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Abstract

In this paper we first derive second order stochastic dominance option bounds from concurrently expiring options. We show that these option bounds are given by piecewise constant pricing kernels. When these option bounds are violated there are second order arbitrage opportunities. We then establish the way to construct arbitrage portfolios to make profits from these opportunities.

Keywords: Option bounds, option pricing, stochastic dominance, risk averse.

JEL Classification Numbers: G13.

Introduction

Since Perrakis and Ryan (1984), Ritchken (1985), and Levy (1985) derived the second order stochastic dominance (hereafter SSD) option bounds, there has developed a rich literature on this topic. For example, Ritchken and Kuo (1989) derived higher order stochastic dominance option bounds. Basso and Pianca (1997) and Mathur and Ritchken (2000) worked on decreasing absolute (relative) risk aversion (hereafter DARA (DRRA)) bounds.

Ryan (2003) tried to improve the SSD option bounds by using the observed prices of concurrently expiring options. However, he used only one observed option at a time because he wrongly concluded that "only the two options with exercise prices closest to the initial option provide binding information."[1]

In this paper we improve the SSD option bounds by using concurrently expiring options. We use a new technique presented by Huang (2004a), which takes the advantage of the distinctive feature of options' payoff functions. We show that given the prices of the underlying stock and n concurrently expiring options, the option bounds are given by piecewise constant pricing kernels.

As explained by Ryan (2003), risk version implies a second order of arbitrage, interpreted as conditional expected return comparison, rather than first order

[1] The option bounds he derived using one observed option price at a time is suboptimal. This can be seen by examining his numerical results. Some of these results are even worse than the first order stochastic dominance option bounds obtained by Bertsimas and Popescu (2002).

arbitrage which involves the comparison of realized returns. When the derived SSD option bounds are violated, there are second order arbitrage opportunities in the market. To take advantage of these opportunities we derive the arbitrage portfolios in this paper.

This paper is closely related to Huang (2004b, 2004c). Huang (2004b, 2004c) improved higher order stochastic dominance option bounds and DARA (DRRA) option bounds by using the observed prices of concurrently expiring options. The methodology used in this paper is the same as the one used there.

This work is also related to the recent important works by Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000). Cochrane and Saa-Requejo (2000) derived option bounds using restrictions on the volatility of the pricing kernel, while Bernardo and Ledoit (2000) derived option bounds using restrictions on the deviation of the pricing kernel from a benchmark pricing kernel. Other related works include Lo (1987), Grundy (1991), and Constantinides and Zariphopoulou (1999, 2001) who all derived option bounds under different conditions.

The structure of the paper is as follows: In Section 1 we introduce the problem. In Section 2 we discuss the case where there is only one observed option. In Section 3 we discuss the case where there are two observed options. In section 4 we discuss the general case where we have n observed options. In Section 5 we derive the arbitrage portfolios which will be used to make profits when the option bounds are violated. The final section concludes the paper.

1 The SSD Option Bound Problem

We assume that there is a stock in an economy on which option contracts are written. We do not have sufficient information to obtain the exact prices of the options. Thus we are interested in deriving option bounds. Assume the second order stochastic dominance rule applies in the economy. Also assume that we observe n options which will expire at the same time as those options we are interested in.

1.1 In a Discrete State Space

Ryan (2003) modeled the problem in a discrete state space as follows. Assume a two-date economy starts at time 0 and will end at time 1. Assume there are Λ states of the economy at time 1 indexed by $j = 1, 2, ..., \Lambda$. The probability of state j is π_j. Denote the value of the stock in state j by s_j. Assuming s_j, $j = 1, ..., \Lambda$, are in an ascending order. The state discount factor (pricing kernel) is denoted by d; its value in state j is d_j, $j = 1, ..., \Lambda$. Let $x_j = d_j - d_{j+1}$, $j = 1, 2, ..., \Lambda - 1$ and $x_\Lambda = d_\Lambda$. Let $y_j = (\sum_1^j \pi_i)x_j$, $j = 1, ..., \Lambda$.

Assume there are n observed options indexed by 1, ..., n expiring at the same time as the target option. The jth observed option has strike price K_i while the target option (which we have to price) has strike price X. Denote the payoff of the ith observed option in state j by c_j^i. Denote the payoff of the target option in state j by c_j^X. For $k = 1, ..., n$ and $j = 1, ..., \Lambda$, write

$$\bar{c}_j^X = \sum_1^j c_i^X \pi_i / \sum_1^j \pi_i, \quad \bar{s}_j = \sum_1^j s_i \pi_i / \sum_1^j \pi_i, \quad \bar{c}_j^k = \sum_1^j c_i^k \pi_i / \sum_1^j \pi_i.$$

Then the SSD option bound problem with n observed options is as follows:[2]

Problem P1 (P2)

$$C^{**} = \min \quad (\text{or } \max) \quad \sum_{j=1}^{\Lambda} \bar{c}_j^X y_j$$

s.t.

$$\sum_{j=1}^{\Lambda} y_j = B_0$$

$$\sum_{j=1}^{\Lambda} \bar{s}_j y_j = S_0$$

$$\sum_{j=1}^{\Lambda} \bar{c}_j^i y_j = c_0^i, \quad i = 1, ..., n$$

$$y_j \geq 0, \quad j = 1, ..., \Lambda$$

Its dual problem is

Problem D1 (D2)

$$\max \quad (\text{or} \min) \;\; C_0^X = \alpha_1 B_0 + \alpha_2 S_0 + \sum_{i=1}^{n} \alpha_{i+2} c_0^i$$

s.t.

$$\alpha_1 + \alpha_2 \bar{s}_j + \sum_{i=1}^{n} \alpha_{i+2} \bar{c}_j^i \leq \;(\text{or} \geq)\; \bar{c}_j^X, \quad j = 1, ..., \Lambda$$

1.2 In a Continuous State Space

In a continuous state space, the presentation is simpler. Let S denote time 1 value of the stock. Let $\phi(S)$ denote the pricing kernel. Denote time t value of a contingent claim by $c(S)$, which is dependent on S; denote its time 0 value

[2] See Ryan (2003) for detailed explanations.

by c_0. Then we have $c_0 = B_0 E(\phi(S)c(S))$, which is obviously dependent on S_0. Assume the lowest possible time 1 value of the stock is s_1.

Since second order stochastic dominance rule applies we have a positive and decreasing pricing kernel. Thus the problem is:

$$\max B_0 E(\phi(S)c^X(S)),$$

where $c^X(S)$ is the payoff of an option with strike price X at time t. subject to

$$E(\phi(S) = 1,$$

$$B_0 E(\phi(S)S) = S_0,$$

$$B_0 E(\phi(S)c^i(S)) = c_0^i, \quad i = 1, 2, ...n,$$

where $\phi(S)$ is positive and decreasing.

To solve the option bound problem, we first solve a similar but more general problem in which we assume that not only the second order stochastic dominance rule applies but also the pricing kernel is bounded from above and below.

We will show in this paper that under this condition, the option bounds are given by a piecewise constant pricing kernel, where the number of segments of the pricing kernel depends on the number of observed option prices.

Moreover, we will see that for an even number of observed option prices the pricing kernel that gives the option bounds has a certain pattern while for an odd number of observed option prices the pricing kernel that gives the option bounds has a different pattern. Thus in order to explain the solutions more clearly we start with the two cases where we have only one or two observed options. Then we explore the general case where we have n observed options.

