**Lancaster University Management School
Working Paper
2002/015**

Impact on Option Prices of Divergent Consumer Confidence

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June 25, 2002

Abstract

This paper investigates the impact on option prices of divergent consumer confidence. To model this, we assume that consumers disagree on the expected growth rate of aggregate consumption. With other conditions unchanged in the discrete-time Black-Scholes option-pricing model, we show that the representative consumer will have declining relative risk aversion instead of the assumed constant relative risk aversion. In this case all options will be underpriced by the Black-Scholes model under the assumption of bivariate lognormality. We also extend Benninga and Mayshar's (2000) results about impact on option prices of heterogeneous beliefs and preferences to an N-agent economy.

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Impact on Option Prices of Divergent Consumer Confidence

Introduction

Rubinstein (1976) and Brennan (1979) derived the discrete-time Black-Scholes option pricing model (hereafter the B-S model) under the assumption that investors have identical beliefs, identical constant coefficients of relative risk aversion and that the stock price and aggregate consumption follow a bivariate log-normal distribution. The assumption that investors have identical beliefs and identical constant coefficients of relative risk aversion has been questioned by some authors. For example, Benninga and Mayshar (2000) (hereafter B & M) have argued that "the assumption that all investors have identical homothetic tastes and identical expectations seems particularly unreasonable" since "this implies that all investors have identical wealth composition", which is far from reality[1].

In the real world, people's opinions about the economy are typically divergent. Some may be optimistic while some may be pessimistic, leaving the rest somewhere in between. For example, a typical consumer confidence survey on August 28, 2001 by the Consumer Research Center showed that among the 5000 U.S. households surveyed, 14.9 percent stated that current business conditions were bad while 28.2 percent stated that business conditions were good[2].

In this paper we investigate the impact on option prices of consumers' divergent confidence in the economy. We study the issue in a two-period economy. As in the B-S model, the consumers have identical constant relative risk aversion and believe that future aggregate consumption is log-normally distributed. We assume that they agree on the variance of the growth rate of aggregate consumption but disagree on its expected value, which reflects their divergent confidence in the economy. We show that in this case the representative investor will have declining relative risk aversion instead of the constant relative risk aversion assumed in the B-S model. Because of this, the actual prices of options on aggregate consumption will be higher than the Black-Scholes prices. Moreover, under the assumption of bivariate lognormality the pricing kernel for contingent claims on stocks will have declining elasticity instead of the constant elasticity assumed in the B-S model. Because of this, the actual prices of options on stocks will be higher than the Black-Scholes prices.

Intuitively, when consumers have divergent confidence in the economy, their optimal consumption plans will be non-linear; to meet these non-linear plans,

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[1] See B & M (2000) p 7.

[2] The Consumer Research Center of the Conference Board makes monthly press releases on the Consumer Confidence Index. Reports on this can be found at its CRC/Confidence survey website at http://www.consumerresearchcenter.org.

they demand more options on aggregate consumption and/or options on stocks. This must push up option prices. Thus it is not surprising that the B-S model, which ignores divergent consumer confidence, underprices options.

Some empirical research has suggested that options are underpriced by the B-S model, i.e., the implied volatility of options typically exceeds the historical volatility of the price of the underlying asset (see, for example, Canina and Figlewski (1993)). Some other empirical research documented so-called volatility smile (see, for example, Mayhew (1995)), which, as suggested by Franke, Stapleton and Subrahmanyam (1998) (hereafter FSS), may be well explained by a pricing kernel with declining elasticity. Thus our result is consistent with the above empirical evidence. Nevertheless, we stress that our result only shows the impact on option prices of divergent consumer confidence. It does not mean that we should necessarily find that all options are underpriced by the B-S model since there are other factors that may cause biases in the Black-Scholes option prices in different directions.

The analysis in this paper is closely related to that of B & M (2000). They investigated how heterogeneity among investors in tastes and beliefs affects the prices of options on aggregate consumption. Since divergence of consumer confidence is one type of heterogeneity in beliefs, our work is closely related to theirs. However, our work is different from theirs in several aspects. First, they focused on options on aggregate consumption while we are more interested in stock options. Second, the methods used to calculate the Black-Scholes prices with which we and B & M compare actual option prices are different. We calculate the Black-Scholes prices in the actual economy taking the spot value of the underlying asset and the interest rate as given while they calculated them in a related but different economy.[3] Both methods may have their advantages. Their method is better to show how option prices change with changes in relevant parameter values, while our method is better to show how heterogeneity leads to systematic biases in Black-Scholes option prices.

