (s_{i-1}^{**}, s_i^{**})$,
 $i = 1, \dots, m+1$, where $s_0^{**} = 0$, $s_{m+1}^{**} = +\infty$, $f_{n(m+1)}^{**}(x) = b$,

$$f_{nm}^{**}(x) = b + \frac{\delta_m^{**}}{(N-2)!} (s_m^{**} - x)^{N-2}, \text{ and for } i = 1, \dots, m-1,$$

$$\begin{aligned} f_{ni}^{**}(x) &= f_{n(i+2)}^{**}(s_{i+1}^{**}) + \delta_i^{**} \frac{(s_i^{**} - x)^{N-2}}{(N-2)!} \\ &\quad + \delta_{i+1}^{**} \sum_1^{N-2} \frac{(s_{i+1}^{**} - s_i^{**})^j (s_i^{**} - x)^{(N-2-j)}}{j! (N-2-j)!}, \end{aligned}$$

where $a_0, b, s_1^{**}, \dots, s_m^{**}, \delta_1^{**}, \dots, \delta_m^{**}$ are to be decided such that $E(\phi_n^{**}(S_t)) = 1$, $E(S_t \phi_n^{**}(S_t))/B_0 = S_0$, and $E(c_i(S_t) \phi_n^{**}(S_t))/B_0 = c_{i0}$, $i = 1, 2, \dots, n$.

- (b) The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, is given by a pricing kernel $\varphi_n^*(x) = \phi_{en}^*$, which is derived in the above proposition.

3 The Arbitrage Portfolios

Using a method similar to the one use by Huang (2000b) we can derive the arbitrage portfolios when the N th order stochastic dominance option bounds are violated. We first introduce the following notation: Let $k = N - 3$. We use $\bar{s}_j^{(N)}$ to denote

$$\frac{\int_0^{s_j} \int_0^{x_k} \dots \int_0^{x_1} E(S_t | S_t < x_0) Pr(S_t < x_0) dx_0 \dots dx_k}{\int_0^{s_j} \int_0^{x_k} \dots \int_0^{x_1} Pr(S_t < x_0) dx_0 \dots dx_k}$$

Given a function $f(x)$, we use $\bar{f}_{i_j}^{(N)}$ to denote the following

$$\frac{\int_0^{s_j} \int_0^{x_k} \dots \int_0^{x_1} E(f(S_t) | S_t < x_0) Pr(S_t < x_0) dx_0 \dots dx_k}{\int_0^{s_j} \int_0^{x_k} \dots \int_0^{x_1} Pr(S_t < x_0) dx_0 \dots dx_k}$$

We also use $(\bar{c}_{i_j}^{(N)X})'$ to denote

$$\frac{d}{dy} \frac{\int_0^y \int_0^{x_k} \dots \int_0^{x_1} E(c_{i_j}^X(S_t) | S_t < x_0) Pr(S_t < x_0) dx_0 \dots dx_k}{\int_0^y \int_0^{x_k} \dots \int_0^{x_1} Pr(S_t < x_0) dx_0 \dots dx_k} \Big|_{y=s_j}.$$

We have the following result.

Proposition 7 *Assume the pricing kernel is decreasing in S_t . Assume the price of a unit bond is B_0 , the underlying stock price is S_0 , and the prices of n options with strike prices K_1, K_2, \dots, K_n are $c_0^1, c_0^2, \dots, c_0^n$ respectively.*

- Assume n is odd. Let $m = (n + 1)/2$.

- For options with strike prices X between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$,

when its lower bound is violated the arbitrage portfolio is given by

$$\alpha_i = (-1)^i \left[\sum_1^m ((\bar{c}_{l_v}^{(N)X})' A_{2v,i} - \bar{c}_{l_v}^{(N)X} A_{2v-1,i}) - \bar{c}_\Lambda^{(N)X} A_{n+2,i} \right] / |A|, \quad (2)$$

$i=1, \dots, n+2$, where s_{l_1}, \dots, s_{l_m} , are determined by 1(b) in Proposition

6 and A is given by

$$\begin{pmatrix} 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^1 & \dots & \bar{c}_{l_1}^{(N)n} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{(N)1})' & \dots & (\bar{c}_{l_1}^{(N)n})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m}^{(N)} & \bar{c}_{l_m}^{(N)1} & \dots & \bar{c}_{l_m}^{(N)n} \\ 0 & (\bar{s}_{l_m}^{(N)})' & (\bar{c}_{l_m}^{(N)1})' & \dots & (\bar{c}_{l_m}^{(N)n})' \\ 1 & \bar{s}_\Lambda^{(N)} & \bar{c}_\Lambda^{(N)1} & \dots & \bar{c}_\Lambda^{(N)n} \end{pmatrix} \quad (3)$$

- For options with strike prices X between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$,

when its upper bound is violated the arbitrage portfolio is given by

$$\alpha_i = (-1)^i [-\bar{c}_1^{(N)X} B_{1i} + \sum_1^m (\bar{c}_{l_v}^{(N)X} B_{2v,i} - (\bar{c}_{l_v}^{(N)X})' B_{2v+1,i})] / |B|, \quad (4)$$

