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# Risk neutral probabilities and option bounds: a geometric approach

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# Risk Neutral Probabilities and Option Bounds: A Geometric Approach

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**Abstract**

In this paper we first present a geometric approach to option bounds. We show that if two risk neutral probability density functions intersect for certain number of times, then comparing the fatness of their tails we can tell which of them gives higher option prices. Thus we can derive option bounds by identifying the risk neutral probability density function which intersects all admissible ones for certain number of times. Applying this approach we tighten the first order stochastic dominance option bounds from concurrently expiring options when the maximum value of the risk neutral density are known.

Keywords: Option bounds, option pricing, risk neutral density, first order s-tochastic dominance.

JEL Classification Numbers: G13.

# Introduction

There are some excellent techniques which are used to derive option bounds. A natural one is the arbitrage approach. Merton (1973), Garman (1976), and Levy (1985) use this approach to derive the first and second order stochastic dominance option bounds. Grundy (1991) uses it to explore the relation between option prices and the true distributions. The intuition of this approach is to compare different portfolio strategies involving the underlying stock and options and work out the option bounds by excluding the existence of any dominant strategies.

Ritchken (1985) introduce the linear programming approach to this area. Ritchken and Kuo (1989), Basso and Pianca (1997), Mathur and Ritchken (2000), and Ryan (2003) use it to derive important results on option bounds. The key of this approach is to model the option bound problem in a discrete state space as a linear programming problem and work out the solution. The advantage of this technique is that there are many reliable techniques of handling linear programming problems. The disadvantage is that it often brings much complexity to calculations.

Others such as Boyle and Lin (1997) and Bertsimas and Popescu (2002) use convex and, in particular semidefinite optimization approach to derive option bounds while Cochrane and Saa-Requejo (2000) derive option bounds with restrictions on the volatility of the pricing kernel using information from other assets.

In this paper we present a new approach. To derive the option bounds under certain conditions, the question we have to answer is what risk neutral probabilities that satisfy specified conditions give higher option prices. In this paper we show that if the number of intersections between risk neutral probability density functions (hereafter P.D.F) is restricted then the fatness of the P.D.Fs' tails holds the key to the answer and this can be worked out using a geometric approach.

More specifically, we show that assuming two risk neutral P.D.Fs give the same prices of the underlying stock and $n$ options on the stock, if they intersect $n+2$ times, then comparing the fatness of their tails we can tell which one gives higher prices of what options. Thus the key of the technique is to identify the risk neutral P.D.F which intersect all admissible risk neutral P.D.Fs for certain number of times.

Applying the new approach we tighten Bertsimas and Popescu's (2002) first order stochastic dominance option bounds from concurrently expiring options by using the only additional information of the maximum value of the risk neutral densities. Note we can always put reasonable bounds on the risk neutral densities. Thus to assume the knowledge of the maximum value of the risk neutral densities is hardly a strong condition.

Although this approach to option bounds takes the advantage of the distinctive feature of options, as a convenient and useful optimization technique it must have broader implications for similar problems in other areas.

The structure of the paper is as follows: In Section 1 we present a geometry of risk neutral probabilities. In Section 2 we derive the first order stochastic dominance option bounds knowing the maximum and minimum values of the risk neutral probabilities of individual states. Section 3 concludes the paper.

## 1 A Geometry of Risk Neutral Probabilities

We assume that there is a stock in an economy on which option contracts are written. The price of the stock at time $t$ is denoted by $S_t$. We assume that the prices of options as well as other contingent claims on the stock are given by a risk neutral probability measure or an equivalent martingale measure.[1] Thus if we denote the time $t$ price of a contingent by $c(S_t)$, its payoff at time $t$, we have $c_0 = B_0 E^Q(c(S_t))$, where $B_0$ is the time 0 price of a unit zero-coupon bond and $E^Q(.)$ denotes the expectation operator under the risk neutral measure $Q$. This probability measure $Q$ may or may not be unique depending on the market completeness.

We assume that the risk neutral measure $Q$ is represented by a probability density function (hereafter p.d.f) $q(S_t)$. We assume that the support of the p.d.f is a subset of $(0, +\infty)$ although the analysis in this paper is valid for cases where support of the p.d.f is a subset of $(-\infty, +\infty)$ (for example when the underlying asset is some cash flow). It follows that

$$c_0 = B_0 \int_0^{\infty} c(S_t) q(S_t) dS_t. \tag{1}$$

[1] We refer readers to Harris and Kreps (1979) and Harris and Pliska (1981) for the theory of equivalent martingale measures and to Cox and Ross (1976) for risk neutral measures.

If we know exactly what is $q(S_t)$, we would be able to calculate the exact price of options. However, in many cases, we do not know what $q(S_t)$ exactly is although we have some information about what $q(S_t)$ looks like. In these cases, since we cannot work out the exact option prices we are interested in the upper and lower bounds for the option prices. In this paper we present a geometry of risk neutral probabilities which will help to derive option bounds. Before we proceed first clarify a concept.

In this paper we talk about intersections of risk neutral probability density functions. When we say that $q_1(x)$ and $q_2(x)$ intersect $n$ times we mean there exists $s_1$, $s_2$, ..., $s_n$, where $0 = s_0 < s_1 < s_2 < ... < s_n < s_{n+1} = +\infty$ such that $(q_1(x_1) - q_2(x_1))(q_1(x_2) - q_2(x_2)) < 0$ for any $s_{i-1} < x_1 < s_i$ and any $s_i < x_2 < s_{i+1}$, $i = 1, 2, ..., n$. In that case, we also say $q_1(x)$ and $q_2(x)$ intersect at $s_1$, $s_2$, ..., and $s_n$.

## 1.1 Risk Neutral P.D.Fs Intersect Twice

As usual, we assume that the prices of a unit zero-coupon bond and the stock, $B_0$ and $S_0$, are known. Jagannathan (1984) shows that the more "risky" the risk neutral distribution of the underlying stock in the Rothschild-Stiglitz sense is, the more valuable an option on the stock. We have the following result.

