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We derive necessary and sufficient conditions for a Pareto optimal sharing rule to be linear in wealth and for all Pareto optimal sharing rules to be linear in wealth. We also show that when agents' beliefs have a particular feature a Pareto optimal sharing rule is linear in wealth if and only if all Pareto optimal sharing rules are linear. The results obtained in this paper generalize those on linear sharing rules in the existing literature. Unlike the special case when aggregate wealth is dependent on a state variable, even if all Pareto optimal sharing rules are linear it is not necessary that agents' utility functions be of the equicautious HARA class.

1. INTRODUCTION

Linear sharing rules in a special economy where aggregate wealth is dependent on a state variable are now well understood in the literature. Mossin (1973) showed that when agents have homogeneous beliefs, for all weightings in an open set the Pareto optimal sharing rules are linear if and only if agents have HARA class utilities with identical constant cautiousness. Amershi and

Stoeckenius (hereafter A & S)(1983) showed that if agents have heterogeneous beliefs then the following statements are equivalent: (i) There exists a Pareto optimal sharing rule which is linear in wealth. (ii) All the Pareto optimal sharing rules are linear in wealth. (iii) All agents have HARA class utilities with identical cautiousness.[1]

Huang and Litzenberger (1985) (hereafter H & L) showed that when agents have homogeneous beliefs, a Pareto optimal sharing rule is linear if and only if they have identical cautiousness along their optimal schedules. From this they pointed out that "for a fixed weighting λ, for the Pareto optimal sharing rule to be linear it is not necessary that the agents' utility functions be of the equicautious HARA class".

In this paper we study linear sharing rules in a general economy where the aggregate wealth, which is random, may not depend on a state variable. Alternatively we may say that the state of the economy is characterized by two random variables: the aggregate wealth and the state variable. The results derived in this paper generalize those given by A & S (1983), H & L (1985) and Mossin (1973). Interestingly we will see from these results that unlike the special case when aggregate wealth is dependent on a state variable, even if all Pareto optimal sharing rules are linear it is not necessary that agents' utility functions be of the equicautious HARA class.

In the following section we briefly introduce the formulation of the problem. In Section 3 we derive the main results. In Section 4 show conditions for all we discuss extensions of the results obtained in Section 3. The final section concludes the paper.

2. THE FORMULATION

Since the formulation of the problem is similar to the one in A & S (1983) and Wilson (1968), we will not go through the details. We refer readers who are interested in the formality to A & S (1983) and Wilson (1968). Assume there is a random variable s, which together with aggregate wealth X describes the state of an economy. Assume the range of s is $S \in R$, which is compact, and the range of X is R_+. The economy is characterized by a measurable space $(\Omega, \sigma(\Omega))$, where $\Omega = \{(s, X) : s \in S \text{ and } X \in R_+\}$ and $\sigma(\Omega)$ is a σ-algebra of events. Let there be N agents in the economy, indexed by $i = 1, 2, ..., N$. Each agent i is characterized by a strictly increasing and concave Von Neumann-Morgenstern utility function $u_i : R \to R$ and a probability density function $p_i(X, s) : \Omega \to R_+$. We assume that every agent i has infinite marginal utility of zero consumption, i.e., $\lim_{x \to 0} u_i'(x) = +\infty$. Moreover, whenever needed, the differentiability of agents' utility functions and probability density functions is assumed.

A sharing rule is a vector-valued function $x = (x_1, x_2, ..., x_N) : R_+ \times S \to R_+^N$ such that

$$\sum_{i=1}^{N} x_i(X; s) = X, \quad \forall X \in R.$$

A sharing rule x is said to be Pareto-optimal if there is no other sharing rule $\hat{x}$

such that

$$E_i[u_i(\hat{x}_i(X;s))] \geq E_i[u_i(x_i(X;s))] \quad \forall i$$

with at least a strict inequality for some i. Using an argument similar to the one by Wilson (1968), we obtain that a sharing rule x is Pareto-optimal if and only if there exist constants (weighting) $(\lambda_1, \lambda_2, ..., \lambda_N)$ such that for every i and j

