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Cautiousness and Tendency to Buy Options

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Abstract

As is well known, Arrow-Pratt measure of risk aversion explains investors' behavior in stock markets while Kimball's measure of prudence explains investors' behavior when they make precautionary savings. What is missing is a measure of investors' tendency to buy options. In this paper we show that cautiousness, which is equivalent to the ratio of prudence to risk aversion, is the measure. We also discuss some properties of this measure.

Introduction

Investors pursue three important financial activities, namely making savings, buying equity, and trading derivatives. What can measure the strength of their motive in doing these activities?

Pratt (1964) and Arrow (1965) developed the measure of risk aversion, which is defined as the negative ratio of the second derivative to the first derivative of a utility function. Pratt (1964) showed that the higher an investor's measure of risk aversion, the more risk premium he demands and the less investment he makes in equity.

Leland (1968) and Kimball (1990) among some others investigated how to measure the strength of an investor's motive to make precautionary savings. Kimball (1990) developed the measure of prudence, which is defined as the negative ratio of the third derivative to the second derivative of a utility function. The higher an investor's measure of prudence, the more precautionary saving he will make responding to a risk in his wealth.

An investor's tendency to trade derivatives is more complicated to measure. Note when the measures of risk aversion and prudence are developed an investor's activities of making precautionary savings and buying stocks are studied separately. However, when deciding an investor's optimal position in a derivative it makes little sense if his position in the underlying equity is ignored. Hence we cannot separate an investor's activity in the derivative market from that in the underlying equity market. This is the reason that makes it difficult to develop a measure of an investor's tendency to trade derivatives.

In this paper, however, we show that the ratio of prudence to risk aversion (minus one), measures an investor's tendency to buy options. This measure has long been called cautiousness though it is never explained if this is really a measure of cautiousness.[1] We show that an investor with higher cautiousness has a stronger tendency to buy options. More specifically, if investor i's lowest coefficient of cautiousness is higher than j's highest coefficient of cautiousness, then investor j buys an option only if i does, and investor i sells the option only if j does, regardless of their initial wealth, the interest rate, the underlying stock price, the option price, and the distribution of the future stock price; and the reverse is also true.

The idea to use cautiousness to explain the demand for options can be attributed to Leland (1980). He used cautiousness to explain the convexity of an investor's optimal payoff function. However, we have two problems to use his result to explain the demand for options. First, his result relies on an important assumption that in an economy where investors have different constant positive cautiousness the pricing representative investor also has constant cautiousness. This is, as we will show in Section 5, not true. Second, in Leland's framework, an option buyer (seller) will buy (sell) all options with strike prices from zero to infinity. This modeling is, of course, not ideal.

In this paper we also discuss the impact of background risk on an investor's cautiousness hence on his tendency to buy options. We show that if an investor has HARA class utility with positive cautiousness then when he has a background risk, the cautiousness of his derived utility function will be strictly higher, hence he will have a stronger tendency to buy options. This result can be used to give a simple proof of Franke, Stapleton, and Subrahmanyam's (hereafter FSS)(1998) main result. FSS (1998) investigated the impact of background risk on investors' optimal payoff functions in an economy in which investors have identical positive constant cautiousness. They showed that in such an economy the investors without background risk will have globally concave optimal payoff functions. However, since they use the same framework as Leland, to go further to explain the demand for options, they had to rely on the same notion of an option buyer (seller) as Leland's.

This paper is also related to Benninga and Blume (1985), Brennan and Cao (1996), and Carr and Madan (2001). Benninga and Blume investigated the optimality of a certain insurance strategy in which an investor buys a risky asset and a put on that asset. Brennan and Cao (1996) investigated the impact of asymmetric information on the demand for options in an economy with exponential utility and normally distributed returns. They concluded that well informed investors tend to buy options on good news and sell options on bad news. Carr and Madan discussed how investors' preferences and beliefs affect their positions in derivatives.

The structure of the paper is as follows. In the first section we introduce a

[1] See Wilson (1968).

measure of investors' preferences, which is called cautiousness. In the second section we establish an ordering of utility functions by cautiousness and show cautiousness is a measure of an investor's tendency to buy options. In the third section we discuss some properties of the above ordering of utility functions. In the fourth section we discuss the impact of background risk on cautiousness hence on an investor's tendency to buy options. In the fifth section we discuss the notion of increasing cautiousness. The final section concludes the paper.