Third, they derived their main results in a two-agent economy, while we do the analysis in a more general economy with N consumers. In fact their results can be extended to an N-agent economy. This is also done in this paper.

Fourth, due to the above differences, the result obtain in this paper is different from theirs. They concluded that the Black-Scholes prices of deep-out-of-the-money options on the aggregate consumption are biased downwards due to heterogeneity while we conclude that the impact of divergent consumer confidence is that it tends to make B-S model underprice all options (including out-of-the-money and in-the-money options on aggregate consumption and on stocks).

The work is also closely related to that of FSS (1998) who studied the impact on option prices of the convexity of a pricing kernel. Their results help to prove the propositions in this paper.

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[3]B & M (2000) calculated the Black-Scholes option prices in an economy in which the value of the investors' parameter from which heterogeneity arises is somewhere between the largest and the smallest in the actual economy.

The structure of the paper is as follows: In Section I we introduce a two-period economy and the B-S model. Section II deals with the impact of divergent consumer confidence on the prices of options on aggregate consumption. Section III addresses the impact of divergent consumer confidence on the prices of options on stocks under the assumption of bivariate log-normality made in the B-S model. In Section IV we extend B & M's (2000) main results to an N-agent economy. The final section concludes the paper.

I A Two-Period Economy

We assume a two-period Arrow-Debreu economy with a single good. Wealth is measured in units of the good. There are N consumers and every consumer's wealth consists of a portfolio of state-contingent claims on the aggregate consumption good in the second period denoted by X. Assume that there is a complete market for state-contingent claims on X. Thus all investors can buy and sell state-contingent claims on X so that any consumer i can choose her desired contingent consumption plan $x_i(X)$. Assume investors have identical constant relative risk aversion, i.e., for every investor i, her utility function can be written as

$$u_i(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$, which is a crucial assumption in the B-S model[4].

As in the B-S model, we also assume that consumers' beliefs are lognormal; however, they may disagree on the distribution's parameters. Let $f(X)$ be the objective probability density function and $f_i(X)$ investor i's subjective probability density function respectively. Since the distributions are lognormal, they can be written as

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}X}e^{-\frac{(\ln X - \mu)^2}{2\sigma^2}} \quad \text{and} \quad f_i(X) = \frac{1}{\sigma_i\sqrt{2\pi}X}e^{-\frac{(\ln X - \mu_i)^2}{2\sigma_i^2}}. \qquad (1)$$

We further assume that consumers agree on the variance of the growth rate of aggregate consumption.[5] I.e.,

$$\text{for every } i, \quad \sigma_i = \sigma.$$

But to model divergent consumer confidence, we assume that consumers disagree on the mean of the growth rate of aggregate consumption. I.e.,

$$\text{for some } i \neq j, \quad \mu_i \neq \mu_j.$$

We assume that there exists a unique pricing kernel, $\phi(X)$, whose functional form will be determined in an equilibrium. Let w_{i0} be investor i's initial endowment expressed as the fraction of the spot value of total wealth of the economy.

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[4]See Rubinstein (1976) and Brennan (1979).

[5]We mean the continuously compounded growth rate which is equal to $\ln(X/X_0)$. For consumer i, the variance of the (continuously compounded) growth rate of aggregate consumption is $Var_i(\ln(X/X_0)) = \sigma_i^2$.

Let x_{i0} be her consumption in the first period and x_i in the second period respectively. She has the following utility maximization problem:

$$\max_{x_{i0},x_i} u_i(x_{i0}) + \rho_i E_i[u_i(x_i)]. \tag{2}$$

Subject to

$$x_{i0} + E(\phi(X)x_i) = w_{i0}(X_0 + E(X\phi(X))). \tag{3}$$

where ρ_i is investor i's time preference parameter, $E_i(.)$ is the expectation operator under her subjective probability measure with p.d.f. $f_i(X)$, X_0 and X are the total endowment in the first period and second period respectively, and $E(.)$ denotes the expectation operator under the true probability measure with p.d.f. $f(X)$. In equilibrium, the market is cleared for all the state-contingent claims, thus we have

$$\sum_i x_{i0}(X_0) = X_0 \quad \text{and} \quad \sum_i x_i(X) = X. \tag{4}$$

Since negative consumption is not allowed we require that for every i, $x_{i0} \geq 0$ and $x_i \geq 0$. In this paper all utility functions are strictly increasing and concave and have infinite marginal utility at zero consumption. This guarantees that the first order condition is an equality, which can be written as

$$u_i'(x_i) = \lambda_i g_i(X)\phi(X), \tag{5}$$

where

$$g_i(X) = f(X)/f_i(X) \quad \text{and} \quad \lambda_i = u_i'(x_{i0})/\rho_i.$$

According to the results in Huang (2001), given the above assumptions there exists a pricing kernel under which every consumer's utility maximization problem (2) is well defined and has an optimal interior solution, i.e., there exists an equilibrium with interior solutions. Moreover, the pricing kernel $\phi(X)$ and every consumer's optimal contingent consumption plan $x_i(X)$ are infinitely differentiable[6].