$$\lambda_i u_i'(x_i) p_i(X,s) = \lambda_j u_j'(x_j) p_j(X,s) \tag{1}$$

Before we move forward we first clarify on notation. Through out this paper we always use $\gamma_i(x_i)$ to denote agent i's coefficient of relative risk aversion and $\varepsilon_i(X,s)$ the elasticity of agent i's probability density function w.r.t X. That is,

$$\gamma_i(x_i) = -x_i u_i''(x_i)/u_i'(x_i) \quad \text{and} \quad \varepsilon_i(X,s) = -\partial \ln p_i(X,s)/\partial \ln X.$$

3. THE MAIN RESULTS

As noticed by Huang and Litzenberger (1985), when agents have homogeneous beliefs, for a fixed weighting λ, for the optimal sharing rule to be linear, it is not necessary that utilities be of the HARA class at all. Instead they concluded that it is necessary and sufficient that agents have identical cautiousness along their optimal schedules. Their result is generalized in the following proposition.

Proposition 1 *A Pareto optimal sharing rule x is linear in wealth if and only if for every i and j*

$$\gamma_i(x_i) + \varepsilon_i(X,s) = \gamma_j(x_j) + \varepsilon_j(X,s). \tag{2}$$

Proof:

Necessity: Taking the derivative of the logarithm of both sides of eq. (1) w.r.t X, we obtain that for every i and j

$$R_i(x_i)\frac{\partial x_i}{\partial X} - \frac{\partial \ln p_i(X,s)}{\partial X} = R_j(x_j)\frac{\partial x_j}{\partial X} - \frac{\partial \ln p_j(X,s)}{\partial X},$$

where $R_i(x_i) = -u_i''(x_i)/u_i'(x_i)$ denotes agent i's coefficient of risk aversion. This equation can be written as

$$\gamma_i(x_i)\frac{\partial \ln x_i}{\partial \ln X} - \frac{\partial \ln p_i(X,s)}{\partial \ln X} = \gamma_j(x_j)\frac{\partial \ln x_j}{\partial \ln X} - \frac{\partial \ln p_j(X,s)}{\partial \ln X}. \tag{3}$$

If the sharing rule is linear in wealth, since $x_i(0,s) = 0$ we must have $x_i = a_i(s)X$, where $a_i(s)$ is a function of s only. Thus eq. (3) can be written as eq. (2).

Sufficiency: Equations (2) and (3) imply that for every i and j

$$\gamma_i(x_i)[\frac{\partial \ln x_i}{\partial \ln X} - 1] = \gamma_i(x_j)[\frac{\partial \ln x_j}{\partial \ln X} - 1],$$

which can be written as

$$\frac{\gamma_i(x_i)}{\gamma_j(x_j)}x_j[\frac{\partial \ln x_i}{\partial \ln X} - 1] = \frac{\partial x_j}{\partial \ln X} - x_j.$$

Noting that $\sum_j x_j = X$ and $\sum_j \partial x_j/\partial X = 1$, from the above equation we obtain for every i

$$\sum_j \frac{x_j}{\gamma_j(x_j)}\gamma_i(x_i)[\frac{\partial \ln x_i}{\partial \ln X} - 1] = 0.$$

It follows that for every i

$$\frac{\partial \ln x_i}{\partial \ln X} - 1 = 0.$$

From this we conclude that x is linear in X. Q.E.D.

This proposition generalizes the result in H & L (1985) which states that when agents have homogeneous beliefs a Pareto optimal sharing rule is linear if and only if agents have identical cautiousness along their optimal schedules.

Corollary 1.1 *Assume agents' beliefs have the following feature*

$$p_i(X, s) = f_i(X)Q(X, s), \tag{4}$$

where $Q(X, s)$ is common across all agents. Then a Pareto optimal sharing rule x is linear in wealth if and only if for every i and j

$$\gamma_i(x_i) + \varepsilon_i(X) = \gamma_j(x_j) + \varepsilon_j(X), \tag{5}$$

where for any i, $\varepsilon_i(X) = -Xf_i'(X)/f_i(X)$.