1 Cautiousness

To introduce the concept of cautiousness we first have to explain the concepts of risk aversion and prudence. Pratt (1964) and Arrow (1965) developed the concept of risk aversion to explain investors' behavior in the equity market. As interpreted by Pratt (1964), given utility function $u(x)$, the function $R(x) = -u''(x)/u'(x)$ is a measure of risk aversion.[2] The higher risk aversion an investor has the larger risk premium he demands for a small and actuarially neutral risk. Precisely, the risk premium demanded by an investor with utility $u(x)$ will be approximately the function $R(x)$ times half the variance of the risk. It is also shown to be a global measure of risk aversion in the sense that if the function $R(x)$ of an investor is always larger than that of the other, then the former will demand a larger risk premium than the latter for any risk, large or small, at any wealth level.

Kimball (1990) developed a theory regarding investors' precautionary savings analogous to Pratt's (1964) theory of risk aversion. Absolute prudence is defined as $P(x) = -u'''(x)/u''(x)$. The higher prudence an investor has, the more equivalent precautionary premium he demands for a risk and the more precautionary savings he makes in response to a risk in his wealth.

The first derivative of risk tolerance, where risk tolerance is the inverse of absolute risk aversion, was called cautiousness by Wilson (1968)[3]. Given a utility function, $u(x)$, its cautiousness is $C(x) \equiv (1/R(x))' = (-u'(x)/u''(x))'$. Equivalently it can be defined as the ratio of absolute prudence to absolute risk aversion minus one. This can be shown as follows.

Given an increasing and concave utility function $u(x)$, we have

$$(\ln R(x))' = (\ln -u''(x))' - (\ln u'(x))' = -(P(x) - R(x))$$

which can be written as

$$R'(x) = -R(x)(P(x) - R(x)).$$

It follows that

$$(1/R(x))' = -R'(x)/R^2(x) = P(x)/R(x) - 1.$$

[2]Throughout this paper we assume all utility functions are strictly increasing, strictly concave, and three times differentiable.

[3]See Wilson (1968).

Thus we have $C(x) = P(x)/R(x) - 1$.

More explicitly we can write it as

$$C(x) = u'''(x)u'(x)/u''^2(x) - 1.$$

Note that

$$(R(x))' = -R^2(x)(1/R(x))' = -R^2(x)C(x).$$

Thus decreasing absolute risk aversion (hereafter DARA) is equivalent to positive cautiousness and constant absolute risk aversion (hereafter CARA) is equivalent to zero cautiousness.

It is well known that exponential utility functions have zero cautiousness while other HARA utility functions have constant positive cautiousness. For example, given a HARA utility function $u(x) = (x+a)^{1-\gamma}/(1-\gamma)$, we have $C(x) = 1/\gamma$.

Note in the rest of the paper, when we say increasing (decreasing) we mean non-decreasing (non-increasing), and when we say higher (lower) we mean no lower (no higher).

2 Measuring the Tendency to Buy Options

Assume a two-date economy with starting time 0 and ending time 1. Assume there is a stock available in the market whose prices at time 0 and 1 are denoted by S_0 and S respectively. Assume there is a derivative written on the stock that is traded in the market. Its payoff at time 1, $c(S)$, is twice differentiable and globally convex in S. Moreover, there exists at least one point, say S^*, at which $c(S)$ is strictly convex[4]. Denote its price at time 0 by c_0. Assume there is also a risk-free bond traded in the market; the risk-free interest rate is denoted by r. S_0, c_0, and r are determined in the equilibrium of the economy. These parameters are exogenously given in an individual investor's investment problem which we study in this paper.

Assume investors are indexed by $i = 1, 2, ...$; and they are all price-takers. Investor i's preference is represented by utility function $u_i(x)$. At time 0 he has initial wealth w_{0i}. Assume investor i invests x_i in the stock and y_i in the derivative, and invests the rest money in the bond, which is $w_{0i} - x_i - y_i$. Denote investor i's wealth at time 1 by $w_i(S; x_i, y_i)$. We have

$$w_i(S; x_i, y_i) = (w_{0i} - x_i - y_i)(1+r) + x_i \frac{S}{S_0} + y_i \frac{c(S)}{c_0}.$$

For brevity we sometimes write $w_i(S; x_i, y_i)$ simply as $w_i(S)$.

[4]To guarantee that the investment problem does not degenerate, the distribution of S in this paper always satisfies the condition that there exists $\epsilon > 0$, such that $prob(S^* - \epsilon < S < S^* + \epsilon) > 0$.

Investor i maximizes the expected utility of his ending time wealth $w_i(S)$. That is,

$$\max_{x_i, y_i} Eu_i(w_i(S)).$$

We obtain the first order conditions

$$Eu_i'(w_i(S))(S - (1+r)S_0) = 0, \quad \text{and} \quad Eu_i'(w_i(S))(c(S) - (1+r)c_0) = 0,$$

which can be written as

$$\frac{E[u_i'(w_i(S))S]}{Eu_i'(w_i(S))} = (1+r)S_0, \quad \text{and} \quad \frac{E[u_i'(w_i(S))c(S)]}{Eu_i'(w_i(S))} = (1+r)c_0. \tag{1}$$

The solutions of x_i and y_i depend on the utility function, S_0, c_0, the interest rate r, the payoff of the derivative, and the distribution of S. We always assume that the solutions of x_i and y_i exist.