Now if for every i, $\mu_i = \mu$, i.e., consumers have identical log-normal beliefs, according to Rubinstein (1974), we have a representative consumer who has constant relative risk aversion. Thus from Rubinstein (1976), the Black-Scholes option pricing formula can be obtained for options on the aggregate consumption. Furthermore, if we assume that a stock's price S and the aggregate consumption X follow a bivariate log-normal distribution, then the Black-Scholes option pricing formula can be also obtained for the options on the stock.

II   Options on Aggregate Consumption

In this section we investigate the impact of divergent consumer confidence on the prices of options on the aggregate consumption. Assume consumers have

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[6]See Proposition 1 in Huang (2001).

identical constant relative risk aversion and have log-normal beliefs as in the discrete-time Black-Scholes option-pricing model. If consumers disagree on the mean of the growth rate of aggregate consumption, we have the following result.

Lemma 1 *The representative consumer of the economy has declining relative risk aversion.*

Proof: See appendix A.

As shown by Rubinstein (1974), in this economy, in which consumers have identical risk-averse power utility functions, if they have homogeneous beliefs then an aggregate consumer in Rubinstein's (1974) sense will exist and she will have constant relative risk aversion. However, Lemma 1 tells us that when consumers have heterogeneous lognormal beliefs, the representative consumer will have declining relative risk aversion.

Lemma 1 tells us clearly how the heterogeneity in beliefs affect the representative consumer's relative risk aversion. While all consumers have the same coefficient of constant relative risk aversion, because of the divergence of opinions the representative consumer's coefficient of relative risk aversion is state dependent; more precisely, it is declining in future aggregate consumption X.

As shown by the lemma, the representative consumer's coefficient of relative risk aversion is not the average value of some relevant variables. Thus it is risky to extend the results obtained under the assumption of homogeneity to an economy in which consumers are heterogeneous with certain values considered to be averages.

Before we proceed to present our main result, we first introduce a lemma which was obtained by FSS (1998):

Lemma 2 (FSS) *Given two pricing kernels $\phi_1(x)$ and $\phi_2(x)$ which have declining elasticity $\gamma_1(x)$ and constant elasticity $\gamma_2(x)$ respectively, taking the interest rate and the spot price of x as given, $\phi_1(x)$ gives higher prices of convex-payoff contingent claims, i.e., $E[v(x)\phi_1(x)] > E[v(x)\phi_2(x)]$, where $v(x)$ is any convex payoff.*

Proof: See Appendix B.

From Lemma 1 and Lemma 2, we have the following result.

Proposition 1 *In the economy taking the interest rate and the spot value of future aggregate consumption as given, the B-S model underprices all options on the future aggregate consumption.*

Proof: According to Rubinstein (1976) and Brennan (1979), if we assume that the conditions for a B-S model are satisfied then option prices can be directly obtained as functions of the actual interest rate, the actual spot value of future aggregate consumption and the variance of future aggregate consumption. And these Black-Scholes option prices imply a pricing kernel. We denote it as $\hat{\phi}(X)$ which is different from the actual pricing kernel denoted by $\phi(X)$.

Let $\gamma_e(X)$ be the actual representative consumer's coefficient of relative risk aversion and $\hat{\gamma}_e(X)$ the coefficient of relative risk aversion of the representative consumer implied by the B-S model respectively. As is well known, $\gamma_e(X)$ is also the elasticity of the actual pricing kernel $\phi(X)$ for contingent claims on aggregate consumption and $\hat{\gamma}_e(X)$ the elasticity of the implied pricing kernel $\hat{\phi}(X)$ respectively. According to Lemma 1, $\gamma_e(X)$ is declining. But $\hat{\gamma}_e(X)$ is constant. Applying Lemma 2, we immediately conclude that the actual pricing kernel $\phi(X)$ gives higher option prices than the implied pricing kernel $\hat{\phi}(X)$. Hence the actual option prices are higher than the Black-Scholes prices. Q.E.D.