In the rest of the paper we will frequently use conditional expectations. We use the following notation to denote these expectations:

$$E(f(S)|a < S < b) = \int_a^b f(S)p(S)dS$$

$$\hat{E}(f(S)|a < S < b) = E(f(S)|a < S < b)/Pr(a < S < b)$$

where $p(S)$ is the true probability density and Pr denotes probability, i.e.,

$$Pr(a < S < b) = \int_a^b p(S)dS.$$

2 Option Bounds With One Observed Option

In this section we examine the case where we have only one observed option. This case has been studied by Ryan (2003) we derive it using our new method. Before we proceed, we introduce two lemmas.

Lemma 1 (FSS (1999)) *Assume two pricing kernels give the same stock price. If they intersect twice, then the pricing kernel with fatter tails gives higher prices of convex-payoff contingent claims written on the stock.*

Proof: See Huang (2004a) or FSS (1999).

Lemma 2 *Assume two pricing kernels give the same prices of the underlying stock and an option with strike price K. If they intersect three times, then the* *pricing kernel with fatter left tail will give higher [lower] prices for all options with strike prices below [above] K than the other.*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the pricing kernel is decreasing in the underlying stock price S its value is bounded above and below.

Lemma 3 *Assume the pricing kernel is decreasing in S and bounded above by $\overline{\phi}$ and below by $\underline{\phi}$. Assume the prices of a unit bond, the underlying stock, and an option with strike price K are B_0, S_0, and c_0 respectively.*

- *The upper bound for an option with strike price below K is given by the pricing kernel $\phi_1^{**}(S) = \overline{\phi}$, $S < s_l$; $\phi_1^{**}(S) = a_1$, $s_l < S < s_u$; $\phi_1^{**}(S) = \underline{\phi}$, $S > s_u$, where a_1, s_l, and s_u are to be determined such that $\overline{\phi} \geq a_1 \geq \underline{\phi}$,*

$$\begin{aligned} \overline{\phi}\ Pr(S < s_l) + a_1\ Pr(s_l < S < s_u) + \underline{\phi}\ Pr(S > s_u) &= 1 \\ \overline{\phi}E(S|S < s_l) + a_1 E(S|s_l < S < s_u) + \underline{\phi}E(S|S > s_u) &= \frac{S_0}{B_0} \\ \overline{\phi}E(c(S)|S < s_l) + a_1 E(c(S)|s_l < S < s_u) + \underline{\phi}E(c(S)|S > s_u) &= \frac{c_0}{B_0} \end{aligned}$$

- *The lower bound for an option with strike price below K is given by the pricing kernel $\phi_1^{*}(S) = b_1$, $S < s_l$; $\phi_1^{*}(S) = b_2$, $S > s_l$, where b_1, b_2, and s_l are to be determined such that $b_1 \geq b_2$,*

$$\begin{aligned} b_1\ Pr(S < s_l) + b_2\ Pr(S > s_l) &= 1 \\ b_1 E(S|S < s_l) + b_2 E(S|S > s_l) &= \frac{S_0}{B_0} \\ b_1 E(c(S)|S < s_l) + b_2 E(c(S)|S > s_l) &= \frac{c_0}{B_0} \end{aligned}$$

- *The upper (lower) bound for options with strike prices above K is given by the pricing kernel $\phi_1^*(S)$ ($\phi_1^{**}(S)$).*

Proof: From Lemma 2 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly three times and then examine the fatness of their left tails.

We first examine ϕ_1^{**}. Note it is three-segmented and piecewise constant. More precisely $\phi_1^{**} = \overline{\phi}$, $S < s_l$; $\phi_1^{**}(S) = a_1$, $s_l < S < s_u$; $\phi_1^{**}(S) = \underline{\phi}$, $S > s_u$, $\overline{\phi} \geq a_1 \geq \underline{\phi}$. Obviously this pricing kernel intersects any decreasing pricing kernel at most three times. However from Lemma 1, it must intersect all admissible pricing kernels at least three times; otherwise they cannot give the same observed option price. Hence ϕ_1^{**} intersects all admissible pricing kernel exactly three times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

Ryan (2003) derived the optimal SSD option bounds using one observed option price. Here follow his result.

[Ryan] Assume the pricing kernel is decreasing in S. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the price of an option with strike price K is c_0. Assume the lowest possible time 1 value of the stock is zero, i.e., $s_1 = 0$.

- Then the upper bound for options with strike prices below K is given by the pricing kernel $\phi_1^{**}(S) = a_0 \frac{\delta(S-s_1)}{p(S)} + f(S)$, where $p(S)$ is the probability density function, $\delta(S)$ is the Dirac function, $a_0 = 1 - \frac{S_0}{B_0 \hat{E}(S|S<s_l)}$, and $f(S) = a$, $S < s_l$; $f(S) = 0$, $S > s_l$, where $a = \frac{S_0}{B_0 E(S|S<s_l)}$ and s_l is to

be determined such that

$$E((S/S_0)|S < s_l) = E((c(S)/c_0)|S < s_l).$$

- The lower bound for options with strike prices below K is given by the pricing kernel $\phi_1^*(S) = b_1$, $S < s_l$; $\phi(S) = b_2$, $S > s_l$, where

$$\begin{aligned} b_1 &= \frac{\hat{E}(S|S > s_l) - (S_0/B_0)}{(\hat{E}(S|S > s_l) - \hat{E}(S|S < s_l))\ Pr(S < s_l)} \\ b_2 &= \frac{(S_0/B_0) - \hat{E}(S|S < s_l)}{(\hat{E}(S|S > s_l) - \hat{E}(S|S < s_l))\ Pr(S > s_l)} \end{aligned}$$

  and s_l is to be determined such that

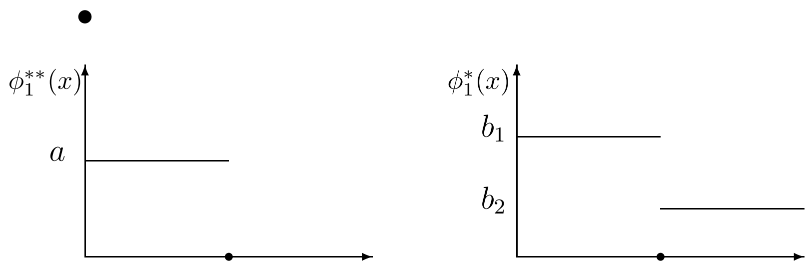
$$\frac{E(S|S < s_l) - (S_0/B_0)}{\hat{E}(S|S > s_l) - \hat{E}(S|S < s_l)} = \frac{E(c(S)|S < s_l) - (c_0/B_0)}{\hat{E}(c(S)|S > s_l) - \hat{E}(c(S)|S < s_l)}.$$

- The upper (lower) bound for options with strike prices above K is given by the pricing kernel $\phi_1^*(S)$ ($\phi_1^{**}(S)$).

Proof: Let $\overline{\phi} \to +\infty$ and $\underline{\phi} \to 0$; from Lemma 3 we immediately conclude that we must have $\phi_1^{**}(S) = a_0 \frac{\delta(S-s_1)}{p(S)} + f(S)$, where $f(S) = a_1$, $S < s_l$; $f(S) = 0$, $S > s_l$, where a_0, a_1, and s_l are to be determined by the three equations about the prices of the unit bond, stock, and the observed option. Noting $s_1 = 0$, solving the three equations we obtain $\phi_1^{**}(S)$. $\phi_1^*(S)$ can be similarly obtained. Q.E.D.