Let $R_i(x)$ denote the absolute risk aversion of $u_i(x)$, i.e., $R_i(x) \equiv -u_i''(x)/u_i'(x)$. Let $C_i(x)$ denote the cautiousness of $u_i(x)$, i.e., $C_i(x) \equiv (1/R_i(x))'$.

Let $\phi_i(S) \equiv u_i'(w_i(S))/Eu_i'(w_i(S))$. Let $\delta_i(S) = -\phi_i'(S)/\phi_i(S)$. We have

$$\delta_i(S) = R_i(w_i(S))w_i'(S).$$

Hence

$$(\frac{1}{\delta_i(S)})' = C_i(w_i(S)) - \frac{1}{R_i(w_i(S))}\frac{w_i''(S)}{w_i'^2(S)}. \tag{2}$$

Before we proceed to present our main result, we first introduce a lemma:

Lemma 1 *Assume two pricing kernels intersect twice. Taking the interest rate and the spot stock price as given, the pricing kernel with fatter tails gives higher prices of convex-payoff contingent claims wriiten on the stock.*

Proof: See Huang (2004) or FSS (1999).

We now present our main result.

Proposition 1 *The following two conditions are equivalent:*

1. *There exists a constant $C \geq 0$ such that $C_i(x) \geq C \geq C_j(x)$.*
2. *Investor j buys the derivative only if investor i does, and investor i sells the derivative only if investor j does, regardless of their initial wealth, the interest rate, the stock price, the derivative price, and the distribution of the future stock price.*

Proof: (1) $\Rightarrow$ (2)

Note $c(S)$ is twice differentiable and globally convex in S and there exists at least one point at which $c(S)$ is strictly convex. Thus investor j buys (sells) the derivative if and only if for all S $w_j''(S) \geq (\leq) 0$ and for at least one S $w_j''(S) > (<) 0$. Hence we need only to show that for all S $w_j''(S) \geq 0$ and for at least one S $w_j''(S) > 0$ only if for all S $w_i''(S) \geq 0$ and for at least one S $w_i''(S) > 0$, and for all S $w_i''(S) \leq 0$ and for at least one S $w_i''(S) < 0$ only if for all S $w_j''(S) \leq 0$ and for at least one S $w_j''(S) < 0$.

We have

$$E(\phi_i(S)) = E(\phi_j(S)) = 1$$

$$E(\phi_i(S)S) = E(\phi_j(S)S) = (1+r)S_0$$

and

$$E(\phi_i(S)c(S)) = E(\phi_j(S)c(S)) = (1+r)c_0$$

Applying Lemma 1, from the above equations we conclude that $\phi_i(S)$ and $\phi_j(S)$ must intersect at least three times; otherwise the two pricing kernels, $\phi_i(S)$ and $\phi_j(S)$, cannot give the same option price. It follows that $\delta_i(S)$ and $\delta_j(S)$ must intersect at least twice. Hence $1/\delta_i(S)$ and $1/\delta_j(S)$ must intersect at least twice.

On the other hand, if investor i sells the derivative but investor j does not, then we have for all S $w_i''(S) \leq 0$ and for at least one S $w_i''(S) < 0$ while $w_j''(S) \geq 0$. It follows that since there exists a constant C such that $C_i(x) \geq C \geq C_j(x)$, from (2), we have for all S $(\frac{1}{\delta_i(S)})' \geq (\frac{1}{\delta_j(S)})'$ and for at least one S the inequality is strict. This implies that $1/\delta_i(S)$ and $1/\delta_j(S)$ intersect at most once. This contradicts the previous assertion.

Thus investor i sells the derivative only if investor j does. Similarly we can show that investor j buys the derivative only if investor i does.

(2) $\Rightarrow$ (1)

We need only to show that if there does not exist a constant C such that $C_i(x) \geq C \geq C_j(x)$ then there is a set of w_{i0}, w_{j0}, S_0, c_0, r, and distribution of S such that investor i optimally holds a short position in the derivative while j does not.

Obviously when S_0 is very low, both investors will buy the stock. When S_0 rises investors will buy less stock. In particular, let S_0 be close (but not equal) to $\overline{S}_0 \equiv \frac{E(S)}{1+r}$, which is the highest possible stock price in a risk averse world. Similarly when c_0 is very low, both investors will buy the derivative. When c_0 rises, they will buy less the derivative. At some point investor j will not buy neither sell the derivative. At this point if we manage to set up a set of w_{i0}, w_{j0}, S_0, c_0, r, and distribution of S such that investor i optimally holds a short position in the derivative then the proof is done. Note since S_0 is close (but not equal) to $\overline{S}_0$, we must have $|x_i| + |y_i|$ close (but not equal) to 0 and $|x_j| + |y_j|$ close (but not equal) to 0.

