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Abstract

In this paper we discuss the existence of an optimal portfolio for every investor in a two-period Arrow-Debreu economy in which risky assets are contingent claims on aggregate consumption. Since we derive an optimal portfolio for every investor, the pricing kernel is endogenously determined. Hence the sufficient conditions for the existence of optimal portfolios given in this paper do not involve the pricing kernel; instead they are directly on investors' preferences and beliefs. We also present a new approach to the equilibrium, which works with the space of investors' first-period consumption. The case where investors have background risk is also discussed.

Introduction

The existence of optimal portfolios given the equilibrium of the economy has been thoroughly discussed (see for example, He and Pearson (1989), Cox and Huang (1990), and Back and Dybvig (1993)). Since the equilibrium is exogenously given, the main concern is under what conditions an investor's expected utility of consumption is bounded given the pricing kernel. For example, Back and Dybvig (1993) have derived sufficient conditions for the boundedness of investors' expected utilities of their optimal portfolios which involve the given pricing kernel. In this paper we discuss the existence of an optimal portfolio for every investor in an Arrow-Debreu economy where risky assets are contingent claims on aggregate consumption. Note that there is a subtle difference between our problem and Back and Dybvig's problem. In our problem the equilibrium of the economy is still to be derived, which implies that the pricing kernel is to be derived. Because of this, it would be inappropriate for us to give conditions

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involving the to-be-derived pricing kernel; rather we have to derive the pricing kernel first.

In this paper we show how to derive the pricing kernel in the two-period Arrow-Debreu economy. We give conditions directly on investors' preferences and beliefs to guarantee the boundedness of the equilibrium present value of aggregate consumption and investors' expected utilities of their optimal consumption so that every investor's optimal portfolio is well defined. We also present a new approach to the equilibrium. Unlike the demand approach and Negishi Method, which work with the space of assets and space of utility weights respectively, our approach works with the space of investors' first-period consumption. The case where investors have background risk is also discussed in this paper.

This paper is closely related to He and Pearson (1989) and Cox and Huang (1990) who also derive conditions for the existence of optimal portfolios given the equilibrium. It is also related to those on existence of equilibrium such as Karatzas, Lehoczky and Shreve (1990) and Mas-Colell and Zame (1991).

The structure of this paper is as follows. In Section one, we introduce the formulation of the problem. In Section two we show how to derive the pricing kernel and give a new approach to the equilibrium. In Section three we discuss the existence of an optimal portfolio for every investor when investors have heterogeneous beliefs and preferences. In Section four we discuss the case where investors also have background risk. The final section concludes the paper.

1 A Two-Period Economy

In this section we introduce a two-period Arrow-Debreu economy, in which there are N investors indexed by $i = 1, 2, ..., N$. Let X_0 and X be the aggregate consumption in the first and second period respectively. We denote $u_{i0}(x)$ and $u_i(x)$ as investor i's first period and second period utility functions respectively. In this paper we assume that all utility functions have positive first derivatives and negative continuous second derivatives, i.e., investors are non-satiate and risk averse. Let $f(X)$ be the objective probability density function and $f_i(X)$ investor i's subjective probability density function respectively. We assume that all these probability density functions are positive and differentiable in $X \in (0, +\infty)$ almost surely. We assume that there exists a unique pricing kernel, ϕ, with unspecified form, which will be determined in equilibrium of the economy.