Figure 2: One Observed Option: Exercise Price = K

Upper Bound (Exercise Price < K) Lower Bound (Exercise Price < K)



3 With Two Observed Options

Ryan (2003) argues that given more than one observed options, only two options with strike prices closest to the interested option provide binding information for the bounds of the option price. However, this is not true. In this section we derive SSD option bounds from two concurrently expiring options. We first introduce a lemma.

Lemma 4 *Assume two pricing kernels give the same prices of the underlying stock and two options with strike prices* K_1 *and* K_2*, where* $K_1 < K_2$*. If they intersect four times, then the pricing kernel with fatter left tail will give higher (lower) prices for options with strike prices outside (inside)* (K_1, K_2)*.*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the second order stochastic dominance rule applies and the value of the pricing kernel is bounded from above and below.

Lemma 5 *Assume the pricing kernel is decreasing in S and bounded above by $\overline{\phi}$ and below by $\underline{\phi}$. Assume the prices of a unit bond, the underlying stock, and two options with strike prices K_1 and K_2, where $K_1 < K_2$, are B_0, S_0, c_0^1, and c_0^2 respectively.*

Then the upper bound for an option with strike price below K_1 or above K_2 is given by the pricing kernel $\phi_2^{**}(S) = \overline{\phi}$, $S < s_l$; $\phi_2^{**}(S) = a_1$, $s_l < S < s_u$; $\phi_2^{**}(S) = a_2$, $S > s_u$, where a_1, a_2, s_l, and s_u are to be determined such that $\overline{\phi} \geq a_1 \geq a_2$,

$$\begin{aligned}
\overline{\phi}\ Pr(S < s_l) + a_1\ Pr(s_l < S < s_u) + a_2\ Pr(S > s_u) &= 1 \\
\overline{\phi} E(S|S < s_l) + a_1 E(S|s_l < S < s_u) + a_2 E(S|S > s_u) &= \frac{S_0}{B_0} \\
\overline{\phi} E(c^i(S)|S < s_l) + a_1 E(c^i(S)|s_l < S < s_u) + a_2 E(c^i(S)|S > s_u) &= \frac{c_0^i}{B_0}
\end{aligned}$$

$i = 1, 2.$

The lower bound for an option with strike price below K_1 or above K_2 is given by the pricing kernel $\phi_2^{*}(S) = b_1$, $S < s_l$; $\phi_2^{*}(S) = b_2$, $s_l < S < s_u$, $\phi_2^{*}(S) = \underline{\phi}$, $S > s_u$, where b_1, b_2, s_l, and s_u are to be determined such that $b_1 \geq b_2$,

$$\begin{aligned}
b_1\ Pr(S < s_l) + b_2\ Pr(s_l < S < s_u) + \underline{\phi}\ Pr(S > s_u) &= 1 \\
b_1 E(S|S < s_l) + b_2 E(S|s_l < S < s_u) + \underline{\phi} E(S|S > s_u) &= \frac{S_0}{B_0} \\
b_1 E(c^i(S)|S < s_l) + b_2 E(c^i(S)|s_l < S < s_u) + \underline{\phi} E(c^i(S)|S > s_u) &= \frac{c_0^i}{B_0}
\end{aligned}$$

$i = 1, 2.$

The upper (lower) bound for an option with strike price between K_1 and K_2 is given by the pricing kernel $\phi_2^*(S)$ ($\phi_2^{**}(S)$).

Proof: From Lemma 4 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly four times and then examine the fatness of their left tails.

We first examine ϕ_2^{**}. Note it is three-segmented and piecewise constant. More precisely $\phi_2^{**}(S) = \overline{\phi}$, $S < s_l$; $\phi_2^{**}(S) = a_1$, $s_l < S < s_u$; $\phi_2^{**}(S) = a_2$, $S > s_u$, where $\overline{\phi} \geq a_1 \geq a_2$.

Obviously this pricing kernel intersects any decreasing pricing kernel at most four times. However from Lemma 1, it must intersect all admissible pricing kernels at least four times; otherwise they cannot give the same observed option prices. Hence ϕ_1^{**} intersects all admissible pricing kernel exactly four times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. For ϕ_1^* the proof is similar. Q.E.D.

Proposition 1 *Assume the pricing kernel is decreasing in S. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of two options with strike prices K_1 and K_2 are c_0^1 and c_0^2 respectively.*

Then the upper bound for options with strike prices below K_1 or above K_2 is given by the pricing kernel $\phi_2^{**}(S) = a_0\frac{\delta(S-s_1)}{p(S)} + f(S)$, where $p(S)$ is the probability density function and $\delta(S)$ is the Dirac function and $f(S) = a_1$, $S < s_l$; $f(S) = a_2$, $S > s_l$, where a_0, a_1, a_2, and s_l are to be determined such *that* $a_0 > 0$, $a_1 \geq a_2$,

$$\begin{aligned} a_0 + a_1\ Pr(S < s_l) + a_2\ Pr(S > s_l) &= 1 \\ a_0 s_1 + a_1 \hat{E}(S|S < s_l) + a_2 \hat{E}(S|S > s_l) &= \frac{S_0}{B_0} \\ a_0 c^i(s_1) + a_1 \hat{E}(c^i(S)|S < s_l) + a_2 \hat{E}(c^i(S)|S > s_l) &= \frac{c_0^i}{B_0} \end{aligned}$$

$i = 1, 2.$

The lower bound for options with strike prices below K_1 *or above* K_2 *is given by the pricing kernel* $\phi_2^*(S) = b_1$, $S < s_l$; $\phi_2^*(S) = b_2$, $s_l < S < s_u$; $\phi_2^*(S) = 0$, $S > s_u$, *where* b_1, b_2, s_l, *and* s_u *are to be determined such that* $b_1 \geq b_2$,

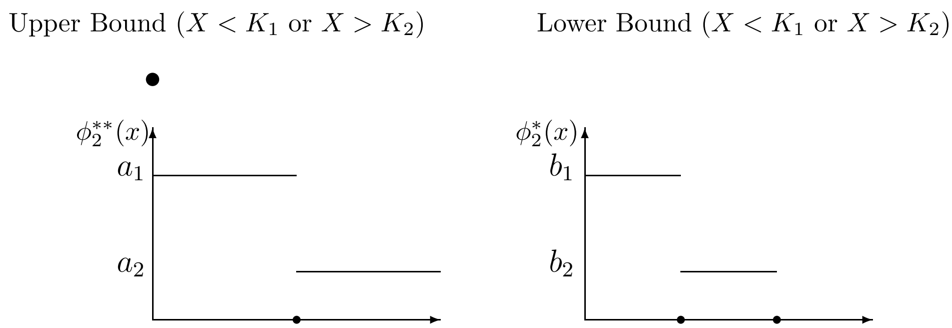
$$\begin{aligned} b_1\ Pr(S < s_l) + b_2\ Pr(s_l < S < s_u) &= 1, \\ b_1 \hat{E}(S|S < s_l) + b_2 \hat{E}(S|s_l < S < s_u) &= \frac{S_0}{B_0} \\ b_1 \hat{E}(c^i(S)|S < s_l) + b_2 \hat{E}(c^i(S)|s_l < S < s_u) &= \frac{c_0^i}{B_0}. \end{aligned}$$

$i = 1, 2.$

The upper (lower) bound for options with strike prices between K_1 *and* K_2 *is given by the pricing kernel* $\phi_2^*(S)$ $(\phi_2^{**}(S))$.

Proof: Let $\overline{\phi} \to +\infty$ and $\underline{\phi} \to 0$; from Lemma 5 we immediately conclude that we must have $\phi_2^{**}(S) = a\frac{\delta(S-s_1)}{p(S)} + f(S)$, where $f(S) = a_1$, $S < s_l$; $f(S) = a_2$, $S > s_l$, where a_0, a_1, a_2, and s_l are subject to $a_0 > 0$, $a_1 \geq a_2$, and the three equations. $\phi_1^*(S)$ can be similarly obtained. Q.E.D.