Since there does not exist a constant C such that $C_i(x) \geq C \geq C_j(x)$, that is, for some x_0 and y_0, $C_i(x_0) < C_j(y_0)$, then there is a neighborhood of x_0, A, and a neighbor hood of y_0, B, such that for all $x \in A$ and all $y \in B$, $C_i(x) < C_j(y)$. Let $w_{i0} = x_0/(1+r)$ and $w_{j0} = y_0/(1+r)$. Then for S_0 sufficiently close (but not equal) to $\overline{S}_0$, we must have almost probability one that $w_i(S) \in A$ and $w_j(S) \in B$.

This implies that for almost probability one $C_i(w_i(S)) < C_j(w_j(S))$. Hence as in the proof of Proposition 1, from (2), we must have for almost probability one

$$\frac{1}{R_i(w_i(S))}\frac{w_i''(S)}{w_i'^2(S)} < \frac{1}{R_j(w_j(S))}\frac{w_j''(S)}{w_j'^2(S)}.$$

Since $w_j''(S) = 0$ we have $w_i''(S) < 0$, i.e., investor i sells the derivative while j does not. Q.E.D.

The above result shows that if investor i's lowest coefficient of cautiousness is higher than j's highest coefficient of cautiousness, then investor i always has a stronger tendency to buy the derivative regardless of their initial wealth, the interest rate, the stock price, the derivative price, and the distribution of the future stock price; and the reverse is also true.

Although when deriving Proposition 1, we require that the derivative's payoff is not only convex but also twice differentiable, this result is valid to options the payoffs of which are convex but not differentiable.

Corollary 1 *Let the derivative be an option written on the stock. The following two conditions are equivalent:*

1. *There exists a constant C such that $C_i(x) \geq C \geq C_j(x)$.*
2. *Investor j buys the option only if investor i does, and investor i sells the option only if investor j does, regardless of their initial wealth, the interest rate, the stock price, the option price, and the distribution of the future stock price.*

Proof: Let the option be a call option with strike price K. Although the payoff function of the option, $c(S)$, is not differentiable, we can always construct a series of derivatives which have convex and twice differentiable payoff function $c_n(S)$, $n = 1, 2, ...$, such that $c_n(S) = 0$, when $S < K$; $c_n(S)$ is increasing and convex, when $K \geq S \geq K + 1/n$; $c_n(S) = S - K = c(S)$, when $S > K + 1/n$; and the payoff function of the option is the limit of the series of payoff functions, i.e., $\lim_{n\to\infty} c_n(S) = c(S)$.

Note that Proposition 1 is valid for the whole series of the derivatives. In limit, it must be also valid for the option.

For a put option, the conclusion can be proved in the same way. Q.E.D.

The above result gives an ordering of utility functions in terms of the tendency to buy options. This ordering is not complete since not all functions can be ordered in such a way. However, HARA class utility functions can be perfectly ordered in such a way since they have constant cautiousness.

Note it is strong that we require one investor's lowest coefficient of cautiousness is higher than j's highest coefficient of cautiousness. The reason that we need this strong condition is because we have to deal with the situation where the investors have arbitrarily different optimal positions in the stock market and the bond market.

3 Properties of the Ordering of Utility Functions

We have given an ordering of utility functions in terms of their cautiousness. Utility functions can be ordered in such a way are of special interest when we compare investors' tendency to buy options. Note since HARA class utility functions have constant cautiousness thus they are ideal candidates for this purpose. Indeed we will see that this ordering of utility functions is closely related to HARA utility functions. We have the following result.

Proposition 2

1. *There exists a constant $C \geq 0$ such that $C_i(x) \geq C \geq C_j(x)$.*
2. *We have $u_i'(x) = t^{-1/C}(x)$, where $t(x)$ is concave, and $u_j'(x) = s^{-1/C}(x)$, where $s(x)$ is convex, unless $C = 0$ and $u_j(x)$ is exponential.*

Proof: Obviously if $C = 0$ then $u_j(x)$ must be exponential since it is DARA. Now assume $C > 0$. Let $v(x) = x^{1-1/C}/(1-1/C)$. Let $u_i'(x) = v'(t(x))$. Then we have

$$C_i(x) = \frac{v'(t(x))(v'''(t(x))t'^2(x) + v''(t(x))t''(x)}{v''^2(t(x))t'^2(x)}.$$

This can be rewritten as

$$C_i(x) = C - \frac{1}{Cx}\frac{t''(x)}{t'^2(x)}.$$

Hence $C_i(x) \geq C$ is equivalent to $t''(x) < 0$.