The fact that consumers disagree on the mean of the growth rate of aggregate consumption implies that they have divergent confidence in the economy. Those who have higher mean parameters are more optimistic about the economy while those who have lower mean parameters are more pessimistic about the economy. The proposition tells us that the divergence of the consumers' opinions about the economy affects option prices. If this divergence is ignored, as is the case in the discrete-time Black-Scholes option-pricing model, all options on aggregate consumption will be underpriced.

Intuitively, if consumers have divergent opinions about future aggregate consumption, in order to satisfy their needs for divergent patterns of optimal consumption, they demand more options on the aggregate consumption[7]. Thus the prices of options become higher. When we apply an option-pricing model such as the B-S model which ignores divergence of consumer confidence, all options on aggregate consumption will be underpriced.

Technically speaking, heterogeneous beliefs lead to a more convex pricing kernel than homogeneous beliefs. And a more convex pricing kernel gives higher option prices if the interest rate and spot value of future aggregate consumption are taken as give. Thus option prices will be underpriced if heterogeneity of beliefs is ignored.

This situation is similar to the case when consumers have identical beliefs but different coefficients of relative risk aversion. According to B & M (2000), when consumers have identical beliefs but different constant coefficients of relative risk aversion, the representative consumer has declining relative risk aversion. Applying Lemma 2, we can obtain a result similar to Proposition 1, which can be stated as follows:

In a two-period economy with N consumers, assume consumers have identical lognormal beliefs but have different constant coefficients of relative risk aversion. Taking the interest rate and the spot value of aggregate consumption as given, the B-S model underprices all options on aggregate consumption.

In a similar framework, B & M (2000) showed that (far) out-of-the-money options on aggregate consumption are underpriced by the Black-Scholes option pricing formula when investors have heterogeneous preferences or heterogeneous

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[7]See, for example, Leland (1980).

beliefs. But as mentioned in the introduction of the paper, their methodology is different from ours. They calculated the Black-Scholes prices in a related but different economy assuming the value of the investors' parameter from which heterogeneity arises is somewhere between the largest and the smallest in the actual economy; while we calculate them in the actual economy taking the spot value of the aggregate consumption and the interest rate as given. Our methodology enables to derive a clearer result about the impact on option prices of divergent consumer confidence.

III Options on Stocks

In this section we investigate the impact of heterogeneity in beliefs on the prices of options on stocks.

According to FSS (1998), we have the following result:

Lemma 3 (FSS) *Assume a stock's price S and the future aggregate consumption follow a bivariate log-normal distribution. If the representative consumer has declining relative risk aversion, then the pricing kernel for contingent claims on the stock has declining elasticity.*

Proof: See Appendix C.

Proposition 2 *In the economy assume a stock's price and the future aggregate consumption follow a bivariate log-normal distribution. Taking the interest rate and the spot stock price as given, the B-S model underprices all options on the stock.*

Proof: From Lemma 1, the representative consumer has declining relative risk aversion. From Lemma 3, the true pricing kernel (denoted by $\varphi(S)$) for contingent claims on the stock has declining elasticity. But the pricing kernel (denoted by $\hat{\varphi}(S)$) implied by the Black-Scholes option prices has constant elasticity. Applying Lemma 2, we immediately conclude that the actual pricing kernel $\varphi(S)$ gives higher option prices than the implied pricing kernel $\hat{\varphi}(S)$. Hence the actual option prices are higher than the Black-Scholes prices. Q.E.D.

This proposition tells us clearly the impact of divergent consumer confidence on stock option prices. Consumers typically have divergent opinions about future aggregate consumption. If this divergence of consumer confidence is ignored, as is the case in the B-S model, all options will be underpriced under the assumption of bivariate lognormality.

The intuition behind the result is similar to that mentioned in the last section. Since the stock options are also designed to help consumers to satisfy their desired consumption patterns, when they have divergent opinions about future aggregate consumption, they demand more stock options to satisfy their needs for divergent patterns of optimal consumption. This pushes up stock option prices. Thus ignoring heterogeneity may lead to systematic biases in stock option prices.

We also have a similar result when consumers have identical beliefs but different coefficients of relative risk aversion. When consumers have identical beliefs but different constant coefficients of relative risk aversion, as shown by B & M (2000), the representative consumer has declining relative risk aversion. Applying Lemma 3, we conclude that the pricing kernel for contingent claims on the stock, whose price and aggregate consumption follow a bivariate log-normal distribution, has declining elasticity. Applying Lemma 2, we obtain the following result:

In a two-period economy with N consumers, assume consumers have identical lognormal beliefs but have different constant coefficients of relative risk aversion. Taking the interest rate and the spot stock price as given, the B-S model underprices all options on any stock whose price and aggregate consumption follow a bivariate log-normal distribution.