Figure 2: Two Observed Options: Exercise Prices = K_1, K_2



4 The General Case

In this section we deal with the case where we have n observed concurrently expiring options. We first introduce a lemma.

Lemma 6 *Assume two pricing kernels give the same prices of the underlying stock and options with strike prices K_1, K_2, ..., K_n, where $K_1 < K_2 < ... < K_n$. Let $K_0 = 0$ and $K_{n+1} = +\infty$. If the two pricing kernels intersect $n + 2$ times then the one with fatter left tail will give higher (lower) prices for all options with strike prices between (K_{2i-2}, K_{2i-1}) $((K_{2i-1}, K_{2i}))$, $i = 1, 2, ...$*

Proof: See Huang (2004a).

We now derive the option bounds under the assumption that the second order stochastic dominance rule applies and the value of the pricing kernel is bounded from above and below.

Lemma 7 *Assume the pricing kernel is decreasing in S and bounded above by $\overline{\phi}$ and below by $\underline{\phi}$. Assume the prices of a unit bond, the underlying stock, and n options with strike prices K_1, ..., K_n, where $K_1 < ... < K_n$, are B_0, S_0, c_0^1, ..., c_0^n respectively.*

Assume n is odd. Let $m = (n+1)/2$.

Then the upper bound for an option with strike price between K_{2i-2} and K_{2i-1}, $i = 1, 2, ...,$ is given by the pricing kernel $\phi_n^{**}(S) = \overline{\phi}$, $S < s_{l_1}$; $\phi_n^{**}(S) = a_1$, $s_{l_1} < S < s_{l_2}$; ...; $\phi_n^{**}(S) = a_m$, $s_{l_m} < S < s_{l_{m+1}}$; $\phi_n^{**}(S) = \underline{\phi}$, $S > s_{l_m+1}$, where a_1,..., a_m, s_{l_1}, ..., $s_{l_{m+1}}$ are to be determined such that $\overline{\phi} \geq a_1 \geq ... \geq a_m \geq \underline{\phi}$,

$$\begin{aligned}
\overline{\phi}\ Pr(S < s_{l_1}) + \sum_1^m a_j Pr(s_{l_j} < S < s_{l_{j+1}}) + \underline{\phi}\ Pr(S > s_{l_{m+1}}) &= 1 \\
\overline{\phi} E(S|S < s_{l_1}) + \sum_1^m a_j E(S|s_{l_j} < S < s_{l_{j+1}}) + \underline{\phi} E(S|S > s_{l_{m+1}}) &= \frac{S_0}{B_0} \\
\overline{\phi} E(c^i(S)|S < s_{l_1}) + \sum_1^m a_j E(c^i(S)|s_{l_j} < S < s_{l_{j+1}}) & \\
+ \underline{\phi} E(c^i(S)|S > s_{l_{m+1}})) &= \frac{c_0^i}{B_0}
\end{aligned}$$

$i = 1, 2, ..., n.$

The lower bound for an option with strike price below K_1 or above K_2 is given by the pricing kernel $\phi_n^*(S) = b_1$, $S < s_{l_1}$; ...; $\phi_n^*(S) = b_{m+1}$, $S > s_{l_m}$, where b_1, ..., b_{m+1}, s_{l_1}, ..., s_{l_m} are to be determined such that $b_1 \geq ... \geq b_{m+1}$,

$$\begin{aligned}
\sum_1^{m+1} b_j\ Pr(s_{l_{j-1}} < S < s_{l_j}) &= 1 \\
\sum_1^{m+1} b_j\ E(S|s_{l_{j-1}} < S < s_{l_j}) &= \frac{S_0}{B_0}
\end{aligned}$$

$$\sum_{1}^{m+1} b_j \; E(c^i(S)|s_{l_{j-1}} < S < s_{l_j}) \;\; = \;\; \frac{c_0^i}{B_0}$$

where $s_{l_{m+1}} = +\infty$ *and* $i = 1, 2, ..., n$.

The upper (lower) bound for options with strike prices between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...,$ *is given by the pricing kernel* $\phi_n^*(S)$ $(\phi_n^{**}(S))$.

Assume n *is even. Let* $m = n/2$.

- *Then the upper bound for an option with strike price between* K_{2i-2} *and* K_{2i-1}, $i = 1, 2, ...,$ *is given by the pricing kernel* $\phi_n^{**}(S) = \overline{\phi}$, $S < s_{l_1}$; $\phi_n^{**}(S) = a_1$, $s_{l_1} < S < s_{l_2}$; ...; $\phi_n^{**}(S) = a_{m+1}$, $S > s_{l_{m+1}}$, *where* a_1, ..., a_{m+1}, s_{l_1}, ..., $s_{l_{m+1}}$ *are to be determined such that*

$$\begin{aligned}
\overline{\phi}\; Pr(S < s_{l_1}) + \sum_{1}^{m+1} a_j Pr(s_{l_j} < S < s_{l_{j+1}}) &= 1, \\
\overline{\phi} E(S|S < s_{l_1}) + \sum_{1}^{m+1} a_j E(S|s_{l_j} < S < s_{l_{j+1}}) &= \frac{S_0}{B_0}, \\
\overline{\phi} E(c^i(S)|S < s_{l_1}) + \sum_{1}^{m+1} a_j E(c^i(S)|s_{l_j} < S < s_{l_{j+1}}) &= \frac{c_0^i}{B_0},
\end{aligned}$$

where $s_{l_{m+2}} = +\infty$ *and* $i = 1, 2, ..., n$.

The lower bound for an option with strike price between K_{2i-2} *and* K_{2i-1}, $i = 1, 2, ...,$ *is given by the pricing kernel* $\phi_n^*(S) = b_1$, $S < s_{l_1}$; ...; $\phi_n^*(S) = b_{m+1}$, $s_{l_1} < S < s_{l_{m+1}}$, $\phi_n^*(S) = \underline{\phi}$, $S > s_{l_{m+1}}$, *where* b_1,..., b_{m+1}, s_{l_1}, ..., $s_{l_{m+1}}$ *are to be determined such that* $b_1 \geq ... \geq b_{m+1}$,

$$\begin{aligned}
\sum_{1}^{m+1} b_j \; Pr(s_{l_{j-1}} < S < s_{l_j}) + \underline{\phi}\; Pr(S > s_{l_{m+1}}) &= 1 \\
\sum_{1}^{m+1} b_j \; E(S|s_{l_{j-1}} < S < s_{l_j}) + \underline{\phi} E(S|S > s_{l_{m+1}}) &= \frac{S_0}{B_0},
\end{aligned}$$

$$\sum_{1}^{m+1} b_j \ E(c^i(S)|s_{l_{j-1}} < S < s_{l_j}) + \underline{\phi} E(c^i(S)|S > s_{l_{m+1}}) \quad = \quad \frac{c_0^i}{B_0},$$

where $s_{l_0} = s_1$ *and* $i = 1, 2, ..., n$.

Proof:From Lemma 6 we need only prove that the pricing kernels described in the lemma intersect all admissible pricing kernels exactly $(n+2)$ times and then examine the fatness of their left tails.