The result about $u_j(x)$ can be proved in the same way. Q.E.D.

Surely HARA utility functions are not the only utility functions can be ordered in such a way. Assume there are a set of ordered utility functions; the question is: do the popular operations on utility functions preserve the ordering?

Proposition 3 *The operation $u(x) :\longrightarrow u(ax+b)$ preserves the ordering of utility functions.*

Proof: Let $u_1(x)$ and $u_2(x)$ are two of a set of ordered utility functions such that $C_1(x) \geq C \geq C_2(x)$, where $C_i(x)$ is the cautiousness of $u_i(x)$, $i = 1, 2$. We have

$$C_i(ax+b) = \frac{(au_i'(ax+b))(a^3u_i'''(ax+b))}{a^4u_i''^2(ax+b)} - 1 = \frac{u_i'(ax+b)u_i'''(ax+b)}{u_i''^2(ax+b)} - 1.$$

It follows that $C_1(ax+b) \geq C \geq C_2(ax+b)$. Hence the ordering is preserved. Q.E.D.

While the above operation completely preserve the ordering, some operations may partially preserve it. For example, we have the following result.

Proposition 4 *If $u_1((x), u_2(x), ..., u_n(x)$ all have cautiousness higher than a constant then the cautiousness of $\sum_1^n a_i u_i(x)$ is also higher than the constant.*

Proof: The general statement follows from the case $u(x) = u_1(x) + u_2(x)$. For this case,

$$C(x) = \frac{(u_1'(x) + u_2'(x))(u_1'''(x) + u_2'''(x))}{(u_1''(x) + u_2''(x))^2} - 1.$$

It follows that

$$C(x) = \frac{(u_1'(x) + u_2'(x))((C_1(x) + 1)\frac{u_1''^2(x)}{u'(x)} + (C_2(x) + 1)\frac{u_2''^2(x)}{u'(x)})}{(u_1''(x) + u_2''(x))^2} - 1.$$

Suppose $C_i(x) \geq C$, $i = 1, 2$, then

$$C(x) \geq (C+1)\frac{(u_1'(x) + u_2'(x))(\frac{u_1''^2(x)}{u'(x)} + \frac{u_2''^2(x)}{u'(x)})}{(u_1''(x) + u_2''(x))^2} - 1 \geq C.$$

Q.E.D.

We also have the following result.

Proposition 5 *Given utility function $u_1((x)$ and $u_2(x)$, if they both have cautiousness higher than constant $C \leq 0.5$, then $u(x) \equiv u_1(u_2(x))$ also has cautiousness higher than C; if they both have cautiousness lower than constant $C \geq 0.5$, then $u(x) \equiv u_1(u_2(x))$ also has cautiousness lower than C.*

Proof: Let $C(x)$ be the cautiousness of $u(x)$. Then

$$C(x) = \frac{u_1'(u_2)u_2'(u_1'''(u_2)u_2'^3 + 3u_1''(u_2)u_2''u_2' + u_1'(u_2)u_2''')}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1,$$

where for brevity the argument of $u_2(x)$ is omitted. It can be written as

$$C(x) = \frac{(C_1(u_2) + 1)u_1''^2(u_2)u_2'^4 + 3u_1''(u_2)u_1'(u_2)u_2'^2u_2'' + (C_2 + 1)u_1'^2(u_2)u_2''^2}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1$$

If $C_i(y) \geq C$, $i = 1, 2$, where $C \leq 0.5$, then

$$C(x) \geq (C+1)\frac{u_1''^2(u_2)u_2'^4 + 2u_1''(u_2)u_1'(u_2)u_2'^2u_2'' + u_1'^2(u_2)u_2''^2}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1 = C$$

If $C_i(y) \leq C$, $i = 1, 2$, where $C \geq 0.5$, then

$$C(x) \leq (C+1)\frac{u_1''^2(u_2)u_2'^4 + 2u_1''(u_2)u_1'(u_2)u_2'^2u_2'' + u_1'^2(u_2)u_2''^2}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1 = C$$

Q.E.D.

4 Impact of Background Risk

We have shown that cautiousness measures an investor's tendency to buy options. In this section we investigate the impact of background risk on an investor's cautiousness hence on his tendency to buy options. Given a utility function, $u(x)$, when there is a background risk ϵ, as usual, we call $\hat{u}(x) = Eu(x+\epsilon)$ the derived utility function. We have the following result.