IV   An Extension of B & M's (2000) Results

B & M (2000) derived their main results in a two-agent economy. We now extend their Proposition 6 and 7 to an economy with N agents.

We first introduce a lemma.

Lemma 4 *Assume $\phi_1 > 0$ and $\phi_2 > 0$, we will have the following results:*

- *If $\lim_{x \to 0^+}(-x\frac{\phi_1'(x)}{\phi_1(x)}) - (-x\frac{\phi_2'(x)}{\phi_2(x)}) > \alpha_0 > 0$, then $\lim_{x \to 0^+} \frac{\phi_1}{\phi_2} = +\infty$;*

- *If $\lim_{x \to +\infty}(-x\frac{\phi_1'(x)}{\phi_1(x)}) - (-x\frac{\phi_2'(x)}{\phi_2(x)}) < -\beta_0 < 0$, then $\lim_{x \to +\infty} \frac{\phi_1}{\phi_2} = +\infty$.*

Proof: Since there exists $x_1 > 0$, such that for $x < x_1$, $(-x\frac{\phi_1'(x)}{\phi_1(x)}) - (-x\frac{\phi_2'(x)}{\phi_2(x)}) \geq \alpha_0$, it follows that for $x < x_1$, $(\ln\frac{\phi_2}{\phi_1})' \geq \frac{\alpha_0}{x}$. Then for any $x < x_1$, it holds that $\ln\frac{\phi_2(x_1)}{\phi_1(x_1)} - \ln\frac{\phi_2(x)}{\phi_1(x)} \geq \alpha_0(\ln x_1 - \ln x)$. Let $x \to 0^+$, we obtain $\lim_{x \to 0^+} \frac{\phi_1}{\phi_2} = +\infty$. Since there exists $x_2 > 0$, such that for $x > x_2$, $(-x\frac{\phi_2'(x)}{\phi_2(x)}) - (-x\frac{\phi_1'(x)}{\phi_1(x)}) \geq \beta_0$, it follows that for $x > x_2$, $(\ln\frac{\phi_1}{\phi_2})' \geq \frac{\beta_0}{x}$. Then for any $x > x_2$, it holds that $\ln\frac{\phi_1(x)}{\phi_2(x)} - \ln\frac{\phi_1(x_2)}{\phi_2(x_2)} \geq \beta_0(\ln x - \ln x_2)$. Let $x \to +\infty$, we obtain $\lim_{x \to 0^+} \frac{\phi_1}{\phi_2} = +\infty$. Q.E.D.

We first extend their Proposition 6. Assume in the actual economy investors have homogeneous lognormal beliefs but have different constant coefficients of relative risk aversion. Let γ_N be the largest constant coefficients of relative risk aversion and γ_1 the smallest one respectively. Using B & M's (2000) methodology, we compare the actual prices of options on aggregate consumption with those in a Black-Scholes economy which is almost the same as the actual economy except that investors all have the same constant coefficient of relative risk

aversion γ_0. Let $\phi(X)$ denote the pricing kernel in the actual economy and $\phi_0(X)$ the pricing kernel in the Black-Scholes economy respectively. Let $c(K)$ $(p(K))$ denote the price of a call (put) option with strike price K in the actual economy and $c_0(K)$ $(p_0(K))$ in the Black-Scholes economy respectively. We have the following result.

Proposition 3 *For any γ_0, such that $\gamma_1 < \gamma_0 < \gamma_N$, there are two positive values X_{high} and X_{low} so that $\phi(X) > \phi_0(X)$ if either $X > X_{high}$ or $0 < X < X_{low}$. As a result,*

- *For sufficiently high K, $c(K) > c_0(K)$.*

- *For sufficiently low $K > 0$, $p(K) > p_0(K)$.*

Proof: According to B & M's (2000) Proposition 3, we have

$$\lim_{X \to +\infty} \gamma_e(X) = \gamma_1$$

and

$$\lim_{X \to 0} \gamma_e(X) = \gamma_N,$$

where $\gamma_e(X)$ is the relative risk aversion of the representative agent in the actual economy.

It follows that

$$\lim_{X \to +\infty} \gamma_e(X) - \gamma_0 = \gamma_1 - \gamma_0 < 0$$

and

$$\lim_{X \to 0} \gamma_e(X) - \gamma_0 = \gamma_N - \gamma_0 > 0,$$

where γ_0 is the coefficient of relative risk aversion of the representative investor in the Black-Scholes economy. Applying Lemma 4, we immediately obtain the conclusion. Q.E.D.