**Proposition 1** *Assume two risk neutral P.D.Fs give the same bond price and stock price. If they intersect twice, then the one with fatter tails give higher option prices.*

Proof: Assume two risk neutral P.D.Fs $q_1(x)$ and $q_2(x)$ give the same stock price. Assume they intersect twice and $q_2(x)$ has fatter tails than $q_1(x)$.

Without loss of generality, assume the two P.D.Fs intersect at $x_1$ and $x_2$, where $x_1 < x_2$. Then we have

$$\begin{aligned} q_1(x) - q_2(x) &\leq 0, && x < x_1 \\ q_1(x) - q_2(x) &\geq 0, && x_1 < x < x_2 \\ q_1(x) - q_2(x) &\leq 0, && x > x_2 \end{aligned} \tag{2}$$

Now construct an arbitrage portfolio of the bond and the stock such that it has the same payoffs as the derivative at $x_1$ and $x_2$. Denote its payoff by $L(x)$. Then $L(x_i) = c(x_i)$, $i = 1, 2$. Since $L(x)$ is linear while $c(x)$ is convex we must have

$$\begin{aligned} c_X(x) - L(x) &\geq 0, && x < x_1 \\ c_X(x) - L(x) &\leq 0, && x_1 < x < x_2 \\ c_X(x) - L(x) &\geq 0, && x > x_2 \end{aligned} \tag{3}$$

From (2) and (3), we conclude that $(c(x) - L(x))$ and $(q_1(x) - q_2(x))$ always have opposite signs. It follows that

$$\int c(x)q_1(x)dx - \int c(x)q_2(x)dx = \int (c(x) - L(x))(q_1(x) - q_2(x))dx \leq 0.$$

Q.E.D.

The essential idea of the above proposition is present in Franke, Stapleton, and Subrahmanyam (1999). In their Theorem 1 they state that assuming two pricing kernels give the same prices of a unit bond and the underlying stock, then the one with declining elasticity gives higher prices to derivatives with convex payoffs than the one with constant elasticity.

## 1.2 Risk Neutral P.D.Fs Intersect Three Times

Now we deal with the case where two risk neutral P.D.Fs intersect three times. We have the following result.

**Proposition 2** *Assume two risk neutral P.D.Fs give the same prices of the bond, the stock, and an option with strike price $K$. If they intersect three times, then the one with fatter left tail give higher (lower) prices to options with strike prices below (above) $K$ .*

Proof: Assume the two P.D.Fs $q_1(x)$ and $q_2(x)$ intersect at $s_1 < s_2 < s_3$ such that

$$\begin{aligned} q^{**}(x) - q_2(x) &> 0, \qquad x < s_1 \\ q^{**}(x) - q_2(x) &< 0, \quad s_1 < x < s_2 \\ q^{**}(x) - q_2(x) &> 0, \quad s_2 < x < s_3 \\ q^{**}(x) - q_2(x) &< 0, \qquad x > s_3. \end{aligned} \tag{4}$$

Obviously given an option with strike price $X$, if $X \leq s_1$ or $X \geq s_3$, the proposition holds. Thus we need only prove it for $s_1 < X < s_3$.

Note since $q^{**}(x)$ and $q_2(x)$ give the same price of the observed option with strike price $K$, we must have $s_1 < K < s_3$.

Denote the option with strike price $X$ by $c_X$. Construct a portfolio of the unit bond, the underlying stock, and the observed option such that the payoff of the portfolio is equal to the payoff of $c_X$ at $x = s_1, s_2, s_3$. Denote the payoff of the portfolio by $L(x)$. Then we have $L(s_i) = c_X(s_i)$, $i = 1, 2, 3$.

Since $q_1(x)$ and $q_2(x)$ give the same prices of the unit bond, underlying stock, and observed option, we have $\int(q_1(x) - q_2(x))c_X(x)dx = \int((q_1(x) -$

$q_2(x))(c_X(x) - L(x))dx$. Because of (7), to prove $\int(q_1(x) - q_2(x))c_X(x)dx \geq 0$, we need only show for $s_1 < X < K$,

$$
\begin{aligned}
c_X(x) - L(x) &\geq 0, & x &< s_1 \\
c_X(x) - L(x) &\leq 0, & s_1 < x &< s_2 \\
c_X(x) - L(x) &\geq 0, & s_2 < x &< s_3 \\
c_X(x) - L(x) &\leq 0, & x &> s_3;
\end{aligned}
\tag{5}
$$

and for $s_3 > x > K$

$$
\begin{aligned}
c_X(x) - L(x) &\leq 0, & x &< s_1 \\
c_X(x) - L(x) &\geq 0, & s_1 < x &< s_2 \\
c_X(x) - L(x) &\leq 0, & s_2 < x &< s_3 \\
c_X(x) - L(x) &\geq 0, & x &> s_3;
\end{aligned}
\tag{6}
$$

Consider the space in which the horizontal axis is $x$ and the vertical axis is the payoff of a derivative. Suppose $s_1 < K \leq s_2$. Assume $s_1 < X < K$. In the space $L(x)$ is two-segmented and piecewise linear. From the right to the left its first linear segment passes through $(s_3, c_X(s_3))$ and $(s_2, c_X(s_2))$ and stops at $(K, L(K))$, where $L(K) > 0$; its second linear segment starts from $(K, L(K))$ and passes through $(s_1, 0)$. Obviously (9) holds.

Assume $s_3 > X > K$. In the space $L(x)$ is two-segmented and piecewise linear. From the right to the left its first linear segment passes through $(s_3, c_X(s_3))$ and $(s_2, c_X(s_2))$ and stops at $(K, L(K))$, where $L(K) < 0$; its second linear segment starts from $(K, L(K))$ and passes through $(s_1, c_X(s_1))$. Obviously (8) holds. Q.E.D.