Proposition 6 *Assume $u(x)$ has cautiousness higher than a constant. Then given a background risk, the cautiousness of the derived utility function will also be higher than the constant.*

Proof: We denote the background risk as ϵ, a random variable. Assume the cautiousness of $u(x)$ is higher than constant C. Let $R(x)$ and $P(x)$ be the risk aversion and prudence of the utility function. Then we have $P(x)/R(x) \geq C+1$.

Note for positive a and b we have $a+b \geq 2\sqrt{ab}$. Thus for any e_1 and e_2,

$$\frac{P(x+e_1)}{R(x+e_2)} + \frac{P(x+e_2)}{R(x+e_1)} \geq 2\sqrt{\frac{P(x+e_1)P(x+e_2)}{R(x+e_1)R(x+e_2)}} \geq 2(C+1) \tag{3}$$

Rearranging the terms in (3), we have, for any e_1 and e_2,

$$u'''(x+e_1)u'(x+e_2)+u'''(x+e_2)u'(x+e_1) \geq 2(C+1)u''(x+e_1)u''(x+e_2) \tag{4}$$

Assuming e_1 and e_2 are independent and have identical distributions as ϵ and taking the expectation of (4) with respect to e_1 and e_2, we obtain

$$2E(u'''(x+\epsilon))E(u'(x+\epsilon)) \geq 2(C+1)(E(u''(x+\epsilon)))^2 \tag{5}$$

Rearranging the terms in (5), we have $\hat{P}(x)/\hat{R}(x) \geq C+1$, where $\hat{R}(x)$ and $\hat{P}(x)$ are the risk aversion and prudence of the derived utility. Hence the cautiousness of the derived utility, $\hat{C}(x) = \hat{P}(x)/\hat{R}(x) - 1 \geq C$. Q.E.D.

Corollary 2 *Assume a utility function is HARA class with positive cautiousness. Given a background risk, the cautiousness of the derived utility function will be strictly higher.*

Proof: Note that a HARA utility function has constant cautiousness, say C. It follows from Proposition 6 that the cautiousness of the derived utility function is higher than C. Note in the proof of Proposition 6 the inequalities are strict unless the utility function has constant $R(x)$ and $P(x)$, that is, it is power utility which has zero cautiousness. Hence if a utility function is HARA class with positive cautiousness then given a background risk, the cautiousness of the derived utility function will be strictly higher. Q.E.D.

The above result shows that if an investor has HARA class utility with positive cautiousness then when he has a background risk, the cautiousness of his derived utility function will be strictly higher, hence he will have a stronger tendency to buy options. This result can be used to give a simple proof of FSS's (1998) main result that in an economy in which investors have identical constant positive cautiousness the investors without background risk will have globally concave optimal payoff functions. The proof is shown in the appendix.

5 Increasing Cautiousness

Huang (2000) showed that if the marginal utility of zero wealth is infinity then increasing (decreasing) cautiousness implies decreasing (increasing) relative risk aversion. Since decreasing relative risk aversion is more popularly accepted, so should be increasing cautiousness. In the following we give another argument for increasing cautiousness. We show that when all investors have constant cautiousness then the pricing representative investor will have increasing cautiousness.

The framework used is similar to the one given by Leland (1980) and FSS (1998). Assume in a one-period economy there are N investors and every investor's wealth consists of a portfolio of state-contingent claims on the market portfolio. Let X be the payoff of the market portfolio at the end of the period. Assume that there is a complete market for state-contingent claims on X. Thus all investors can buy and sell state-contingent claims on X so that, as discussed in Leland (1980), any investor i can choose a payoff function $x_i(X)$. Let $u_i(x)$ be investor i's utility function. We assume that there exists a pricing kernel, $\phi(X)$, whose functional form will be determined in the equilibrium of the economy.

Let w_{i0} be investor i's initial endowment, expressed as the fraction of the spot value of the total wealth in the economy. Let x_i be his optimal payoff function respectively. Then the investor has the following utility maximization problem:

$$\max_{x_i} Eu_i(x_i). \tag{6}$$

Subject to

$$E(\phi x_i) = w_{i0}E(\phi X). \tag{7}$$

where $E(.)$ denotes the expectation operator. In equilibrium, the market is cleared, thus we have

$$\sum_i x_i(X) = X. \tag{8}$$

We have the first order condition

$$u_i'(x_i) = \lambda_i \phi(X). \tag{9}$$

Differentiating both sides of (9) will lead to the following result:

$$x_i'(X) = R_e(X)/R_i(x_i), \tag{10}$$

where $R_i(x) = -u_i''/u_i'(x)$ is investor i's absolute risk aversion and $R_e(X) = -\phi'(X)/\phi(X)$ is the pricing representative investor's absolute risk aversion.