$i=1, \dots, n+2$, where s_{l_1}, \dots, s_{l_m} are determined by 1(a) in Proposition

6 and B is given by

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_1^{(N)} & \bar{c}_1^{(N)1} & \dots & \bar{c}_1^{(N)n} \\ 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{(N)1} & \dots & \bar{c}_{l_1}^{(N)n} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{(N)1})' & \dots & (\bar{c}_{l_1}^{(N)n})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m}^{(N)} & \bar{c}_{l_m}^{(N)1} & \dots & \bar{c}_{l_m}^{(N)n} \\ 0 & (\bar{s}_{l_m}^{(N)})' & (\bar{c}_{l_m}^{(N)1})' & \dots & (\bar{c}_{l_m}^{(N)n})' \end{array} \right\} \quad (5)$$

- For options with strike prices X between (K_{2i-1}, K_{2i}) , $i = 1, 2, \dots$, when its lower bound is violated the arbitrage portfolio is given by (4); when its upper bound is violated the arbitrage portfolio is given by (2)

- Assume n is even. Let $m = n/2$.

- For options with strike prices X between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$, when its lower bound is violated the arbitrage portfolio is given by

$$\alpha_i = (-1)^i \sum_1^{m+1} ((\bar{c}_{l_v}^{(N)X})' U_{2v,i} - \bar{c}_{l_v}^{(N)X} U_{2v-1,i}) / |U|, \quad (6)$$

$i=1, \dots, n+2$, where s_{l_1}, \dots, s_{l_m} , are determined by 2(b) in Proposition

6 and U is given by

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{(N)1} & \dots & \bar{c}_{l_1}^{(N)n} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{(N)1})' & \dots & (\bar{c}_{l_1}^{(N)n})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_{m+1}}^{(N)} & \bar{c}_{l_{m+1}}^{(N)1} & \dots & \bar{c}_{l_{m+1}}^{(N)n} \\ 0 & (\bar{s}_{l_{m+1}}^{(N)})' & (\bar{c}_{l_{m+1}}^{(N)1})' & \dots & (\bar{c}_{l_{m+1}}^{(N)n})' \end{array} \right\} \quad (7)$$

– For options with strike prices X between (K_{2i-2}, K_{2i-1}) , $i = 1, 2, \dots$,

when its upper bound is violated the arbitrage portfolio is given by

$$\alpha_i = (-1)^i [-\bar{c}_1^{(N)X} V_{1i} + \sum_1^m (\bar{c}_{l_v}^{(N)X} V_{2v,i} - (\bar{c}_{l_v}^{(N)X})' V_{2v+1,i}) + \bar{c}_\Lambda^{(N)X} V_{n+2,i}] / |V|, \quad (8)$$

$i=1, \dots, n+2$, where s_{l_1}, \dots, s_{l_m} , are determined by 2(a) in Proposition

6 and V is given by

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_1^{(N)} & \bar{c}_1^{(N)1} & \dots & \bar{c}_1^{(N)n} \\ 1 & \bar{s}_{l_1}^{(N)} & \bar{c}_{l_1}^{(N)1} & \dots & \bar{c}_{l_1}^{(N)n} \\ 0 & (\bar{s}_{l_1}^{(N)})' & (\bar{c}_{l_1}^{(N)1})' & \dots & (\bar{c}_{l_1}^{(N)n})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m}^{(N)} & \bar{c}_{l_m}^{(N)1} & \dots & \bar{c}_{l_m}^{(N)n} \\ 0 & (\bar{s}_{l_m}^{(N)})' & (\bar{c}_{l_m}^{(N)1})' & \dots & (\bar{c}_{l_m}^{(N)n})' \\ 1 & \bar{s}_\Lambda^{(N)} & \bar{c}_\Lambda^{(N)1} & \dots & \bar{c}_\Lambda^{(N)n} \end{array} \right\} \quad (9)$$

– For options with strike prices X between (K_{2i-1}, K_{2i}) , $i = 1, 2, \dots$,

when its lower bound is violated the arbitrage portfolio is given by

(8); when its upper bound is violated the arbitrage portfolio is given

by (6).

Proof: The proof is very similar to the case of second stochastic dominance. For brevity, it is omitted. For details, see Huang (2004b).

4 Conclusions

In this paper we derive N th order stochastic dominance option bounds from concurrently expiring options. We show that given the prices of a unit bond, underlying stock, and n option prices, the k th order stochastic dominance option bounds are given by a pricing kernel the $(N - 2)$ th derivative of which is $(n/2)$ -segmented and piecewise constant if n is even or $((n+1)/2)$ -segmented and piecewise constant if n is odd. Since stochastic dominance rules are generally accepted, the derived option bounds in this paper are practically meaningful.

The results have important implications for arbitrage opportunities in the markets of concurrently expiring options. When the option bounds are violated we can construct arbitrage portfolios to take the advantage.

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