Let w_{i0} be investor i's initial endowment, expressed as the fraction of the spot value of the total wealth in the economy. Let x_{i0} be investor i's first period consumption and x_i his/her second period consumption respectively. We assume that there is a complete market for state-contingent claims on X. Every investor can buy and sell state-contingent claims on X to form a desired portfolio whose second-period payoff is a function of X, i.e., $x_i = x_i(X)$, $i = 1, 2, ..., N$. Then the investor has the following utility maximization problem:

$$\max_{x_{i0},x_i} u_{i0}(x_{i0}) + E_i[u_i(x_i)]. \tag{1}$$

Subject to

$$x_{i0} + E(\phi x_i) = w_{i0}(X_0 + E(X\phi)). \tag{2}$$

where $E_i(.)$ denotes the expectation operator under the subjective probability measure with p.d.f. $f_i(X)$, X_0 is the aggregate consumption in the first period, and $E(.)$ denotes the expectation operator under the true probability measure with p.d.f. $f(X)$. In equilibrium, the market is cleared, thus we have

$$\sum_i x_{i0}(X_0) = X_0 \quad \text{and} \quad \sum_i x_i(X) = X. \tag{3}$$

Since negative consumption is not allowed we have

$$x_{i0} \geq 0 \text{ and } x_i \geq 0. \tag{4}$$

We assume that all utility functions have infinite marginal utility of zero consumption. This implies that there is no corner solutions and the first order condition is an equality as follows

$$u_i'(x_i) = \lambda_i g_i(X)\phi, \tag{5}$$

where $g_i(X) = f(X)/f_i(X)$ and $\lambda_i = 1/u_{i0}'(x_{i0})$.

As is well known, given the pricing kernel ϕ, if investor i is non-satiate and strictly risk averse, i.e., $u_i(x)$ is strictly increasing and concave and the marginal utility of zero consumption is infinity, then a solution to the first order condition (5) subject to (2) will be the optimal solution the utility maximization problem (1).

If given ϕ, we can solve (5) and obtain $x_i(X)$, then the first-period consumption can be obtained by solving

$$u_{i0}'(x_{i0}) = E_i(u_i'(x_i)).$$

But unfortunately, the pricing kernel cannot be given exogenously. Since the first order conditions for investors utility maximization problems are all equalities, the pricing kernel is determined by these equalities. In the next section we show how the pricing kernel is determined.

2 Deriving the Pricing Kernel

In this section we show how to derive the pricing kernel. Now for every investor i, we take his/her first period consumption x_{i0} as given. Write $x_0 = (x_{10}, x_{20}, ..., x_{N0})$. Since $u_i'(x)$ is monotonic in x, from (5), we obtain

$$x_i = u_i'^{-1}(\lambda_i g_i(X)\phi), \tag{6}$$

where $\lambda_i = u'_{i0}(x_{i0})$. Since $\sum_i x_i = X$, we obtain

$$\sum_i u_i'^{-1}(\lambda_i g_i(X)\phi) = X. \tag{7}$$

Assume for every i, $u'_i(0) = +\infty$ and $u'_i(\infty) = 0$. Then we have

$$\{u'_i(x) | x \in (0, +\infty)\} = (0, \infty). \tag{8}$$

From (8), it is clear that given any $X > 0$ there is a solution of ϕ to Equation (7). Since for every i, $u'_i(x)$ is monotonic, this solution is unique. We write the solution as

$$\phi = \phi(X; x_0). \tag{9}$$

Substituting ϕ into (6), we obtain

$$x_i = x_i(X; x_0). \tag{10}$$

Now it remains to be shown that there exists x_0 such that the obtained $\phi(X; x_0)$ and $x_i(X; x_0)$ satisfy the wealth constraint (2) for every i.

Given $u_{i0}(x)$ and $u_i(x)$, investor i's first period and second period utility function, let $R_i(x) = -u''_i(x)/u'_i(x)$ and $\gamma_i(x) = xR_i(x)$ which are investor i's absolute risk aversion and relative risk aversion in the second period respectively. We define two subsets of the N-dimension real space:

$$C = \{y = (y_1, y_2, ..., y_N) : \ \sum_i^N y_i = X_0 \ and, \ \forall i, \ y_i \geq 0\} \tag{11}$$

and

$$C^+ = \{y = (y_1, y_2, ..., y_N) : \ y \in C \ and, \ \forall i, \ y_i > 0\}. \tag{12}$$

We have the following lemma.