Differentiating both sides of (10), we obtain

$$x_i''(X) = R_e^2(X)[C_i(x_i) - C_e(X)]/R_i(x_i), \tag{11}$$

where $C_i(x)$ is investor i's cautiousness and $C_e(X) = (1/R_e(X))'$ is the pricing representative investor's cautiousness.

From (8) and (10) we obtain

$$R_e(X) = (\sum_i R_i^{-1}(x_i))^{-1} \tag{12}$$

From (8) and (11), we obtain

$$C_e(X) = \sum_i s_i C_i(x_i). \tag{13}$$

where $s_i = R_i^{-1}(x_i)/\sum_i R_i^{-1}(x_i)$.

We have the following result.

Proposition 7 *Assume all investors have increasing cautiousness. Then the pricing representative investor also has increasing cautiousness. Moreover, the cautiousness of the pricing representative investor is strictly increasing unless all investors have identical constant cautiousness.*

Proof:

Differentiating both sides of (11), we have

$$\frac{x_i'''(X)}{x_i''(X)} = -2\frac{R_e'(X)}{R_e(X)} - \frac{R_i'(x_i)}{R_i(x_i)}x_i'(X) + \frac{C_i'(x_i)x_i'(X) - C_e'(X)}{C_i(x_i) - C_e(X)}.$$

It can be rewritten as:

$$x_i'''/x_i'' = 2(P_e - R_e) + (P_i - R_i)x_i' + (C_i'x_i' - C_e')/(C_i - C_e),$$

where we have omitted the arguments of the functions. Applying (10) and (11) to the above equation and rearranging the terms, we obtain

$$x_i'''(X) = \frac{R_e^2}{R_i}(C_i - C_e)(2(P_e - R_e) + (P_i - R_i)\frac{R_e}{R_i}) - \frac{R_e^2}{R_i}C_e' + C_i'x_i'\frac{R_e^2}{R_i}.$$

Since $\sum_i x_i''' = 0$ we have

$$R_e \sum_i R_i^{-1}(C_i - C_e)(2(P_e - R_e) + (P_i - R_i)\frac{R_e}{R_i}) + \sum_i C_i'x_i'^2 - C_e' = 0.$$

Since $C_i' \geq 0$, we have

$$C_e' \geq 2R_e(P_e - R_e)\sum_i (C_i - C_e)/R_i + R_e^2 \sum_i (C_i - C_e)C_i/R_i.$$

From (13) we obtain $\sum_i (C_i - C_e)/R_i = 0$. Thus we have

$$C_e' \geq R_e^2 \sum_i C_i(C_i - C_e)/R_i.$$

Applying (12) and (13) we can rewrite it as

$$C_e' \geq R_e^2(\sum_i C_i^2/R_i - R_e(\sum_i C_i/R_i)^2).$$

Rearranging the terms, we obtain

$$C_e' \geq R_e^3(\sum_i R_i^{-1} \sum_i C_i^2/R_i - (\sum_i C_i/R_i)^2).$$

Applying the Cauchy inequality, we obtain $C_e' \geq 0$. $C_e' = 0$ if and only if $C_i(x) = C_j(x) = C$ is a constant for any i and j. Q.E.D.

This shows if every investor has increasing cautiousness then so does the pricing representative investor. As a special case, when every investor has constant cautiousness we have the following corollary.

Corollary 3 *Assume all investors have constant cautiousness. Then the pricing representative investor has strictly increasing cautiousness unless all investors have identical constant cautiousness.*

Proof: It directly follows from Proposition 7.

The above result shows clearly that if investors have different positive constant cautiousness, then the pricing representative investor will not have constant cautiousness. Instead he will have strictly increasing cautiousness.

6 Conclusions

In this paper we have established that cautiousness, which is equivalent to the ratio of prudence to risk aversion, is a measure of an investor's tendency to buy options. It is interesting to see that the measure of an investor's tendency to buy options is closely related to the measures of risk aversion and prudence which explain investors' activities in the bond market and stock market. Regarding the latter two activities, it is now widely accepted that investors should have decreasing absolute risk aversion while Kimball (1993) proposed decreasing absolute prudence. It is also said that investors are more likely to have decreasing relative risk aversion. As Huang (2000) showed that increasing (decreasing) cautiousness implies decreasing (increasing) relative risk aversion given that marginal utility of zero wealth is infinity, then increasing cautiousness may be more likely.

An investor's cautiousness can only tell if he has a stronger tendency to buy options than others. However, if two investors both buy options, their coefficients of cautiousness cannot tell if one buys more options than the other.

We have also showed that background risk will increase the cautiousness of HARA class utility. Further research will be interesting to show the impact of background risk on the cautiousness of a general utility function.

Appendix A Proof of Theorem 3 in FSS (1998)

In the economy is described in Section 5, we have the following result.