### 1.3 Risk Neutral P.D.Fs Intersect Four Times

Now we deal with the case where two risk neutral P.D.Fs intersect four times. We have the following result.

**Proposition 3** *Assume two risk neutral P.D.Fs give the same prices of the bond, the stock, and two options with strike price $K_1$ and $K_2$, where $K_1 < K_2$. If they intersect four times, then the one with fatter left tail give lower (higher) prices to options with strike prices (not) in $(K_1, K_2)$ .*

Proof: Assume the two P.D.Fs $q_1(x)$ and $q_2(x)$ intersect at $s_1 < s_2 < s_3 < s_4$. Then we have

$$\begin{aligned}
q_1(x) - q_2(x) &> 0, & x &< s_1 \\
q_1(x) - q_2(x) &< 0, & s_1 < x &< s_2 \\
q_1(x) - q_2(x) &> 0, & s_2 < x &< s_3 \\
q_1(x) - q_2(x) &< 0, & s_3 < x &< s_4. \\
q_1(x) - q_2(x) &> 0, & x &> s_4
\end{aligned} \tag{7}$$

Note since $q_1(x)$ and $q_2(x)$ give the same price of the observed option with strike price $K$, we must have $s_1 < K1 < K_2 < s_4$. First assume $K_1$ and $K_2$ are not separated by either $s_2$ or $s_3$. Then we must have $K_i \in [s_2, s_3]$; otherwise using the method of constructing arbitrage portfolios as in the proof of Proposition 2, we can show that $q_1(x) = q_2(x)$, when $K_1 < x < K_2$.

Given an option with strike price $X$, assume $K_1 < X < K_2$. Now we again construct a portfolio of the unit bond, the underlying stock, and the two observed options such that the payoff of the portfolio is equal to the payoff of $c_X$ at $x = s_1, s_2, s_3, s_4$. Denote the payoff of the portfolio by $L(x)$. Then we have

$L(s_i) = c_X(s_i)$, $i = 1, 2, 3, 4$. Because of (7), if we can show for $K_1 < X < K_2$

$$\begin{aligned} c_X(x) - L(x) &\leq 0, & x &< s_1 \\ c_X(x) - L(x) &\geq 0, & s_1 < x &< s_2 \\ c_X(x) - L(x) &\leq 0, & s_2 < x &< s_3 \\ c_X(x) - L(x) &\geq 0, & s_3 < x &< s_4 \\ c_X(x) - L(x) &\leq 0, & x &> s_4, \end{aligned} \tag{8}$$

then the proof is done.

In the space $L(x)$ is three-segmented and piecewise linear. From the right to the left its first linear segment passes through $(s_4, c_X(s_4))$ and $(s_3, c_X(s_3))$ and stops at $(K_2, c_X(K_2))$, where $c_X(K_2) > 0$; its second linear segment starts from $(K_2, c_X(K_2))$ and stops at $(K_1, L(K_1))$, where $L(K_1) = 0$; its third linear segment starts from $(K_1, 0)$ and passes through origin. Obviously (8) holds.

For $X < K_1$ or $X > K_2$ we similarly construct portfolio $L(x)$ and show that

$$\begin{aligned} c_X(x) - L(x) &\geq 0, & x &< s_1 \\ c_X(x) - L(x) &\leq 0, & s_1 < x &< s_2 \\ c_X(x) - L(x) &\geq 0, & s_2 < x &< s_3 \\ c_X(x) - L(x) &\leq 0, & s_3 < x &< s_4 \\ c_X(x) - L(x) &\geq 0, & x &> s_4; \end{aligned} \tag{9}$$

thus the proof is done for the case where $K_1$ and $K_2$ are not separated by $s_2$ or $s_3$.

For the case where $K_1$ and $K_2$ are separated by $s_2$ or $s_3$, the proof is similar.

Q.E.D.

## 1.4 The General Case

Now we deal with the case where two risk neutral P.D.Fs intersect $n+2$ times. We have the following result.

**Proposition 4** *Assume two risk neutral P.D.Fs give the same prices of the bond, the stock, and $n$ options with strike price $K_1$, ..., $K_n$, where $0 = K_0 < K_1 < ... < K_n < K_{n+1} = +\infty$. If they intersect $n+2$ times, then the one with fatter left tail gives higher (lower) prices to options with strike prices (not) in $(K_{2i}, K_{2i+1})$, $i = 0, 1, ...$.*

Proof: Assume the two P.D.Fs intersect at $s_1$, ..., $s_{n+2}$. Now we only consider $k_i$, $i = 1, ..., n$ and $s_j$, $j = 1, ..., n+2$ unless stated otherwise. Without loss of generality assume there is no $K_i = s_j$.[2] We call an interval $(s_i, s_{i+1})$ a zero if there is no $k_j$ in this interval. We call it a single if there is just one $k_j$. Similarly, we define a double and a triple. Before we proceed, we need the following lemma, which is proved in Appendix 1.

**Lemma 1** *The following patterns are impossible.*

1. *A triple.*
2. *Adjacent doubles.*
3. *Two doubles linked by singles.*
4. *A double without a zero to its right (left).*

[2] Suppose for some $i$, $K_i = s_j$. If $(s_{j-1}, s_j)$ has more $K_q$'s than $(s_j, s_{j+1})$, treat it as if it is in $(s_{j-1}, s_j)$. Otherwise treat it as if it is $(s_j, s_{j+1})$. Then the rest of the proof will be the same.

*5. Adjacent zeros.*

*6. Two zeros linked by singles.*

With the help of the above lemma we now prove the proposition. Applying Lemma 1, we conclude that every chain of singles must be sandwiched by a zero and a double. That is, the general pattern of the intervals is as follows

> Single, ..., single, zero, single,..., single, double, single,..., single, zero,
> single,..., single, double, ..., single, ..., single, zero, single, ..., single,

where the chains of singles can have zero length.