We now extend their Proposition 7. Assume in the actual economy investors have the same constant coefficient of relative risk aversion, γ, and they agree on the variance of logarithm of (lognormally distributed) aggregate consumption, σ, but disagree on its mean. Let μ_N be the largest mean and μ_1 the smallest respectively. Assume the objective mean of logarithm of (lognormally distributed) aggregate consumption is μ and $\mu_1 < \mu < \mu_N$. Using B & M's (2000) methodology, we compare the actual prices of options on aggregate consumption with those in a Black-Scholes economy which is almost the same as the actual economy except that investors all agree on the mean of logarithm of (lognormally distributed) aggregate consumption, μ. Let $\phi(X)$ denote the pricing kernel in the actual economy and $\phi_0(X)$ the pricing kernel in the Black-Scholes economy respectively. We first have a lemma.

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Lemma 5 *Let $\varepsilon_i = (\mu_i - \mu)/\sigma^2$, $i = 1, ..., N$. We have*

$$\lim_{X \to +\infty} \gamma_e(X) = \gamma - \varepsilon_N \tag{6}$$

and

$$\lim_{X \to 0} \gamma_e(X) = \gamma - \varepsilon_1, \tag{7}$$

where $\gamma_e(X)$ is the relative risk aversion of the representative agent in the actual economy.

Proof: Let $w_i = \frac{x_i}{X}$. We have

$$(\ln w_i)' = \frac{x_i'}{x_i} - \frac{1}{X}.$$

Substituting (9) into the above equation, we obtain

$$(\ln w_i)' = \frac{1}{\gamma X}(\gamma_e(X) + \varepsilon_i) - \frac{1}{X}.$$

Substituting (10) into the above equation, we obtain

$$(\ln w_i)' = \frac{1}{\gamma X}\left(\gamma + \varepsilon_i - \sum_i w_i \varepsilon_i\right) - \frac{1}{X}.$$

It can be rewritten as:

$$d \ln w_i = \frac{\varepsilon_i - \sum_i w_i \varepsilon_i(X)}{\gamma} d \ln X. \tag{8}$$

Now we assert that

- for any i if $\varepsilon_i < \varepsilon_N$ which is equivalent to $\mu_i < \mu_N$, then $\lim_{X \to +\infty} w_i(X) = 0$;

- for any i if $\varepsilon_i > \varepsilon_1$ which is equivalent to $\mu_i > \mu_1$, then $\lim_{X \to 0^+} w_i(X) = 0$.

Suppose the first statement is not true. There must exist $\alpha_0 > 0$ such that for sufficiently large $X_0 > 0$, when $X > X_0$,

$$\varepsilon_N - \sum_i w_i(X)\varepsilon_i(x_i) > \alpha_0.$$

From the above equation and (8), we have when $X > X_0$,

$$d \ln w_N > \frac{\alpha_0}{\gamma} d \ln X.$$

It follows that when $X > X_0$,

$$\ln w_N(X) - \ln w_N(X_0) > \frac{\alpha_0}{\gamma} \ln \frac{X}{X_0}.$$

Now in the above equation let $X \to +\infty$. We have $\lim_{X \to +\infty} w_N(X) = \infty$. This is impossible. Analogously, we can prove the second statement.

From the above two statements and (10), we obtain

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- $\gamma_e(\infty) = \lim_{X \to \infty} \gamma_e(X) = \gamma - \varepsilon_N,$

- $\gamma_e(0) = \lim_{X \to 0^+} \gamma_e(X) = \gamma - \varepsilon_1.$

Q.E.D.

As before, let $c(K)$ ($p(K)$) denote the price of a call (put) option with strike price K in the actual economy and $c_0(K)$ ($p_0(K)$) in the Black-Scholes economy respectively. We now present the following result.

Proposition 4 *For any μ, such that $\mu_1 < \mu < \mu_N$, there are two positive values X_{high} and X_{low} so that $\phi(X) > \phi_0(X)$ if either $X > X_{high}$ or $0 < X < X_{low}$. As a result,*

- *For sufficiently high K, $c(K) > c_0(K)$.*

- *For sufficiently low $K > 0$, $p(K) > p_0(K)$.*

Proof: Applying Lemma 5, we obtain (6) and (7). It follows that

$$\lim_{X \to +\infty} \gamma_e(X) - \gamma = -\varepsilon_N < 0$$

and

$$\lim_{X \to 0} \gamma_e(X) - \gamma = -\varepsilon_1 > 0,$$

where γ is the coefficient of relative risk aversion of the representative investor in the Black-Scholes economy. Applying Lemma 4, we immediately obtain the conclusion. Q.E.D.