> [Theorem 3, FSS (1998)] Assume all investors have identical positive constant cautiousness and some investors have uninsurable background risk. Then the investors without background risk have concave optimal payoff functions.

Proof: Assume all investors have identical positive constant cautiousness C. When investor i is exposed to background risk ϵ_i, the utility function $u_i(x_i)$ in the utility maximization problem (6) is replaced by the indirect utility function $\hat{u}_i(x_i) = E_{\epsilon_i}(u_i(x_i + \epsilon_i))$. Thus on the right hand side of (13) $C_i(x_i)$ is replaced by the cautiousness of investor i's derived utility function, $\hat{C}_i(x_i)$, if investor i has background risk. For the investors without background risk, $C_i(x_i) = C$ is a positive constant.

From Corollary 2, we know that for every investor i who has background risk, $\hat{C}_i(x_i) > C$. Thus from Equation (13), we easily verify that the pricing representative investor's cautiousness is strictly higher than those of the investors without background risk. From (11) the optimal payoff functions of those without background risk are strictly concave. Q.E.D.

REFERENCES

1. Amershi, A. H. and J. H. W. Stoeckenius, The Theory of Syndicates and Linear Sharing Rules, *Econometrica* 51 (1983), 1407-1416.

2. Arrow, Kenneth J. (1965), Aspects of a Theory of Risk Bearing, Yrjo Jahnsson Lectures, Helsinki. Reprinted in Essays in the theory of Risk Bearing (1971). Chicago: Markham Publishing Co.

3. Benninga S. and M. Blume (1985), On the Optimality of Portfolio Insurance, *Journal of Finance* 40, No. 5, 1341-1352.

4. Benninga, S. and J. Mayshar, (2000), Heterogeneity and Option Pricing, *Review of Derivatives Research* 4, 7-27.

5. Borch, K. (1962), Equilibrium in a Reinsurance Market, *Econometrica* 30, 424-444.

6. Brennan, M. J. and H. H. Cao (1996), Information, trade, and derivative securities, Review of Financial Studies 9, No.1, 163-208.

7. Brennan, M. J. and R. Solanki (1981), Optimal Portfolio Insurance, *Journal of Financial and Quantitative Analysis* 16, 279-300.

8. Carr P. and D. Madan, 2001, Optimal Positioning in Derivative Securities, *Quantitative Finance* 1, 19-37.

9. Eeckhoudt L. and H. Schlesinger (1994), A Precautionary Tale of Risk Aversion and Prudence. B. Munier and M. J. Machina (eds.), Models and Experiments in Risk and Rationality, 75-90, Kluwer Academic Publishers. Printed in the Netherlands.

10. Franke, G., R. C. Stapleton, and M. G. Subrahmanyam (1999), When are Options Overpriced: The Black-Scholes Model and Alternative Characterizations of the Pricing Kernel, *European Finance Review* 3, 79-102.

11. Franke, G., R. C. Stapleton and M. G. Subrahmanyam (1998), Who Buys and Who Sells Options, *Journal of Economic Theory* 82, 89-109.

12. Gollier, C. and C. J. W. Pratt (1996), Risk Vulnerability and the tempering Effect of Background Risk, *Econometrica* 49, 1109-1123.

13. Huang C. and R. Litzenberger (1988), Foundations for Financial Economics. Prentice-Hall Canada, Incorporated.

14. Huang, James (2000). "Relationships between Risk Aversion, Prudence, and Cautiousness." Lancaster University Management School working paper.

15. Huang, James (2004). "Risk Neutral Probabilities and Option Bounds: A Geometric Approach." Lancaster University Management School working paper.

16. Kihlstrome, Richard E., David Rome, and Steve Williams (1981): Risk Aversion with Random Initial Wealth, *Econometrica* 49, 911-920.

17. Kimball, Miles S. (1990), Precautionary Saving in the Small and in the Large, *Econometrica* 58, 53-73.

18. Kimball, Miles S. (1993), Standard Risk Aversion, *Econometrica* 61, 589-611.

19. Leland, H. E. (1980), Who Should Buy Portfolio Insurance? *Journal of Finance* 35, 581-594.

20. Mas-Colell, A. (1985), *The Theory of General Economic Equilibrium: A Differentiable Approach.* Econometric Society Monograph, Cambridge, Cambridge University Press.

21. Pratt, J. W. (1964), Risk Aversion in the Small and in the Large, *Econometrica* 32, 122-136.

22. Pratt, J. W., and R. Zeckhauser (1987), Proper Risk Aversion, *Econometrica* 55, 143-154.

23. Rubinstein, M. E. (1976), An Aggregation Theorem for Securities Markets, *Journal of Financial Economics* 1, 225-244.

24. Wilson, R. (1968), The Theory of Syndicates, *Econometrica* 36, 119-132.