Now given any option, using the technique which we apply in the previous propositions, i.e., construct arbitrage portfolios, we can directly verify that the proposition holds.

We prove it for every interval from the left to right by induction.

For the first chain of singles (if there are), it is easy to verify that the result holds.

Given any zero, if for all intervals to its left side the result holds, then the result must also hold for this zero; otherwise either the zero will become a single or one of its adjacent singles will become a double, which Lemma 1 forbids.

Given any single, assume for all intervals to its left side the result holds. Then with the help with the nearest double, we can construct an arbitrage portfolio to show that the result must also hold for this single.

Given any double, assume for all intervals to its left side the result holds. Then with the help of itself, we can construct an arbitrage portfolio to show that the result must also hold for this double. Q.E.D.

## 1.5 Two Extensions

### 1.5.1 Pricing Kernels

Now assume correspoding to the risk neutral probability measure $Q$, the true probability measure is $P$. Assume the risk neutral measure is represented by probability density function $q(S_t)$ while the true measure is represented by the true probability density function $p(S_t)$. Let $\phi(S_t) = q(S_t)/p(S_t)$. $\phi(S_t)$ is often called the pricing kernel.

All propositions derived in this section will hold if we replace risk neutral densities with pricing kernels and replace integration operators with expectation operators under the true probability measure $P$. The proofs are virtually the same.

### 1.5.2 The Discrete Case

If the state space is discrete, we can show that the results obtained previously in this section still hold. When the state space is continuous we use risk neutral probability density functions; when it is discrete, we have to use risk neutral probabilities.

Given two sets of risk neutral probabilities, $\{Q_i(S_j); j = 1, 2, ...\}$, $i = 1, 2$, assume $S_j$, $j = 1, 2...$, are in ascending order. When we say that the two sets intersect $n$ times we mean there exists $i_1$, $i_2$, ..., $i_n$, where $0 = i_0 < i_1 < i_2 < ... < i_n < i_{n+1} = +\infty$ such that $(Q_1(S_i) - Q_2(S_i))(Q_1(S_j) - Q_2(S_j)) < 0$ for any $i_{t-1} < i < i_t$ and any $i_t < j < s_{t+1}$, $t = 1, 2, ..., n$. In that case, we also say $Q_1(x)$ and $Q_2(x)$ intersect at $S_1$, $S_2$, ..., and $S_n$.

With the above explanantion, all propositions derived in this section will hold in a discrete state space if we replace risk neutral densities with risk neutral probabilities. The proofs are virtually the same.

# 2 Risk Neutral Probability and Option Bounds

There are arguments against using log-normal risk neutral probabilities to price options. The reason is that the actual risk neutral probabilities can be very abnormal. In this section we propose a method to derive option bounds when we have such a problem.

In this section we will use integrals very often, for brevity when no confusion is caused we will write $\int_a^b f(S_t)dS_t$ simply as $\int_a^b f(S_t)$.

## 2.1 With No Observed Option

**Proposition 5** *Assume the risk neutral probability density is bounded above by $\overline{q}$ and below by $0$. Assume the prices of a unit bond and the underlying stock are $B_0$ and $S_0$ respectively.*

- *The upper bound for all options is given by the risk neutral probability density $q^{**}(S_t) = \overline{q}$, $S_t < s_1$; $q^{**}(S_t) = 0$, $s_1 < S_t < s_2$; $q^{**}(S_t) = \overline{q}$, $S_t > s_2$, where $s_1$ and $s_2$ are to be decided such that*

$$\overline{q}(s_1 - s_2 + s_N) = 1$$

$$B_0\overline{q}(\int_{s_0}^{s_1} S_t + \int_{s_2}^{s_N} S_t) = S_0$$

- *The lower bound for all options is given by the risk neutral probability density* $q^*(S_t) = 0$, $S_t < s_1$; $q^*(S_t) = \overline{q}$, $s_1 < S_t < s_2$; $q^*(S_t) = 0$, $S_t > s_2$, *where* $s_1$ *and* $s_2$ *are to be decided such that*

$$\overline{q}(-s_1 + s_2) = 1$$

$$B_0\overline{q}\int_{s_1}^{s_2} S_t = S_0$$

Proof: Construct the following set of risk neutral probability density: $q^{**}(S_t) = \overline{q}$, $S_t < s_1$; $q^{**}(S_t) = 0$, $s_1 < S_t < s_2$; $q^{**}(S_t) = \overline{q}$, $S_t > s_2$, where $s_1$ and $s_2$ are to be decided such that

$$\begin{aligned} \int_{s_0}^{s_N} q^{**}(S_t) &= 1 \\ B_0\int_{s_0}^{s_N} S_t q^{**}(S_t) &= S_0 \\ B_0\int_{s_0}^{s_N} q^{**}(S_t)c_K(S_t) &= c_{K0} \end{aligned}$$

Note this set of risk neutral probability density can be regarded as three-segmented and piecewise constant, where at the odd segments, their values are equal to $\overline{q}$ while at the even segments, their values are equal to 0. It is clear that this set of risk neutral probability density intersects all admissible sets of risk neutral probabilities at most twice. But because it gives the same price of the stock, it must intersect all the admissible ones at least twice. Thus it intersects all the admissible ones exactly twice. Applying Proposition 2, we conclude that it gives the upper bounds for all options. From the above three equations we obtain the three equations in the proposition. Hence the first result is proved. The other result can be similarly proved. Q.E.D.