V   Conclusions

Consumers usually have divergent confidence in the economy. Because of this, consumers demand more options to satisfy their needs for non-linear optimal consumption plans. This pushes up option prices. Thus if this divergence of consumer confidence is ignored, as in the case of the discrete-time Black-Scholes option-pricing model, all options on future aggregate consumption, or on any stock whose price and aggregate consumption follow a bivariate log-normal distribution, will be underpriced (provided that other things are consistent with the B-S model). The result is consistent with the documented downward biases in Black-Scholes option prices (see, for example Canina and Figlewski (1993)) and so-called volatility smile (see, for example Mayhew (1995)). This, however, does not mean that we should always find empirically that all option prices are biased downwards since there are other factors that may cause biases in the Black-Scholes option prices in different directions.

Nevertheless, we can reasonably conjecture that in a period when consumers have more divergent opinions about the growth rate of aggregate consumption, the difference between the true option prices and those implied by the B-S model will be larger (given that other conditions do not change drastically). It will be interesting if we can propose a proper measure of divergence of consumer confidence and show if the above conjecture is true.

A Proof of Lemma 1

Differentiating both sides of Equation (5), we obtain

$$x_i'(X) = R_i^{-1}(x_i)[R_e(X) - g_i'(X)/g_i(X)].$$

It can be written as

$$x_i'(X) = \frac{x_i}{\gamma X}[\gamma_e(X) + \varepsilon_i], \tag{9}$$

where $\varepsilon_i = -Xg_i'(X)/g_i(X)$ is a constant. Since $\sum_i x_i(X) = X$, from the above equation we obtain

$$\gamma_e(X) = \sum_i [s_i(X)(\gamma - \varepsilon_i)],$$

where $s_i(X) = R_i^{-1}(x_i)/\sum_i R_i^{-1}(x_i) = x_i/X$. It can be rewritten as

$$\gamma_e(X) = \gamma - \sum_i s_i(X)\varepsilon_i. \tag{10}$$

Since $s_i(X) = x_i/X$, we have

$$(\ln s_i(X))' = x_i'/x_i - 1/X.$$

Substituting (9) into the above equation, we obtain

$$(\ln s_i(X))' = [\frac{1}{\gamma}(\gamma_e(X) + \varepsilon_i) - 1]/X.$$

Substituting (10) into the above equation, we have

$$(\ln s_i(X))' = \frac{1}{\gamma X}[\varepsilon_i - \sum_i s_i(X)\varepsilon_i]. \tag{11}$$

Differentiating both sides of (10), we obtain

$$\gamma_e'(X) = -\sum_i s_i'(X)\varepsilon_i.$$

Substituting (11) into the above equation, we obtain

$$\gamma_e'(X) = -\frac{1}{\gamma X}[\sum_i s_i(X)\varepsilon_i^2 - (\sum_i s_i\varepsilon_i)^2]. \tag{12}$$

Applying Cauchy's inequality, we immediately conclude that $\gamma'(X) \leq 0$. The equality will hold if and only if for every i and j, $\varepsilon_i = \varepsilon_j$, which is equivalent to the condition that for every i and j, $\mu_i = \mu_j$. Q.E.D.

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B   Proof of Lemma 2

Since $\gamma_1(x)$ and $\gamma_2(x)$ intersect only once, it follows that $(\ln \phi_1(x)/\phi_2(x))' = 0$ has a unique solution, or $\phi_1(x)/\phi_2(x)$ has a unique critical point. Thus $\phi_1(x)$ and $\phi_2(x)$ intersect at most twice. Otherwise suppose they intersect more than twice and x_1, x_2 and x_3 are the three consecutive points at which they intersect. Then there are at least two critical points, one between x_1 and x_2 and the other between x_2 and x_3, which contradicts the given condition. Since $E[\phi_1(x)] = E[\phi_2(x)]$, the two pricing kernels must intersect. But since $E(x\phi_1(x)) = E(x\phi_2(x))$, they will intersect exactly twice. Otherwise suppose they intersect once. Without loss of generality, assume $\phi_1(x)$ intersects $\phi_2(x)$ from below at x_0. Following $E[\phi_1(x)] = E[\phi_2(x)]$, it holds that

$$E[x(\phi_1(x) - (\phi_2(x))] = E[(x - x_0)(\phi_1(x) - \phi_2(x))].$$

Since the terms $(x - x_0)$ and $(\phi_1(x) - \phi_2(x))$ always have the same sign, we conclude that

$$E(x\phi_1(x)) > E(x\phi_2(x)),$$

which contradicts the given condition. Following the fact that $\gamma_1(x)$ intersects $\gamma_2(x)$ from above, it is easy to verify that $\phi_1(x)$ intersects $\phi_2(x)$ from above at the first intersection.