We first examine ϕ_n^{**}. Assume n is odd. Note it is (m+2)-segmented and piecewise constant, where $m = (n+1)/2$. More precisely, $\phi_n^{**}(S) = \overline{\phi}$, $S < s_{l_1}$; $\phi_n^{**}(S) = a_1$, $s_{l_1} < S < s_{l_2}$; ...; $\phi_n^{**}(S) = a_m$, $s_{l_m} < S < s_{l_{m+1}}$; $\phi_n^{**}(S) = \underline{\phi}$, $S > s_{l_{m+1}}$, where $\overline{\phi} \geq a_1 \geq ... \geq a_m \geq \underline{\phi}$.

Obviously this pricing kernel intersects any decreasing pricing kernel at most $(n+2)$ times. However from Lemma 6, it must intersect all admissible pricing kernels at least $(n+2)$ times; otherwise they cannot give the same observed option prices. Hence ϕ_n^{**} intersects all admissible pricing kernel exactly $(n+2)$ times. It is not difficult to verify that ϕ_1^{**} has fatter left tail. This proves the first result. Other results can similarly proved. Q.E.D.

Proposition 2 *Assume the pricing kernel is decreasing in S. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of n options with strike prices K_1, K_2, ..., K_n are c_0^1, c_0^2, ..., and c_0^n respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

1. *Assume n is odd. Let $m = (n+1)/2$.*

   (a) *Then the upper bound for options with strike prices between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ is given by the pricing kernel $\phi_n^{**}(S) = a_0 \frac{\delta(S-s_1)}{p(S)} +$*

$f(S)$, where $p(S)$ is the probability density function and $\delta(S)$ is the Dirac function and $f(S) = a_1$, $S < s_{l_1}$; $f(S) = a_2$, $s_{l_1} < S < s_{l_2}$; ...; $f(S) = a_m$, $s_{l_{m-1}} < S < s_{l_m}$; $f(S) = 0$, $S > s_{l_m}$, where a_0, a_1, ..., a_m, s_{l_1}, s_{l_2}, ..., and s_{l_m} are to be determined such that $a_0 > 0$, $a_1 \geq ... \geq a_m$,

$$\begin{aligned} a_0 + \sum_1^m a_j \; Pr(s_{l_{j-1}} < S < s_{l_j}) &= 1 \\ a_0 s_1 + \sum_1^m a_j \; E(S|s_{l_{j-1}} < S < s_{l_j}) &= \frac{S_0}{B_0} \\ a_0 c^i(s_1) + \sum_1^m a_j \; E(c^i(S)|s_{l_{j-1}} < S < s_{l_j}) &= \frac{c_0^i}{B_0} \end{aligned}$$

where $s_{l_0} = 0$ and $i = 1, 2, ..., n$.

(b) *The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, is given by the pricing kernel $\phi_n^*(S) = b_1$, $S < s_{l_1}$; $\phi_n^*(S) = b_2$, $s_{l_1} < S < s_{l_2}$; ...; $\phi_n^*(S) = b_{m+1}$, $S > s_{l_m}$, where b_1, b_2,..., b_{m+1}, s_{l_1}, s_{l_2},..., and s_{l_m} are to be determined such that $b_1 \geq b_2 \geq ... \geq b_{m+1}$,*

$$\begin{aligned} \sum_1^{m+1} b_j \; Pr(s_{l_{j-1}} < S < s_{l_j}) &= 1 \\ \sum_1^{m+1} b_j \; E(S|s_{l_{j-1}} < S < s_{l_j}) &= \frac{S_0}{B_0} \\ \sum_1^{m+1} b_j \; E(c^i(S)|s_{l_{j-1}} < S < s_{l_j}) &= \frac{c_0^i}{B_0} \end{aligned}$$

where $s_{l_0} = 0$, $s_{l_{m+1}} = +\infty$, and $i = 1, 2, ..., n$.

(c) *The upper (lower) bound for options with strike prices between $(K_{2i-1}$, $K_{2i})$, $i = 1, 2, ...$, is given by the pricing kernel $\phi_n^*(S)$ $(\phi_n^{**}(S))$.*

2. *Assume n is even. Let $m = n/2$.*

   (a) *Then the upper bound for options with strike prices between $(K_{2i-2}$, $K_{2i-1})$, $i = 1, 2, ...,$ is given by the pricing kernel $\phi_n^{**}(S) = a_0 \frac{\delta(S-s_1)}{p(S)} +$ $f(S)$, where $p(S)$ is the probability density function and $\delta(S)$ is the Dirac function and $f(S) = a_1$, $S < s_{l_1}$; $f(S) = a_2$, $s_{l_1} < S < s_{l_2}$; ...; $f(S) = a_m$, $s_{l_{m-1}} < S < s_{l_m}$; $f(S) = a_{m+1}$, $S > s_{l_m}$, where a_0, a_1, ..., a_{m+1}, s_{l_1}, s_{l_2}, ..., and s_{l_m} are to be determined such that $a_0 > 0$, $a_1 \geq ... \geq a_{m+1}$,*

$$\begin{aligned} a_0 + \sum_1^{m+1} a_j \ Pr(s_{l_{j-1}} < S < s_{l_j}) &= 1 \\ a_0 s_1 + \sum_1^{m+1} a_j \ E(S|s_{l_{j-1}} < S < s_{l_j}) &= \frac{S_0}{B_0} \\ a_0 c^i(s_1) + \sum_1^{m+1} a_j \ E(c^i(S)|s_{l_{j-1}} < S < s_{l_j}) &= \frac{c_0^i}{B_0} \end{aligned}$$

   where $s_{l_0} = 0$, $s_{l_{m+1}} = +\infty$, and $i = 1, 2, ..., n$.

   (b) *The lower bound for options with strike prices between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ is given by the pricing kernel $\phi_n^*(S) = b_1$, $S < s_{l_1}$; $\phi_n^*(S) = b_2$, $s_{l_1} < S < s_{l_2}$; ...; $\phi_n^*(S) = b_{m+1}$, $s_{l_m} < S < s_{l_{m+1}}$; $\phi_n^*(S) = 0$, $S > s_{l_{m+1}}$, where b_1, b_2,..., b_{m+1}, s_{l_1}, s_{l_2},..., and $s_{l_{m+1}}$ are to be determined such that $b_1 \geq b_2 \geq ... \geq b_{m+1}$,*

$$\begin{aligned} \sum_1^{m+1} b_j \ Pr(s_{l_{j-1}} < S < s_{l_j}) &= 1 \\ \sum_1^{m+1} b_j \ E(S|s_{l_{j-1}} < S < s_{l_j}) &= \frac{S_0}{B_0} \\ \sum_1^{m+1} b_j \ E(c^i(S)|s_{l_{j-1}} < S < s_{l_j}) &= \frac{c_0^i}{B_0} \end{aligned}$$

where $s_{l_0} = 0$ and $i = 1, 2, ..., n$.

(c) *The upper (lower) bound for options with strike prices between (K_{2i-1}, K_{2i}), $i = 1, 2, ...,$ is given by the pricing kernel $\phi_n^*(S)$ $(\phi_n^{**}(S))$.*

Proof: Assume n is odd. Let $\overline{\phi} \to +\infty$ and $\underline{\phi} \to 0$; from Lemma 7 we immediately conclude that we must have $\phi_n^{**}(S) = a\frac{\delta(S-s_1)}{p(S)} + f(S)$, where $f(S) = a_1$, $S < s_{l_1}$; $f(S) = a_2$, $s_{l_1} < S < s_{l_2}$; ...; $f(S) = a_m$, $s_{l_{m-1}} < S < s_{l_m}$; $f(S) = 0$, $S > s_{l_m}$, where a_0, a_1, ..., a_m, s_{l_1}, s_{l_2}, ..., and s_{l_m} are to be determined by the $n + 2$ equations. This proves the first result. Other results can be similarly proved. Q.E.D.