## 2.2 With One Observed Option

**Proposition 6** *Assume the risk neutral probability density are bounded above by $\overline{q}$ and below by 0. Assume the prices of a unit bond, the underlying stock, and an option with strike price $K$ are $B_0$, $S_0$, and $c_{K0}$ respectively.*

- *The upper bound for an option with strike price below $K$ is given by the risk neutral probability density $q_1^{**}(S_t) = \overline{q}$, $S_t < s_1$; $q_1^{**}(S_t) = 0$, $s_1 < S_t < s_2$; $q_1^{**}(S_t) = \overline{q}$, $s_2 < S_t < s_3$; $q_1^{**}(S_t) = 0$, $S_t > s_3$, where $s_i$, $i = 1, 2, 3$, are to be decided such that*

$$\begin{aligned} \overline{q}(s_1 - s_2 + s_3) &= 1 \\ \overline{q}\left(\int_{s_0}^{s_1} + \int_{s_2}^{s_3}\right) S_t &= \frac{S_0}{B_0} \\ \overline{q}\left(\int_{s_0}^{s_1} + \int_{s_2}^{s_3}\right) c_K(S_t) &= \frac{c_{K0}}{B_0} \end{aligned}$$

- *The lower bound for an option with strike price below $K$ is given by the risk neutral probability density $q_1^{*}(S_t) = 0$, $S_t < s_1$; $q_1^{*}(S_t) = \overline{q}$, $s_1 < S_t < s_2$; $q_1^{*}(S_t) = 0$, $s_2 < S_t < s_3$; $q_1^{*}(S_t) = \overline{q}$, $S_t > s_3$, where $s_i$, $i = 1, 2, 3$, are to be decided such that*

$$\begin{aligned} (-\overline{q})(s_1 - s_2 + s_3) + \overline{q}s_N &= 1 \\ \overline{q}\left(\int_{s_1}^{s_2} + \int_{s_3}^{s_N}\right) S_t &= \frac{S_0}{B_0} \\ \overline{q}\left(\int_{s_1}^{s_2} + \int_{s_3}^{s_N}\right) c_K(S_t) &= \frac{c_{K0}}{B_0} \end{aligned}$$

- *The upper (lower) bound for an option with strike price above $K$ is given by the risk neutral probability density $q_1^*(S_t)$ $(q_1^{**}(S_t))$.*

Proof: Construct the following set of risk neutral probability density: $q_1^{**}(S_t) = \overline{q}$, $S_t < s_1$; $q_1^{**}(S_t) = 0$, $s_1 < S_t < s_2$; $q_1^{**}(S_t) = \overline{q}$, $s_2 < S_t < s_3$; $q_1^{**}(S_t) = 0$, $S_t > s_3$, where $s_i$, $i = 1, 2, 3$, are to be decided such that

$$\begin{aligned}
\int_{s_0}^{s_N} q_1^{**}(S_t) &= 1 \\
B_0 \int_{s_0}^{s_N} S_t q_1^{**}(S_t) &= S_0 \\
B_0 \int_{s_0}^{s_N} q_1^{**}(S_t) c_K(S_t) &= c_{K0}
\end{aligned}$$

Note this set of risk neutral probability density can be regarded as four-segmented and piecewise constant, where at the odd segments, their values are equal to $\overline{q}$ while at the even segments, their values are equal to 0. It is clear that this set of risk neutral probability density intersects all admissible sets of risk neutral probabilities at most three times. But because it gives the same prices of the stock and option, from Proposition 1, it must intersect all the admissible ones at least three times. Thus it intersects all the admissible ones exactly three times. Applying Proposition 2, we conclude that it gives the upper bounds on the prices of options with strike prices below $K$. The above three equations can be rewritten as the three in the proposition. Hence the first result is proved. The other two results can be similarly proved. Q.E.D.

## 2.3 With Two Observed Options

**Proposition 7** *Assume the risk neutral probability density are bounded above by $\overline{q}$ and below by 0. Assume the prices of a unit bond, the underlying stock, and two options with strike prices $K_1$ and $K_2$ are $B_0$, $S_0$, $c_{10}$, and $c_{20}$ respectively.*

- *The upper bound for an option with strike price below $K_1$ or above $K_2$ is given by the risk neutral probability density $q_2^{**}(S_t) = \overline{q}$, $S_t < s_1$; $q_2^{**}(S_t) = 0$, $s_1 < S_t < s_2$; $q_2^{**}(S_t) = \overline{q}$, $s_2 < S_t < s_3$; $q_2^{**}(S_t) = 0$, $s_3 < S_t < s_4$; $q_2^{**}(S_t) = \overline{q}$, $S_t > s_4$, where $s_i$, $i = 1, 2, 3, 4$, are to be decided such that*

$$\begin{aligned}
\overline{q}(s_1 - s_2 + s_3 - s_4 + s_N) &= 1 \\
\overline{q}(\int_{s_0}^{s_1} + \int_{s_2}^{s_3} + \int_{s_4}^{s_N})S_t &= \frac{S_0}{B_0} \\
\overline{q}(\int_{s_0}^{s_1} + \int_{s_2}^{s_3} + \int_{s_4}^{s_N})c_j(S_t) &= \frac{c_{j0}}{B_0}
\end{aligned}$$

  *where $j = 1, 2$.*

- *The lower bound for an option with strike price below $K_1$ or above $K_2$ is given by the risk neutral probability density $q_2^{*}(S_t) = 0$, $S_t < s_1$; $q_2^{*}(S_t) = \overline{q}$, $s_1 < S_t < s_2$; $q_2^{*}(S_{u2}) = q_2$; $q_2^{*}(S_t) = 0$, $s_2 < S_t < s_3$; $q_2^{*}(S_{u3}) = q_3$; $q_2^{*}(S_t) = \overline{q}$, $S_t > s_3$, where $s_i$, $i = 1, 2, 3$, are to be decided such that*

$$\begin{aligned}
\overline{q}(-s_1 + s_2 - s_3 + s_4) &= 1 \\
\overline{q}(\int_{s_1}^{s_2} + \int_{s_3}^{s_4})S_t &= \frac{S_0}{B_0} \\
\overline{q}(\int_{s_1}^{s_2} + \int_{s_3}^{s_4})c_j(S_t) &= \frac{c_{j0}}{B_0}
\end{aligned}$$

  *where $j = 1, 2$.*

- *The upper (lower) bound for an option with strike price between $K_1$ and $K_2$ is given by the risk neutral probability density $q_2^*(S_t)$ ($q_2^{**}(S_t)$).*