Now assume $\phi_1(x)$ and $\phi_2(x)$ intersect at x_1 and x_2, where $x_1 < x_2$. Given a convex payoff $v(x)$, we construct a linear function $L(x)$ such that $L(x_1) = v(x_1)$ and $L(x_2) = v(x_2)$. Since $v(x)$ is convex, it follows that

$$v(x) > L(x), \ for \ x < x_1 \ or \ x > x_2;$$

and

$$v(x) < L(x), \ for \ x_1 < x < x_2.$$

Then we have

$$E[v(x)\phi_1(x)] - E[v(x)\phi_2(x)] = E[v(x)(\phi_1(x) - \phi_2(x))].$$

Since $E[\phi_1(x)] = E[\phi_2(x)]$ and $E[x\phi_1(x)] = E[x\phi_2(x)]$, we have

$$E[v(x)(\phi_1(x) - \phi_2(x))] = E[(v(x) - L(x))(\phi_1(x) - \phi_2(x))].$$

Since the terms $(v(x) - L(x))$ and $(\phi_1(x) - \phi_2(x))$ have the same sign on interval $(0, +\infty)$, it follows that

$$E[(v(x) - L(x))(\phi_1(x) - \phi_2(x))] > 0.$$

Thus we have $E[v(x)\phi_1(x)] > E[v(x)\phi_2(x)]$. Q.E.D.

C   Proof of Lemma 3

Since X and S follow a bivariate log-normal distribution, they have the following relationship

$$\ln X = \alpha + \beta \ln S + \epsilon, \tag{13}$$

where α and $\beta > 0$ are constant and ϵ is a random variable independent of S, the stock price. Let $\varphi(S)$ denote the pricing kernel for contingent claims on S and $\gamma_e^\varphi(S)$ its elasticity respectively. Since $\varphi(S) = E[\phi(X)|S]$, we have

$$\gamma_e^\varphi(S) = -S\frac{\frac{d}{dS}E[\phi(X)|S]}{E[\phi(X)|S]}. \tag{14}$$

But from (13), we have

$$\frac{d}{dS}E[\phi(X)|S] = \beta E[\phi'(X)X|S]/S \tag{15}$$

Substituting this into (14), we obtain

$$\gamma_e^\varphi(S) = -\beta\frac{E[\phi'(X)X|S]}{E[\phi(X)|S]}.$$

It follows that

$$\frac{d}{dS}\ln\gamma_e^\varphi(S) = \beta\frac{\frac{d}{dS}E[\phi'(X)X|S]}{E[\phi'(X)X|S]} - \frac{\frac{d}{dS}E[\phi(X)|S]}{E[\phi(X)|S]} \tag{16}$$

But from (15) and (13), we obtain

$$\frac{d}{dS}E[\phi'(X)X|S] = \beta E[\phi''(X)X^2|S]/S + \beta E[\phi'(X)X|S]/S$$

Substituting this and (15) into (16), we have

$$\frac{d}{dS}\ln\gamma_e^\varphi(S) = \frac{\beta^2}{S}(\frac{E[\phi''(X)X^2|S] + E[\phi'(X)X|S]}{E[\phi'(X)X|S]} - \frac{E[\phi'(X)X|S]}{E[\phi(X)|S]}). \tag{17}$$

Since $\frac{d}{dX}\gamma_e(X) < 0$, we have

$$-\frac{\phi''(X)}{\phi'(X)} + \frac{\phi'(X)}{\phi(X)} > \frac{1}{X},$$

which implies

$$\phi''(X)X^2 + \phi'(X)X > \frac{\phi'^2(X)}{\phi(X)}X^2.$$

Substituting this into (17), we obtain

$$\frac{d}{dS}\ln\gamma_e^\varphi(S) < \frac{\beta^2}{S}(\frac{E[\frac{\phi'^2(X)}{\phi(X)}X^2|S]}{E[\phi'(X)X|S]} - \frac{E[\phi'(X)X|S]}{E[\phi(X)|S]}). \tag{18}$$

According to Cauchy's inequality, we have

$$E[\frac{\phi'^2(X)}{\phi(X)}X^2|S]E[\phi(X)|S] > (E[\phi'(X)X|S])^2$$

From the above equation, Equation (18) and the fact that $\phi'(X) < 0$, we conclude that

$$\frac{d}{dS}\ln\gamma_e^\varphi(S) < 0.$$

Q.E.D.

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