5 The Arbitrage Portfolios

When the option bounds derived in this paper are violated, then there are second order arbitrage opportunities in the markets. In this case we can construct arbitrage portfolios to make profits. But first we have to know how to construct such portfolios. In order to get the right solution, we will first work out the arbitrage portfolios in a discrete state space, then pass it to the limit continuous case we will obtain our result.

5.1 Solutions in a Discrete State Space

We directly present the general case where we use n observed concurrently expiring options. We have the following result.

Lemma 8 *Assume d_j is decreasing in j. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of n options with strike prices K_1, K_2, ..., K_n are c_0^1, c_0^2, ..., and c_0^n respectively. Let $K_0 = 0$ and $K_{n+1} = +\infty$.*

1. *Assume n is odd. Let $m = (n+1)/2$.*

   (a) *For options with strike prices X between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ the solution to (P1) is given by*

   $$0 \le (y_{l_1}, y_{l_1+1}, ..., y_{l_m}, y_{l_m+1}, y_\Lambda) = (B_0, S_0, c_{01}, ..., c_{0n})A^{-1} \qquad (1)$$

   where matrix A is given by

   $$\left\{ \begin{array}{ccccc} 1 & \bar{s}_{l_1} & \bar{c}^1_{l_1} & ... & \bar{c}^n_{l_1} \\ 1 & \bar{s}_{l_1+1} & \bar{c}^1_{l_1+1} & ... & \bar{c}^n_{l_1+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}^1_{l_m} & ... & \bar{c}^n_{l_m} \\ 1 & \bar{s}_{l_m+1} & \bar{c}^1_{l_m+1} & ... & \bar{c}^n_{l_m+1} \\ 1 & \bar{s}_\Lambda & \bar{c}^1_\Lambda & ... & \bar{c}^n_\Lambda \end{array} \right\}$$

   $y_j = 0$, $j \ne l_1, l_1+1, ..., l_m, l_m+1, \Lambda$.

   (b) *For options with strike prices X between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ the solution to Problem P2 is given by*

   $$0 \le (y_1, y_{l_1}, y_{l_1+1}, ..., y_{l_m}, y_{l_m+1}) = (B_0, S_0, c_{01}, ..., c_{0n})B^{-1} \qquad (2)$$

where matrix B is given by

$$\left\{\begin{array}{ccccc} 1 & \bar{s}_1 & \bar{c}_1^1 & ... & \bar{c}_1^n \\ 1 & \bar{s}_{l_1} & \bar{c}_{l_1}^1 & ... & \bar{c}_{l_1}^n \\ 1 & \bar{s}_{l_1+1} & \bar{c}_{l_1+1}^1 & ... & \bar{c}_{l_1+1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}_{l_m}^1 & ... & \bar{c}_{l_m}^n \\ 1 & \bar{s}_{l_m+1} & \bar{c}_{l_m+1}^1 & ... & \bar{c}_{l_m+1}^n \end{array}\right\}$$

$y_j = 0$, $j \neq 1, l_1, l_1+1, ..., l_m, l_m+1$.

(c) *For options with strike prices X between* $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...,$ *the solution to Problem P1 is given by (2) and the solution to Problem P2 is given by (1).*

2. *Assume n is even. Let* $m = n/2$.

(a) *For options with strike prices X between* $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...,$ *the solution to (P1) is given by*

$$0 \leq (y_{l_1}, y_{l_1+1}, ..., y_{l_{m+1}}, y_{l_{m+1}+1}) = (B_0, S_0, c_{01}, ..., c_{0n})U^{-1} \qquad (3)$$

where matrix U is given by

$$\left\{\begin{array}{ccccc} 1 & \bar{s}_{l_1} & \bar{c}_{l_1}^1 & ... & \bar{c}_{l_1}^n \\ 1 & \bar{s}_{l_1+1} & \bar{c}_{l_1+1}^1 & ... & \bar{c}_{l_1+1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_{m+1}} & \bar{c}_{l_{m+1}}^1 & ... & \bar{c}_{l_m}^n \\ 1 & \bar{s}_{l_{m+1}+1} & \bar{c}_{l_{m+1}+1}^1 & ... & \bar{c}_{l_{m+1}+1}^n \end{array}\right\}$$

$y_j = 0$, $j \neq l_1, l_1+1, ..., l_{m+1}, l_{m+1}+1$.

(b) For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, \ldots$, the solution to Problem P2 is given by

$$0 \leq (y_1, y_{l_1}, y_{l_1+1}, \ldots, y_{l_m}, y_{l_m+1}, y_\Lambda) = (B_0, S_0, c_{01}, \ldots, c_{0n})V^{-1} \quad (4)$$

where matrix V is given by

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_1 & \bar{c}_1^1 & \ldots & \bar{c}_1^n \\ 1 & \bar{s}_{l_1} & \bar{c}_{l_1}^1 & \ldots & \bar{c}_{l_1}^n \\ 1 & \bar{s}_{l_1+1} & \bar{c}_{l_1+1}^1 & \ldots & \bar{c}_{l_1+1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}_{l_m}^1 & \ldots & \bar{c}_{l_m}^n \\ 1 & \bar{s}_{l_m+1} & \bar{c}_{l_m+1}^1 & \ldots & \bar{c}_{l_m+1}^n \\ 1 & \bar{s}_\Lambda & \bar{c}_\Lambda^1 & \ldots & \bar{c}_\Lambda^n \end{array} \right\}$$

$y_j = 0,\ j \neq 1, l_1, l_1+1, \ldots, l_m, l_m+1, \Lambda.$

(c) For options with strike prices X between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, \ldots$, the solution to Problem P1 is given by (4) and the solution to Problem P2 is given by (3).

Proof: The proof is similar to the continuous case. Note Lemma 6 is valid for discrete state spaces.[3] Thus we need only to find the right pricing kernels that intersect the true pricing kernel exactly $(n+2)$ times. These pricing kernels have similar features as their counterparts in the continuous case. That is they are piecewise constant, and the numbers of their segments are equal to their counterparts in the continuous case. For example, if n is odd the pricing kernel

[3]See Huang (2004a).

that gives an upper bound for options with strike prices X between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, has $m+1$ segments, where $m = (n+1)/2$. Hence we have

$$\begin{aligned} d_1 = & \quad ... \quad = d_{l_1}, \\ d_{l_1+2} = & \quad ... \quad = d_{l_2}, \\ & \quad ... \quad , \\ d_{l_{m-1}+2} = & \quad ... \quad = d_{l_m}, \\ d_{l_m+2} = & \quad ... \quad = d_{\Lambda}. \end{aligned}$$

Note the only difference with the continuous case is that there is freedom for the points between adjacent segments to choose their own values. These here are d_{l_j+1}, $j = 1, ..., m$.

Since $y_j = (\sum_1^j \pi_i)(d_j - d_{j+1})$, we have

$$\begin{aligned} y_1 = & \quad ... \quad = y_{l_1-1} = 0, \\ y_{l_1+2} = & \quad ... \quad = y_{l_2-1} = 0, \\ & \quad ... \quad , \\ y_{l_{m-1}+2} = & \quad ... \quad = y_{l_m-1} = 0, \\ y_{l_m+2} = & \quad ... \quad = y_{\Lambda-1} = 0, \end{aligned}$$

$y_{l_j} \geq 0$, $y_{l_j+1} \geq 0$, $j = 1, ..., m$, and $y_{\Lambda} \geq 0$. This proves the first result. The other results can be similarly proved. Q.E.D.