Proof: We just prove the first result. The other results can be similarly proved. Construct the following set of risk neutral probability density: $q_2^{**}(S_t) = \overline{q}$, $S_t < s_1$; $q_2^{**}(S_t) = 0$, $s_1 < S_t < s_2$; $q_2^{**}(S_t) = \overline{q}$, $s_2 < S_t < s_3$; $q_2^{**}(S_t) = 0$, $s_3 < S_t < s_4$; $q_2^{**}(S_t) = \overline{q}$, $S_t > s_4$, where $s_i$, $i = 1, 2, 3, 4$, are to be decided such that

$$\begin{aligned}\int_{s_0}^{s_N} q_2^{**}(S_t) &= 1\\ B_0\int_{s_0}^{s_N} S_t q_2^{**}(S_t) &= S_0\\ B_0\int_{s_0}^{s_N} c_j(S_t) q_2^{**}(S_t) &= c_{j0}\end{aligned}$$

where $j = 1, 2$. Note this set of risk neutral probability density can be regarded as five-segmented and piecewise constant, where at the odd segments, their values are equal to $\overline{q}$ while at the even segments, their values are equal to 0. It is clear that this set of risk neutral probability density intersects all admissible sets of risk neutral probabilities at most four times. But because it gives the same prices of the stock and option, from Proposition 2, it must intersect all the admissible ones at least four times. Thus it intersects all the admissible ones exactly four times. Applying Proposition 3, we conclude that it gives the upper bounds on the prices of options with strike prices below $K$. With some calculations from the above three equations, we conclude that $s_i$, $i = 1, 2, 3, 4$, are to be decided by the three equations specified in the proposition. This proves the first result. The other two results can be similarly proved. Q.E.D.

## 2.4 The General Case

**Proposition 8** *Assume the risk neutral probability density are bounded above by* $\overline{q}$ *and below by* $0$*. Assume the prices of a unit bond, the underlying stock, and* $n$ *options with strike price* $K_1$*, ...,* $K_n$ *are* $B_0$*,* $S_0$*, and* $c_{10}$*, ...,* $c_{n0}$ *respectively.*

- *Assume* $n$ *is odd. Let* $m = (n+1)/2$*.*
  - *The upper bound for an option with strike price between* $K_{2i}$ *and* $K_{2i+1}$*,* $i = 0, ..., m-1$*, (where* $k_0 = 0$*) is given by the risk neutral probability density* $q_n^{**}(S_t) = \overline{q}$*, for* $s_{2i} < S_t < s_{2i+1}$*,* $i = 0, ..., m$*;* $q_n^{**}(S_t) = 0$*, for* $s_{2i-1} < S_t < s_{2i}$*,* $i = 1, ..., m+1$*, where* $0 = s_0 < s_1 < ... < s_{n+3} = s_N$*, and* $s_i$*,* $i = 1, ..., n+2$*, are to be decided such that*

$$\begin{aligned} \overline{q}\sum_{1}^{n+2}(-1)^{i+1}s_i &= 1 \\ \overline{q}\sum_{0}^{m}\int_{s_{2i}}^{s_{2i+1}} S_t &= \frac{S_0}{B_0} \\ \overline{q}\sum_{0}^{m}\int_{s_{2i}}^{s_{2i+1}} c_j(S_t) &= \frac{c_{j0}}{B_0} \end{aligned}$$

    *where* $j = 1, ..., n$*.*
  - *The lower bound for an option with strike price between* $K_{2i}$ *and* $K_{2i+1}$*,* $i = 0, ..., m-1$*, (where* $k_0 = 0$*) is given by the risk neutral probability density* $q_n^{*}(S_t) = 0$*, for* $s_{2i} < S_t < s_{2i+1}$*,* $i = 0, ..., m$*;* $q_n^{**}(S_t) = \overline{q}$*, for* $s_{2i-1} < S_t < s_{2i}$*,* $i = 1, ..., m+1$*, where* $0 = s_0 < s_1 < ... < s_{n+3} = s_N$*, and* $s_i$*,* $i = 1, ..., n+2$*, are to be decided such*

*that*

$$\begin{aligned}
\overline{q}\sum_{1}^{n+3}(-1)^i s_i &= 1 \\
\overline{q}\sum_{1}^{m+1}\int_{s_{2i-1}}^{s_{2i}} S_t &= \frac{S_0}{B_0} \\
\overline{q}\sum_{1}^{m+1}\int_{s_{2i-1}}^{s_{2i}} c_j(S_t) &= \frac{c_{j0}}{B_0}
\end{aligned}$$

*where* $j = 1, ..., n$.

- *The upper (lower) bound for an option with strike price between* $K_{2i-1}$ *and* $K_{2i}$, $i = 1, ..., m$, *(where* $K_{n+1} = +\infty$*) is given by the risk neutral probability density* $q_n^*(S_t)$ *(*$q_n^{**}(S_t)$*).*
- *Assume* $n$ *is even. Let* $m = n/2$.
  - *The upper bound for an option with strike price between* $K_{2i}$ *and* $K_{2i+1}$, $i = 0, ..., m$, *(where* $k_0 = 0$ *and* $K_{n+1} = +\infty$*) is given by the risk neutral probability density* $q_n^{**}(S_t) = \overline{q}$, *for* $s_{2i} < S_t < s_{2i+1}$, $i = 0, ..., m+1$; $q_n^{**}(S_{ui}) = q_i$, $i = 1, ..., n+2$; $q_n^{**}(S_t) = 0$, *for* $s_{2i-1} < S_t < s_{2i}$, $i = 1, ..., m+1$, *where* $0 = s_0 < s_1 < ... < s_{n+3} = s_N$ *and* $s_i$, $i = 1, ..., n+2$, *are to be decided such that*

$$\begin{aligned}
\overline{q}\sum_{1}^{n+3}(-1)^{i+1} s_i &= 1 \\
\overline{q}\int_{0}^{m+1}\int_{s_{2i}}^{s_{2i+1}} S_t &= \frac{S_0}{B_0} \\
\overline{q}\sum_{0}^{m}\int_{s_{2i}}^{s_{2i+1}} c_j(S_t) &= \frac{c_{j0}}{B_0}
\end{aligned}$$

*where* $j = 1, ..., n$.