Proof: It is an immediate result of Proposition 1.

From the condition it is not difficult to construct examples in which agents do not have HARA utilities while Pareto optimal sharing rules are linear in wealth.

Corollary 1.2 *Assume agents have homogeneous beliefs. Then a Pareto optimal sharing rule is linear in wealth if and only if agents have identical relative risk aversion along their optimal schedules.*

Proof: It is an immediate result of Proposition 1.

H & L (1985) found that under homogeneous beliefs for a Pareto optimal sharing rule to be linear it is necessary and sufficient that agents have identical cautiousness along their optimal schedules. This corollary gives a different necessary and sufficient condition. In fact under the assumption that agents have infinite marginal utility of zero consumption these two conditions are equivalent.

Condition (1) is equivalent to the condition that for every $i = 2, 3, ..., N$

$$\lambda_i u_i'(x_i) p_i(X, s) = u_1'(x_1) p_1(X, s). \tag{6}$$

Let $\lambda \equiv (\lambda_2, \lambda_3, ..., \lambda_N)$. We have the following result:

Proposition 2 *For all weightings λ in an open set, the Pareto optimal sharing rules are linear in wealth if and only if the following conditions are met: (i) All agents have constant relative risk aversion. (ii) Agents' beliefs have the following feature:*

$$p_i(X, s) = f_i(X) Q(X, s) g_i(s), \tag{7}$$

where $Q(X, s)$ is common across all agents. (iii) The sum of an agent's coefficient of relative risk aversion and elasticity of his probability density function w.r.t the aggregate wealth is identical across all agents, i.e., for every i and j

$$\gamma_i + \varepsilon_i(X) = \gamma_j + \varepsilon_j(X), \tag{8}$$

where for any i, γ_i is agent i's coefficient of relative risk aversion and $\varepsilon_i(X) = -X f_i'(X)/f_i(X)$.

Proof: Since

$$\varepsilon_i(X,s) - \varepsilon_j(X,s) = -\partial \ln p_i(X,s)/\partial \ln X + \partial \ln p_j(X,s)/\partial \ln X,$$

the sufficiency is implied by Proposition 1. Thus we need only to show the necessity.

Given a linear sharing rule x, since for every i, $x_i(0,s) = 0$, we must have for every i, $x_i(X,s) = a_i(s)X$, where $a_i(s)$ is a function of s only. Taking the derivative of the logarithm of both sides of eq. (2) w.r.t λ_k, we obtain for every i and$j$

$$\gamma_i'(x_i)\frac{\partial a_i}{\partial \lambda_k} = \gamma_j'(x_j)\frac{\partial a_j}{\partial \lambda_k}. \tag{9}$$

Since for every i, $u_i''(x) < 0$ we must have for every i, $\partial a_i/\partial \lambda_k \neq 0$.[2]

Now we assert that for every i $\gamma_i'(x_i) = 0$. Otherwise we must have some i such that $\gamma_i'(x_i) \neq 0$. Since for every i, $\partial a_i/\partial \lambda_k \neq 0$, from (9) we conclude that for every i, $\gamma_i'(x_i) \neq 0$.

Given any i, since $\partial a_i/\partial \lambda_k \neq 0$, there must exist j such that $\partial(a_j/a_i)/\partial \lambda_k \neq 0$. Let $a_{ij} \equiv a_j/a_i$ and $X = Y/a_i$, where Y is independent of λ_k. Rewrite (9) as

$$\gamma_i'(Y) = \gamma_j'(a_{ij}Y)\frac{\partial a_j}{\partial \lambda_k}/\frac{\partial a_i}{\partial \lambda_k}. \tag{10}$$

Differentiating both sides of eq. (10) w.r.t λ_k, we have

$$\gamma_j''(a_{ij}Y)\frac{\partial a_{ij}}{\partial \lambda_k}Y(\frac{\partial a_j}{\partial \lambda_k}/\frac{\partial a_i}{\partial \lambda_k}) + \gamma_j'(a_{ij}Y)\frac{\partial(\frac{\partial a_j}{\partial \lambda_k}/\frac{\partial a_i}{\partial \lambda_k})}{\partial \lambda_k} = 0.$$

This implies that $-x_j\gamma_j''(x_j)/\gamma_j'(x_j)$ is constant.