5.2 Arbitrage Portfolios in a Discrete State Space

Now we can derive the arbitrage portfolios in the discrete case. We have the following result.

Lemma 9 *Assume d_j is decreasing in j. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of n options with strike prices K_1, K_2, ..., K_n are c_0^1, c_0^2, ..., and c_0^n respectively.*

- *Assume n is odd. Let $m = (n+1)/2$.*
  - *For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...$, when its lower bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n+2}) = (\bar{c}_{l_1}^X, \bar{c}_{l_1+1}^X, ..., \bar{c}_{l_m}^X, \bar{c}_{l_m+1}^X, \bar{c}_{\Lambda}^X)(A^{-1})^T, \quad (5)$$

    where $l_1, ..., l_m$, and A are determined by (1a) in Lemma 8.
  - *For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...$, when its upper bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n+2}) = (\bar{c}_{1}^X, \bar{c}_{l_1}^X, \bar{c}_{l_1+1}^X, ..., \bar{c}_{l_m}^X, \bar{c}_{l_m+1}^X)(B^{-1})^T, \quad (6)$$

    where $l_1, ..., l_m$, and B are determined by (1b) in Lemma 8.
  - *For options with strike prices X between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...$, when its lower bound is violated the arbitrage portfolio is given by (6); when its upper bound is violated the arbitrage portfolio is given by (5).*
- *Assume n is even. Let $m = n/2$.*

- *For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...,$ when its lower bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n+2}) = (\bar{c}^X_{l_1}, \bar{c}^X_{l_1+1}, ..., \bar{c}^X_{l_{m+1}}, \bar{c}^X_{l_{m+1}+1})(U^{-1})^T, \quad (7)$$

where $l_1, ..., l_{m+1}$, and U are determined by (2a) in Lemma 8.

- *For options with strike prices X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...,$ when its upper bound is violated the arbitrage portfolio is given by*

$$(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n+2}) = (\bar{c}^X_1, \bar{c}^X_{l_1}, \bar{c}^X_{l_1+1}, ..., \bar{c}^X_{l_m}, \bar{c}^X_{l_m+1}, \bar{c}^X_\Lambda)(V^{-1})^T, \quad (8)$$

where $l_1, ..., l_m$, and V are determined by (2b) in Lemma 8.

- *For options with strike prices X between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...,$ when its lower bound is violated the arbitrage portfolio is given by (8); when its upper bound is violated the arbitrage portfolio is given by (7).*

Proof: Suppose n is odd. For options with strike prices X between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...,$ applying Lemma 8, we obtain its lower bound

$$\begin{aligned}&(y_{l_1}, y_{l_1+1}, ..., y_{l_m}, y_{l_m+1}, y_\Lambda)(\bar{c}^X_{l_1}, \bar{c}^X_{l_1+1}, ..., \bar{c}^X_{l_m}, \bar{c}^X_{l_m+1}, \bar{c}^X_\Lambda)^T \\ &= (B_0, S_0, c_{01}, ..., c_{0n})A^{-1}(\bar{c}^X_{l_1}, \bar{c}^X_{l_1+1}, ..., \bar{c}^X_{l_m}, \bar{c}^X_{l_m+1}, \bar{c}^X_\Lambda)^T,\end{aligned}$$

which is the optimal value of the objective function of Problem (P1). At optimality, the primal and dual objective functions are equal. Thus the above value must be equal to the optimal value of Problem (D1)'s objective function, i.e.,

$$(B_0, S_0, c_{01}, ..., c_{0n})(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n+2})^T.$$

Note this must hold for all values of B_0, S_0, c_0^1, ..., c_0^n. Hence we obtain (7). Other results can be similarly proved. Q.E.D.

5.3 Arbitrage Portfolios in a Continuous State Space

If we let the differences between adjacent states become smaller and smaller, we come closer and closer to a continuous state space. In the limit we reach the continuous state space and for all j, s_{l_j+1} and s_{l_j} converge to a single s_{l_j}. In a continuous state space, we have

$$\bar{s}_{l_j} = \hat{E}(S|S < s_{l_j}), \quad \bar{c}^i_{l_j} = \hat{E}(c^i(S)|S < s_{l_j}), \quad \bar{c}^X_{l_j} = \hat{E}(c^X(S)|S < s_{l_j}).$$

We will also use the following notation: $(\bar{c}^X_{l_j})' \equiv d\hat{E}(c^X(S)|S < s_{l_j})/dS|_{S=s_{l_j}}$. When we take the limit while for all j $s_{l_j+1} \to s_{l_j}$, for brevity we write $\lim_{\forall j, s_{l_j+1}\to s_{l_j}}$ simply as lim.

From the results given in the last subsection we can obtain the arbitrage portfolios in a continuous state space. We now present our main result.

Proposition 3 *Assume the pricing kernel is decreasing in S. Assume the price of a unit bond is B_0, the underlying stock price is S_0, and the prices of n options with strike prices K_1, K_2, ..., K_n are c_0^1, c_0^2, ..., and c_0^n respectively.*

- *Assume n is odd. Let $m = (n+1)/2$.*
  - *For options with strike prices X between (K_{2i-2}, K_{2i-1}), $i = 1, 2, ...$, when its lower bound is violated the arbitrage portfolio is given by*

$$\alpha_i = (-1)^i [\sum_1^m ((\bar{c}^X_{l_v})' \hat{A}_{2v,i} - \bar{c}^X_{l_v} \hat{A}_{2v-1,i}) - \bar{c}^X_\Lambda \hat{A}_{n+2,i}]/|\hat{A}|, \qquad (9)$$

i=1, ..., n+2, where $s_{l_1}, ..., s_{l_m}$, *are determined by 1(b) in Proposition 2 and* $\hat{A}$ *is given by*

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_{l_1} & \bar{c}^1_{l_1} & ... & \bar{c}^n_{l_1} \\ 0 & (\bar{s}_{l_1})' & (\bar{c}^1_{l_1})' & ... & (\bar{c}^n_{l_1})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}^1_{l_m} & ... & \bar{c}^n_{l_m} \\ 0 & (\bar{s}_{l_m})' & (\bar{c}^1_{l_m})' & ... & (\bar{c}^n_{l_m})' \\ 1 & \bar{s}_{\Lambda} & \bar{c}^1_{\Lambda} & ... & \bar{c}^n_{\Lambda} \end{array} \right\} \tag{10}$$

– *For options with strike prices* X *between* $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...,$ *when its upper bound is violated the arbitrage portfolio is given by*

$$\alpha_i = (-1)^i[-\bar{c}^X_1 \hat{B}_{1i} + \sum_1^m (\bar{c}^X_{l_v} \hat{B}_{2v,i} - (\bar{c}^X_{l_v})' \hat{B}_{2v+1,i})]/|\hat{B}|, \tag{11}$$

i=1, ..., n+2, where $s_{l_1}, ..., s_{l_m}$ *are determined by 1(a) in Proposition 2 and* $\hat{B}$ *is given by*

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_1 & \bar{c}^1_1 & ... & \bar{c}^n_1 \\ 1 & \bar{s}_{l_1} & \bar{c}^1_{l_1} & ... & \bar{c}^n_{l_1} \\ 0 & (\bar{s}_{l_1})' & (\bar{c}^1_{l_1})' & ... & (\bar{c}^n_{l_1})' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}^1_{l_m} & ... & \bar{c}^n_{l_m} \\ 0 & (\bar{s}_{l_m})' & (\bar{c}^1_{l_m})' & ... & (\bar{c}^n_{l_m})' \end{array} \right\} \tag{12}$$