– *The lower bound for an option with strike price between* $K_{2i}$ *and* $K_{2i+1}$, $i = 0, ..., m$, *(where* $k_0 = 0$ *and* $K_{n+1} = +\infty$*) is given by the risk neutral probability density* $q_2^*(S_t) = 0$, *for* $s_{2i} < S_t < s_{2i+1}$, $i = 0, ..., m+1$; $q_n^{**}(S_{ui}) = q_i$, $i = 1, ..., n+2$; $q_n^{**}(S_t) = \overline{q}$, *for* $s_{2i-1} < S_t < s_{2i}$, $i = 1, ..., m+1$, *where* $0 = s_0 < s_1 < ... < s_{n+3} = s_N$ *and* $s_i$, $i = 1, ..., n+2$, *are to be decided such that*

$$\begin{aligned}
\overline{q}\sum_{1}^{n+2}(-1)^i s_i &= 1 \\
\overline{q}\sum_{1}^{m+1}\int_{s_{2i-1}}^{s_{2i}} S_t &= \frac{S_0}{B_0} \\
\overline{q}\sum_{1}^{m+1}\int_{s_{2i-1}}^{s_{2i}} c_j(S_t) &= \frac{c_{j0}}{B_0}
\end{aligned}$$

*where* $j = 1, ..., n$.

– *The upper (lower) bound for an option with strike price between* $K_{2i-1}$ *and* $K_{2i}$, $i = 1, ..., m$, *is given by the risk neutral probability density* $q_2^*(S_t)$ $(q_2^{**}(S_t))$.

Proof: For odd $n$, the proof is similar to the case where $n = 1$ while for even $n$, the proof is similar to the case where $n = 2$. Thus the proof is omitted.

## 2.5 Relation to Bertsimas and Popescu's (2002) Work

Bertsimas and Popescu find that assuming risk neutral probability density is positive the option bounds are given by the convexity of option prices in exercise prices. This result is implied by the results just obtained in this section.

**Corollary 1 (Bertsimas and Popescu (2002))** *The risk neutral probability density is positive if and only if option prices are convex in exercise prices.*

Proof: If we let $\overline{q} \to +\infty$, then the work of Bertsimas and Popescu (2002) becomes our limit case. Thus their result can be obtained from our result. For detailed proof, see Appendix 2.

Note as we let $\overline{q}$ be larger, according to the results in the first section, the option bounds will be loosened. Thus Bertsimas and Popescu's option bounds as a limit case of ours when $\overline{q} \to +\infty$ must be looser than ours.

# 3 Conclusions

In this paper we have presented to a geometric approach to option bounds. By identifying the very set of risk neutral probability density that intersect all admissible ones for the right number of times, we can derive option bounds. The advantage of this method is that it is transparent and simple.

We have used this method to derive the option bounds when we know the bounds of the risk neutral probability density. Unlike higher order stochastic dominance bounds, these bounds do not require the knowledge of the true probability density.

There are concerns about using the log-normal risk neutral probabilities to price options; the reason is that the actual risk neutral probabilities are not that normal. Our result gives a solution to this problem. Assuming only the bounds of the risk neutral probability densities allows potentially very abnormal probability densities.

Note otherwise if we do not impose conditions, then as shown by Bertsimas and Popescu's (2002) excellent work, option prices are only bounded by their convexity in exercise prices, which means we can hardly get useful option bounds from observed option prices.

The method present in this paper works in both discrete and continuous state spaces. As a useful technique to option optimization problems, it must have broader implications for similar problems in other areas.

# Appendix 1 Proof of Lemma 1

First we show a triple is impossible. Otherwise suppose $K_i$, $K_{i+1}$, and $K_{i+2}$ are in one interval, one can show by constructing an arbitrage portfolio that the two P.D.Fs are equal at $x \in [K_i, K_{i+2}]$. Similarly we can show that adjacent doubles, doubles linked by singles, and a double without a zero to its right (left) are all impossible.

Now we show adjacent zeros are impossible. Suppose $(s_{i-1}, s_i)$ and $(s_i, s_{i+1})$ are both zeroes, then the $n-2$ strike prices must be in the other $n-3$ intervals. Note to the left of $s_{i-1}$ if there is a double, there must be a zero to the left of the double (ignoring the singles between them) while to the right of $s_{i+1}$ if there is a double, there must follow a zero (ignoring the singles between them); otherwise the impossible scenarios we have just checked will appear. Thus in average, one interval has (at most) one strike price. This implies that there cannot be $n-2$ strike prices in the other n-3 intervals.

Finally we show zeroes linked by singles are impossible. Without loss of generality, suppose $(s_{i-1}, s_i)$ and $(s_{i+1}, s_{i+2})$ are both zeroes while $(s_i, s_{i+1})$ is single, then the rest $n-3$ strike prices must be in the rest $n-4$ intervals. As we have just argued for adjacent zeroes, this is impossible. Q.E.D.