Differentiating both sides of eq. (10) w.r.t Y, we have

$$\gamma_i''(Y) = \gamma_j''(a_{ij}Y)a_{ij}\frac{\partial a_j}{\partial \lambda_k}/\frac{\partial a_i}{\partial \lambda_k}.$$

From this and eq. (10) we obtain

$$-x_i \frac{\gamma_i''(x_i)}{\gamma_i'(x_i)} = -x_j \frac{\gamma_j''(x_j)}{\gamma_j'(x_j)} (\frac{\partial a_j}{\partial \lambda_k} / \frac{\partial a_i}{\partial \lambda_k}).$$

Since $-x_j \gamma_j''(x_j)/\gamma_j'(x_j)$ is constant, so is $-x_i \gamma_i''(x_i)/\gamma_i'(x_i)$. It follows that

$$\gamma_i'(x_i) = \alpha_i x_i^{\beta_i} \text{ and } \gamma_j'(x_j) = \alpha_j x_j^{\beta_j}, \tag{11}$$

where $\alpha_i (\neq 0)$, $\alpha_j (\neq 0)$, β_i, and β_j are all constant. Substituting this into (11), we conclude that $\beta_i = \beta_j = \beta$, which is constant across all agents.

On the other hand, from eq. (9) and the fact that $\sum_{i=1}^N \partial a_i / \partial \lambda_k = 0$ we must have $\sum_{i=1}^N 1/\gamma_i'(x_i) = 0$, which is equivalent to $\sum_{i=1}^N 1/\alpha_i = 0$. This implies there must exist i and j such that $\alpha_i < 0$ and $\alpha_j > 0$.

Suppose $\beta = -1$, from eq. (11) we have

$$\gamma_i(x_i) = \alpha_i \ln x_i + \alpha_{i0} \text{ and } \gamma_j(x_j) = \alpha_j \ln x_j + \alpha_{j0}.$$

When x_i approaches zero or infinity, since $\alpha_i \alpha_j < 0$, we have $\gamma_i(x_i)\gamma_j(x_j) < 0$. This is contradictory to the condition that agents' utility functions are increasing and concave.

Suppose $\beta \neq -1$, we have

$$\gamma_i(x_i) = \frac{\alpha_i}{1+\beta} x_i^{1+\beta} + \alpha_{i0} \text{ and } \gamma_j(x_j) = \frac{\alpha_j}{1+\beta} x_j^{1+\beta} + \alpha_{j0}.$$

If $\beta > -1$, when x_i approaches infinity we have $\gamma_i(x_i)\gamma_j(x_j) < 0$. If $\beta < -1$, when x_i approaches zero we also have $\gamma_i(x_i)\gamma_j(x_j) < 0$. Both are contradictory to the condition that agents' utility functions are increasing and concave. Hence it is proved that for every i, $\gamma_i(x_i)$ must be constant.

Since for every $\gamma_i(x_i)$ is constant, from eq. (2) we obtain eq. (8).

Moreover, differentiating both sides of eq. (8) w.s.t s, we obtain for every i and j

$$\frac{\partial^2 \ln p_i(X,s)}{\partial \ln X \partial s} = \frac{\partial^2 \ln p_j(X,s)}{\partial \ln X \partial s}.$$

This implies condition (ii). Q.E.D.