– *For options with strike prices* X *between* $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...,$ *when its lower bound is violated the arbitrage portfolio is given by (11); when its upper bound is violated the arbitrage portfolio is given by (9)*

- *Assume n is even. Let $m = n/2$.*
  - *For options with strike prices X between $(K_{2i-2},\, K_{2i-1})$, $i = 1, 2, ...,$ when its lower bound is violated the arbitrage portfolio is given by*

$$\alpha_i = (-1)^i \sum_{1}^{m+1} ((\bar{c}_{l_v}^X)' \hat{U}_{2v,i} - \bar{c}_{l_v}^X \hat{U}_{2v-1,i}) / |\hat{U}|, \tag{13}$$

i=1, ..., n+2, where $s_{l_1}, ..., s_{l_m}$, are determined by 2(b) in Proposition 2 and $\hat{U}$ is given by

$$\left\{ \begin{array}{ccccc} 1 & \bar{s}_{l_1} & \bar{c}_{l_1}^1 & ... & \bar{c}_{l_1}^n \\ 0 & (\bar{s}_{l_1})' & (\bar{c}_{l_1}^1)' & ... & (\bar{c}_{l_1}^n)' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_{m+1}} & \bar{c}_{l_{m+1}}^1 & ... & \bar{c}_{l_{m+1}}^n \\ 0 & (\bar{s}_{l_{m+1}})' & (\bar{c}_{l_{m+1}}^1)' & ... & (\bar{c}_{l_{m+1}}^n)' \end{array} \right\} \tag{14}$$

  - *For options with strike prices X between $(K_{2i-2},\, K_{2i-1})$, $i = 1, 2, ...,$ when its upper bound is violated the arbitrage portfolio is given by*

$$\alpha_i = (-1)^i [-\bar{c}_1^X \hat{V}_{1i} + \sum_{1}^{m} (\bar{c}_{l_v}^X \hat{V}_{2v,i} - (\bar{c}_{l_v}^X)' \hat{V}_{2v+1,i}) + \bar{c}_\Lambda^X \hat{V}_{n+2,i}] / |\hat{V}|, \tag{15}$$

i=1, ..., n+2, where $s_{l_1}, ..., s_{l_m}$, are determined by 2(a) in Proposition

2 and $\hat{V}$ is given by

$$\left\{\begin{array}{ccccc} 1 & \bar{s}_1 & \bar{c}_1^1 & ... & \bar{c}_1^n \\ 1 & \bar{s}_{l_1} & \bar{c}_{l_1}^1 & ... & \bar{c}_{l_1}^n \\ 0 & (\bar{s}_{l_1})' & (\bar{c}_{l_1}^1)' & ... & (\bar{c}_{l_1}^n)' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}_{l_m}^1 & ... & \bar{c}_{l_m}^n \\ 0 & (\bar{s}_{l_m})' & (\bar{c}_{l_m}^1)' & ... & (\bar{c}_{l_m}^n)' \\ 1 & \bar{s}_\Lambda & \bar{c}_\Lambda^1 & ... & \bar{c}_\Lambda^n \end{array}\right\} \tag{16}$$

– *For options with strike prices X between $(K_{2i-1},\ K_{2i})$, $i = 1, 2, ...$, when its lower bound is violated the arbitrage portfolio is given by (15); when its upper bound is violated the arbitrage portfolio is given by (13).*

Proof: Assume n is odd. Let $m = (n + 1)/2$. Consider an option with strike price X between $(K_{2i-2},\ K_{2i-1})$, $i = 1, 2, ...$ If the upper bound of its value is violated, then applying Lemma 9 we know the arbitrage portfolio in a discrete state space is given by (6). We have

$$(B^T)^{-1} = \frac{1}{|B|}((-1)^{i+j}B_{ij})_{(n+2)\times(n+2)}.$$

Hence from (5) we have

$$|B|\alpha_i = \bar{c}_1^X(-1)^{1+i}B_{1i} + \sum_1^m (-1)^{2v+i}(\bar{c}_{l_v}^X B_{2v,i} - \bar{c}_{l_v+1}^X B_{2v+1,i}).$$

Rewrite it as

$$|B|\alpha_i = \bar{c}_1^X(-1)^{1+i}B_{1i} + \sum_1^m (-1)^{2v+i}(\bar{c}_{l_v}^X(B_{2v,i} - B_{2v+1,i}) - (\bar{c}_{l_v+1}^X - \bar{c}_{l_v}^X)B_{2v+1,i}). \tag{17}$$

Let $\hat{B}$ be given by (12).

Obviously we have

$$|B| = \begin{vmatrix} 1 & \bar{s}_1 & \bar{c}_1^1 & \dots & \bar{c}_1^n \\ 1 & \bar{s}_{l_1} & \bar{c}_{l_1}^1 & \dots & \bar{c}_{l_1}^n \\ 0 & \bar{s}_{l_1+1} - \bar{s}_{l_1} & \bar{c}_{l_1+1}^1 - \bar{c}_{l_1}^1 & \dots & \bar{c}_{l_1+1}^n - \bar{c}_{l_1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{s}_{l_m} & \bar{c}_{l_m}^1 & \dots & \bar{c}_{l_m}^n \\ 0 & \bar{s}_{l_m+1} - \bar{s}_{l_m} & \bar{c}_{l_m+1}^1 - \bar{c}_{l_m}^1 & \dots & \bar{c}_{l_m+1}^n - \bar{c}_{l_m}^n \end{vmatrix}$$

Hence

$$\lim\ |B|/[(s_{l_1+1} - s_{l_1})...(s_{l_m+1} - s_{l_m})] = |\hat{B}|.$$

Similarly we have

$$\lim\ B_{1i}/[(s_{l_1+1} - s_{l_1})...(s_{l_m+1} - s_{l_m})] = \hat{B}_{1i},$$

$$\lim\ (B_{2v,i} - B_{2v+1,i})/[(s_{l_1+1} - s_{l_1})...(s_{l_m+1} - s_{l_m})] = \hat{B}_{2v,i},$$

$$\lim\ (\bar{c}_{l_v+1}^X - \bar{c}_{l_v}^X)B_{2v+1,i})/[(s_{l_1+1} - s_{l_1})...(s_{l_m+1} - s_{l_m})] = (\bar{c}_{l_v}^X)'\hat{B}_{2v+1,i}.$$

Substituting the above four equations into (17) while taking the limit we obtain

$$|\hat{B}|\alpha_i = \bar{c}_1^X(-1)^{1+i}\hat{B}_{1i} + \sum_1^m (-1)^{2v+i}(\bar{c}_{l_v}^X\hat{B}_{2v,i} - (\bar{c}_{l_v}^X)'\hat{B}_{2v+1,i}).$$

This proves (11). Other results can be similarly proved. Q.E.D.

6 Conclusions

As argued by Ryan (2003), the pricing information for the pricing kernel contained in the relevant options, as well as in the bond and underlying stock, can significantly improve the previous option bounds. In this paper we have improved the SSD option bounds by using concurrently expiring options. We have shown that given the prices of the underlying stock and n concurrently expiring options, the option bounds are given by piecewise constant pricing kernels.

When these SSD option bounds are violated there are second order arbitrage opportunities in the market. We have also presented the arbitrage portfolios that will be used to make profits from these arbitrage opportunities. Since first order arbitrage opportunities virtually do not exist, our results on second order arbitrage opportunities are particularly useful.

It would be interesting if the results derived in this paper are extended to the continuous time case.

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