# Appendix 2 Proof of Corollary 1

The sufficiency is easy to prove; thus we need only show the necessity. We first examine the case where $n = 2$. From Proposition 7, when $\overline{q} \to +\infty$, the upper bound for an option with strike price between $K_1$ and $K_2$ is given by the pricing kernel $\phi_2^*(S_t) = a_1\delta(S_t - s_1) + a_2\delta(S_t - s_2)$, where $\delta(S_t)$ is the Dirac function, $a_1$, $a_2$, $s_1$, and $s_2$ are to be determined such that

$$a_1 + a_2 = 1, \quad B_0(a_1 s_1 + a_2 s_2) = S_0, \quad B_0(a_1 c_i(s_1) + a_2 c_i(s_2)) = c_{i0},\ i = 1, 2.$$

With some calculation we obtain the upper bound for the option as follows

$$\frac{c_{10} - c_{20}}{(K_2 - K_1)}\left(\frac{K_2 - K_1}{c_{10} - c_{20}} c_{20} + K_2 - X\right),$$

where $K_1 < X < K_2$. This is equivalent to

$$\frac{c_{20} - c_{X0}}{(K_2 - X)} \geq \frac{c_{X0} - c_{10}}{X - K_1}, \tag{10}$$

where the inequality is strict unless the actual risk neutral probability is equal to $\phi_2^*(S_t)$.

Now we examine the general case. Assume $n$ is odd. Let $m = (n+1)/2$. Let $K_0 = 0$, $K_{n+1} = s_N$, $c_{00} = S_0$, and $c_{(n+1)0} = 0$. That is, the underlying stock is viewed as a call option with zero strike price, and, of course, a call option with trike price $s_N$ always has zero value.

Let $\overline{q} \to +\infty$ in Proposition 8; then we conclude that the upper bound for an option with strike price between $K_{2i-2}$ and $K_{2i-1}$, $i = 1, 2, ...$, is given by the pricing kernel

$$q^{**}(S_t) = a_0\delta(S_t - s_1) + a_1\delta(S_t - s_1) + ... + a_m\delta(S_t - s_m),$$

where $\delta(S_t)$ is the Dirac function, $a_0$, ..., $a_m$, $s_1$, ..., and $s_m$ are to be determined such that

$$a_0 + ... + a_m \quad = \quad 1$$

$$
\begin{aligned}
B_0(a_0 s_1 + a_1 s_1 + ... + a_m s_m) &= S_0 \\
B_0(a_0 c_i(s_1) + a_1 c_i(s_1) + ... + a_m c_i(s_m)) &= c_{i0},
\end{aligned}
$$

$i = 1, ..., n$.

Note we must have $K_n < s_m$. Moreover, in any interval $(s_i, s_{i+1})$ there are at most two observed exercise prices $K_j$ and $K_{j+1}$; otherwise suppose there are three observed option prices $K_{j-1}$, $K_j$, and $K_{j+1}$ in $(s_i, s_{i+1})$. Thus we have

$$
\begin{aligned}
B_0(a_{i+1}(s_{i+1} - K_{j-1}) + ... + a_m(s_m - K_{j-1})) &= c_{(j-1)0} \\
B_0(a_{i+1}(s_{i+1} - K_j) + ... + a_m(s_m - K_j)) &= c_{j0} \\
B_0(a_{i+1}(s_{i+1} - K_{j+1}) + ... + a_m(s_m - K_{j+1})) &= c_{(j+1)0}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
B_0(a_{i+1} + ... + a_m)(K_j - K_{j-1})) &= c_{(j-1)0} - c_{j0} \\
B_0(a_{i+1} + ... + a_m)(K_{j+1} - K_j)) &= c_{j0} - c_{(j+1)0}
\end{aligned}
$$

This implies $\frac{c_{(j-1)0} - c_{j0}}{K_j - K_{j-1}} = \frac{c_{j0} - c_{(j+1)0}}{K_{j+1} - K_j}$.

When we derive (10) we have concluded that this happens only if the actual risk neutral probability has the same form as $\phi_2^{**}$, which has infinite values. This is, of course, excluded.

Thus we conclude that $K_1 \in (0, s_1)$, $K_{2i}, K_{2i+1} \in (s_i, s_{i+1})$, $i = 1, ..., m-1$. Hence we obtain

$$
\begin{aligned}
&a_i(s_i - K_{2i-2}) + ... + a_m(s_m - K_{2i-2}) = c_{(2i-2)0}/B_0, \\
&a_i(s_i - K_{2i-1}) + ... + a_m(s_m - K_{2i-1}) = c_{(2i-1)0}/B_0, \qquad (11)
\end{aligned}
$$

where $i = 1, ..., m$.

It follows that

$$B_0(a_i + ... + a_m)(K_{2i-1} - K_{2i-2}) = c_{(2i-2)0} - c_{(2i-1)0}.$$

It follows that

$$a_i + ... + a_m = \frac{c_{(2i-2)0} - c_{(2i-1)0}}{B_0(K_{2i-1} - K_{2i-2})}. \tag{12}$$

On the other hand we have from (11) that

$$a_i s_i + ... + a_m s_m = \frac{c_{(2i-1)0}}{B_0} + (a_i + ... + a_m)K_{2i-1}.$$

Hence we have

$$a_i s_i + ... + a_m s_m = \frac{c_{(2i-1)0}}{B_0} + K_{2i-1}\frac{c_{(2i-2)0} - c_{(2i-1)0}}{B_0(K_{2i-1} - K_{2i-2})}. \tag{13}$$

Recall that for an option with strike price $X \in (K_{2i-2}, K_{2i-1})$, the upper bound on its price is given by

$$B_0(a_0 c_X(s_1) + a_1 c_X(s_2) + ... + a_m c_X(s_m)).$$

Note we have $K_{2i-2}, K_{2i-1} \in (s_{i-1}, s_i)$; this implies $X \in (s_{i-1}, s_i)$. Thus the upper bound is given by

$$B_0(a_i(s_i - X) + ... + a_m(s_m - X)).$$

This together with (12) and (13) implies that the upper bound is give by

$$\frac{c_{(2i-2)0} - c_{(2i-1)0}}{(K_{2i-1} - K_{2i-2})}(\frac{K_{2i-1} - K_{2i-2}}{c_{(2i-2)0} - c_{(2i-1)0}}c_{(2i-1)0} + K_{2i-1} - X).$$

That is, the bound is given by the convexity of option prices in exercise prices.

For other cases, the proof is similar. Q.E.D.

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