The sufficiency can also be verified directly as follows. We have for every i and j

$$\lambda_i \alpha_i x_i^{-\gamma_i} X^{\gamma_i} g_i(s) = \lambda_j \alpha_j x_j^{-\gamma_j} X^{\gamma_j} g_j(s),$$

where α_i and α_j are constant. It follows that $x_i = a_i(s)X$, where $a_i(s)$ is the solution to

$$\sum_{j=1}^{N} a_i^{\frac{\gamma_i}{\gamma_j}} = \sum_{j=1}^{N} (\lambda_i \alpha_i g_i(s))^{\frac{1}{\gamma_j}} / \sum_{i=1}^{N} (\lambda_j \alpha_j g_j(s))^{\frac{1}{\gamma_j}}. \tag{12}$$

The existence of a solution to eq. (12) is evident. Let $\pi(y) = \sum_{j=1}^{N} y^{\gamma_i/\gamma_j}$. Since $\pi(0) = 0$ and $\pi(+\infty) = +\infty$, given any $s \in S$, there always exists $a_i(s)$ satisfies eq. (12). This shows that if the conditions in Proposition 2 is satisfied, then for all weightings there exist Pareto optimal sharing rules which are all linear in wealth.

This proposition generalizes Mossin's (1973, pp. 114) result. Mossin stated that when agents have homogeneous beliefs a necessary and sufficient condition under which optimal sharing rules are linear in wealth for all weightings λ in an open set is that utility functions be of the equicautious HARA class (See Mossin (1973) pp. 114). Proposition 2, however, tells us that the condition is not necessary when agents have heterogeneous beliefs over aggregate wealth (although it is necessary that utility functions be of the HARA class).

Now consider the case when aggregate wealth X is dependent on the state variable s and a non-random decision α as in A & S's (1983) economy. If given α, $X = X(s, \alpha) : S \to R_+$ is one to one, then we have an equivalent probability space on X induced from the one on s. Let $f_i(X, \alpha)$ be agent i's probability density function of X induced from his probability density function of s. Applying Proposition 2, we conclude that all Pareto optimal sharing rules are linear if and only if all agents have constant relative risk aversion and eq. (8) holds. Denote this result as $R\alpha$. Result $R\alpha$ is very different from Theorem 5 in A & S (1983) which states that given that agents have heterogeneous beliefs, all Pareto optimal sharing rules are linear in wealth if and only if all agents have identical relative risk aversion. Note that the linear sharing rules in $R\alpha$ are determinate while those in Theorem 5 are dependent on the random variable s.

From Proposition 2 we have the following corollary:

Corollary 2.1 *Assume agents' beliefs have the following feature*

$$p_i(X, s) = Q(X, s)g_i(s), \tag{13}$$

where $Q(X, s)$ is common across all agents. Then for an open set of λ, the Pareto optimal sharing rules are linear in wealth if and only if agents have identical constant relative risk aversion.

Proof: It is an immediate result of Proposition 2.

This corollary also generalizes Mossin's (1973, pp. 114) result. Loosely speaking, it tells us that Mossin's result holds in a more general economy where agents have homogeneous beliefs regarding aggregate wealth X.

Proposition 3 *Assume $\partial \ln p_i(X,s)/\partial s$ is not identical across all agents on an open subset of S and for every i and j, $\partial \varepsilon_i(X,s)/\partial X = \partial \varepsilon_j(X,s)/\partial X$. Then the following statements are equivalent: (i) There exists a Pareto optimal sharing rule which is linear in wealth on the open subset of S. (ii) All Pareto optimal sharing rules are linear in wealth on all of S. (iii) Agents' beliefs have the feature in eq. (7), every agent i has constant relative risk aversion and for every i and j, eq. (8) holds.*

Proof: (iii)$\Rightarrow$ (ii) is implied by Proposition 1. (ii) $\Rightarrow$ (i) is trivial. Thus we need only to show (i)$\Rightarrow$ (iii).