Proposition 1 (i) *Assume for every i, $u_{i0}(x)$ is strictly increasing and differentiable and* $\lim_{x\to 0} u'_{i0}(x) = +\infty$. *(ii) Assume for every i, $u_i(x)$ is strictly increasing and differentiable,* $\lim_{x\to 0} u'_i(x) = +\infty$ *and* $\lim_{x\to\infty} u'_i(x) = 0$. (iii) *Assume that for any $x_0 \in C$, we have $E(X\phi(X; x_0)) < +\infty$.* (iv) *Assume that for any $x_0 \in C^+$ and every i, $-\infty < E_i(u_i(x_i(X; x_0))) < +\infty$. Then there exists a pricing kernel under which every investor has an optimal portfolio, i.e., there exists equilibrium. Furthermore, if for every i, $u_i(x)$ is $n+1$ $(n > 1)$ times differentiable and $g_i(X)$ is n times differentiable, then $x_i(X)$ and $\phi(X)$ are n times differentiable in X.*

Conditions (i) and (ii) are to guarantee that we have interior solutions. Condition (ii) is to guarantee that given $x_0 \in C$, for any $X > 0$, there is a solution of $\phi = (X; x_0)$ to Equation (7), i.e., $\phi(X; x_0)$ is well defined. Condition (iii) is to guarantee that $E(X\phi)$ in (2) is well defined, so is $E(x_i\phi)$, for every i. And Condition (iv) is to guarantee that given $x_0 \in C^+$, for every i, $E(u_i(x_i))$ in (1) is well defined.

Proof: Existence: Since $\lim_{x\to 0} u_i'(x) = +\infty$ and $\lim_{x\to\infty} u_i'(x) = 0$, following the argument proceeding this lemma, for given x_0, we obtain (9) and (10). It is clear that if for every i, $u_i'(x)$ and $g_i(X)$ are n times differentiable, then $x_i(X; x_0)$ and $\phi(X; x_0)$ are n times differentiable in X. So far we have not used conditions (iii) and (iv).

Substitute ϕ and x_i into (2). Now we only need to show there is a solution of x_0. Consider the set C defined in (11), which is obviously closed and convex. Since it is bounded, it is in fact compact. We define $h : C \to R^N$ by the relation

$$(h(y))_i = w_{i0}(X_0 + E(X\phi(X; y)) - E(x_i(X; y)\phi(X; y)),$$

which is obviously continuous. Although C is not necessarily an invariant set, we can show as follows that for each $y_0 \in C$ there exists a real number t such that $|t| < 1$ and $ty_0 + (1-t)h(y_0) \in C$.

Suppose $y_{i0} = 0$. Since $\lim_{x\to 0} u_i'(x) = +\infty$ and $\lim_{x\to 0} u_{i0}'(x) = +\infty$, from (5), we have $x_{i0}(X; y_0) = 0$, which implies $(h(y_0))_i > 0$. Thus there exists $0 < t < 1$ such that $ty_{i0} + (1-t)(h(y_0))_i > 0$. Suppose $y_{i0} > 0$. Since $E(x_i(X; y)\phi(X; y)) < +\infty$ and $E(X\phi(X; y_0)) < +\infty$, we have $-\infty < (h(y_0))_i < +\infty$. Thus there obviously exists small enough $t > 0$ such that $ty_{i0} + (1-t)(h(y_0))_i > 0$. Noting that $\sum_i^N (h(y))_i = X_0$, we obtain the conclusion immediately.