Taking the derivative of both sides of eq. (5) w.r.t X and applying the condition that for every i and j, $\partial \varepsilon_i(X,s)/\partial X = \partial \varepsilon_j(X,s)/\partial X$, we obtain for every i and j

$$\gamma_i'(x_i)a_i = \gamma_j'(x_j)a_j. \tag{14}$$

Given any i, there must exist j such that $\partial(a_j/a_i)/\partial s \neq 0$ (otherwise we have for every i, $\partial a_i/\partial s = 0$ which is contradictory to the condition that $\partial \ln p_i(X,s)/\partial s$ is not identical across all agents on an open subset of S). The rest of the proof is almost the same as the proof of the necessity part of Proposition 2 except that we replace eq. (10) with eq. (14) and λ_k with s respectively. Q.E.D.

This Proposition generalizes Theorem 5 in A & S (1983) which states that when aggregate wealth is dependent on the state variable (and a non-random decision) then (i) and (ii) are equivalent and they are both equivalent to the condition that all agents have identical constant cautiousness.

Corollary 3.1 *Assume agents' beliefs have the feature as in eq. (13) and* $g_i'(s)/g_i(s)$ *is not identical across* i *on an open subset of* S*. Then the following statements are equivalent: (i) There exists a Pareto optimal sharing rule which is linear in wealth on the open subset of* S*. (ii) All Pareto optimal sharing rules are linear in wealth on all of* S*. (iii) All agents have identical constant relative risk aversion.*

Proof: It is an immediate result of Proposition 3.

This corollary also generalizes Theorem 5 in A & S (1983). Loosely speaking, it tells us that Theorem 5 holds in a more general economy where agents have homogeneous beliefs regarding aggregate wealth X or where aggregate wealth is dependent on the state variable (and a non-random decision).

Proposition 4 *Assume agents' beliefs have the feature in (7). Then there exists a Pareto optimal sharing rule* x *which is linear in wealth and for every* i, $\partial x_i/\partial s \neq 0$ *only if every agent has constant relative risk aversion,* $g_i'(s)/g_i(s)$ *is not identical across all agents, and for every* i *and* j*, eq. (8) holds.*

Proof:

Taking the derivative of the logarithm of both sides of eq. (2) w.r.t s and applying eq. (7), we obtain for every i and$j$

$$\gamma_i'(x_i)\frac{\partial a_i}{\partial s} = \gamma_j'(x_j)\frac{\partial a_j}{\partial s}. \tag{15}$$

Given any i, since $\partial a_i/\partial s \neq 0$, there must exist j such that $\partial(a_j/a_i)/\partial s \neq 0$. The rest of the proof is almost the same as the proof of the necessity part of Proposition 2 except that we replace eq. (10) with eq. (15) and λ_k with s respectively. Q.E.D.

Now assume all agents' beliefs have the feature in eq. (13). Apparently, the syndicate discussed by Wilson (1973) and A & S (1983), where aggregate wealth completely depends on state variable s and a non-random decision, corresponds to a special case of the economy where the common factor in (13) is equal to one. Thus the economy where agents' beliefs have the feature in eq. (13) can be seen as a generalization of the syndicate in Wilson (1973) and A & S (1983).

Although this more general economy is different from a Wilson syndicate, it can be treated exactly the same when we derive the Pareto optimal sharing rules. This can be directly verified by observing the first order conditions for the utility maximization problems in the two economies. In a Wilson syndicate, the first order condition has the form

$$\lambda_i u_i'(x_i)g_i(s) = \lambda_j u_j'(x_j)g_j(s).$$

In an economy with the feature in eq. (13), the first order condition has the form

$$\lambda_i u_i'(x_i)Q(X,s)g_i(s) = \lambda_j u_j'(x_j)Q(X,s)g_j(s),$$

where $Q(X, s)$ is a common factor across all agents. Since the common factor can be canceled, the two first order conditions are essentially the same.

Because of this all results on Pareto optimal sharing rules hold in a Wilson syndicate will also hold in this more general economy.

For example, we have the following proposition, which corresponds to Theorem 6 in A & S (1983).

Proposition 5 *In a two-agent economy assume agents have heterogeneous beliefs and their beliefs have the feature in (13). The following statements are equivalent: (i) There exists a representative agent for some fixed weighting λ who has state-independent utility and whose belief has the same feature as individual agents. (ii) The Pareto optimal sharing rule is linear in wealth (for the same λ). (iii) Both agents have identical constant relative risk aversion.*

Proof: The proof is almost the same as that of Theorem 6 in A & S (1983). Thus it is omitted.

We also have results similar to Theorems 7 and 8 in A & S (1983) in this economy. For the same reason as given by A & S (1983), the conclusion in Proposition 5 does not hold in an economy where there are more than two agents.

4. EXTENSIONS

In Section 2 we have assumed that agents have infinite marginal utility of zero consumption, i.e., for every i

$$\lim_{x \to 0} u_i'(x) = +\infty.$$

This assumption can be relaxed. We now assumed that for every i there exists a_i such that

$$\lim_{x \to a_i} u_i'(x) = +\infty.$$

Let $A \equiv \sum_{i=1}^{N} a_i$. Assume the range of aggregate wealth is $(A, +\infty)$. Let

$$\hat{\gamma}_i(x) = -(x - a_i)u_i''(x)/u_i'(x)$$

and

$$\hat{\varepsilon}_i(X, s) = -(X - A)\partial p_i(X, s)/\partial X.$$

Following the same argument as in the previous section, we can derive similar results without any changes except that we replace γ_i and ε_i with $\hat{\gamma}_i$ and $\hat{\varepsilon}_i$ respectively for every i.

5. CONCLUSIONS

The results obtained in this paper show that even if all Pareto optimal sharing rules are linear it is not necessary that agents' utility functions be of the equicautious HARA class. When Pareto optimal sharing rules are all linear, we have weak aggregation. That is, equilibrium prices are stable with respect to a perturbation of the weighting. Thus the results imply that unlike the case when aggregate wealth is dependent on the state variable, to obtain weak aggregation it is not necessary that agents' utility functions be of the equicautious HARA class.

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Notes

[1]In the economy discussed by A & S (1983) aggregate wealth is completely dependent on a state variable and a non-random decision.

[2] From (6), we have for any $i \neq k > 1$,

$$\frac{u_i''(x_i)}{u_i'(x_i)} \frac{\partial x_i}{\partial \lambda_k} = \frac{u_1''(x_1)}{u_1'(x_1)} \frac{\partial x_1}{\partial \lambda_k}.$$

Suppose for some $i \neq k > 1$, $\partial x_i / \partial \lambda_k = 0$ then from the above equation we have for every $i \neq k$, $\partial x_i / \partial \lambda_k = 0$. This implies that for every i, $\partial x_i / \partial \lambda_k = 0$. But from (6) we have

$$\frac{1}{\lambda_k} + \frac{u_k''(x_k)}{u_k'(x_k)} \frac{\partial x_k}{\partial \lambda_k} = \frac{u_1''(x_1)}{u_1'(x_1)} \frac{\partial x_1}{\partial \lambda_k}.$$

This is impossible. Suppose for every $i \neq k > 1$, $\partial x_i / \partial \lambda_k \neq 0$ but $\partial x_k / \partial \lambda_k = 0$. Then we have $\sum_{j \neq k} \partial x_i / \partial \lambda_k = 0$. But from (6) we obtain for every $i \neq k$,

$$\frac{1}{\lambda_k} = \frac{u_i''(x_i)}{u_i'(x_i)} \frac{\partial x_i}{\partial \lambda_k}.$$

It follows that

$$\frac{1}{\lambda_k} \sum_{j \neq k} \frac{u_i'(x_i)}{u_i''(x_i)} = \sum_{j \neq k} \frac{\partial x_i}{\partial \lambda_k} = 0.$$

Since for every i, $u_i'(x_i) > 0$ and $u_i''(x) < 0$, the above equality is impossible. Hence for every i, $\partial x_i / \partial \lambda_k \neq 0$.