Now we can apply a generalization of Schauder's fixed point theorem and conclude that there exists a fixed point $x_0^* \in C$.[1] It is clear that it must hold that $x_0^* \in C^+$. Otherwise, if for some i, $x_{i0} = 0$, then $x_i(X; x_0) = 0$. Thus for investor i, (2) cannot hold. Hence there exists $x_0^* \in C$ such that $x_i(X; x_0^*)$ and $\phi(X; x_0^*)$ satisfy (2) and (3). From condition (iv), we have for every i, $-\infty < E_i(u_i(x_i(X; x_0^*))) < +\infty$. Thus we conclude that there exists a pricing kernel $\phi = \phi(X; x_0^*)$ under which there is a solution to problem (1) subject to (2), (3) and (4) for every investor i, i.e., every investor has an optimal portfolio and there exists equilibrium. Q.E.D.

3 Existence of an Optimal Portfolio for Every Investor

Assume that for every i, $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative. Assume for every i, g_i has continuous first derivative. From Proposition 1, for any given $x_0 = (x_{10}, x_{20}, ..., x_{N0})$, $x_i = x_i(X; x_0)$

[1] A generalization of Schauder's fixed point theorem states as follows: Let H be a locally convex topological space and let C be a compact convex nonempty subset of H. Suppose that $h : C \to H$ is a continuous mapping satisfying property:

for each $y_0 \in C$ there exists a (real or complex) number z such that $|z| < 1$ and $zy_0 + (1 - z)h(y_0) \in C$.

Then h has a fixed point in C. See Theorem 5.4.16 in V. I. Istratescu (1981).

and $\phi = \phi(X; x_0)$, which is given by (9), have continuous first derivatives w.r.t X. Before we proceed, we remind the readers that for brevity we sometimes suppress the notation x_0 in relevant functions whenever it does not lead to confusion. For example we write $\phi(X; x_0)$ as $\phi(X)$ and $x_i(X; x_0)$ as $x_i(X)$.

Differentiating both sides of (5), we have

$$x_i'(X) = R_i^{-1}(x_i)(R_e - \frac{g_i'}{g_i}), \tag{13}$$

where $R_e(X) = -\phi'(X)/\phi(X)$ is the representative investor's absolute risk aversion.[2]

Applying (3) we obtain

$$XR_e(X) = \sum_i s_i(x_i R_i(x_i) + X\frac{g_i'}{g_i}), \tag{14}$$

where $s_i = R_i^{-1}(x_i)/(\sum_i R_i^{-1}(x_i))$.

Equation (14) can be written as

$$\gamma_e(X) = \sum_i s_i(\gamma_i(x_i) - \epsilon_i), \tag{15}$$

where $\epsilon_i = -Xg_i(X)'/g_i(X)$ and $\gamma_e(X) = XR_e(X)$ is the representative investor's relative risk aversion.

In the following context, we will see that it is helpful to understand the characteristics of the pricing kernel in the extremely bad states and extremely good states. When X approaches zero, x_i approaches zero. From (14), this implies that the value of $\gamma_e(X)$ near $X = 0$ is determined by the values of $\gamma_i(x)$ and $\epsilon_i(x)$ near $x = 0$.

Now consider the situation when X approaches infinity. Assume for every i, $\lim_{x\to\infty} R_i(x) = 0$. When X approaches infinity, if x_i is bounded, then $s_i = R_i^{-1}(x_i)/(\sum_i R_i^{-1}(x_i))$ will approach zero. From (14), this implies that the value of $\gamma_e(X)$ when X approaches infinity is determined by the values of $\gamma_i(x)$ and $\epsilon_i(x)$ when x approaches infinity.

Let $\underline{\gamma_i}(0) = \liminf_{x\to 0} \gamma_i(x)$, $\overline{\gamma_i}(0) = \limsup_{x\to 0} \gamma_i(x)$, $\underline{\gamma_i}(\infty) = \liminf_{x\to\infty} \gamma_i(x)$, $\overline{\delta_i}(0) = \overline{\gamma_i}(0) - \underline{\epsilon_i}(0)$, $\overline{\delta}(0) = \max_i\{\overline{\delta_i}(0)\}$, $\underline{\delta_i}(\infty) = \underline{\gamma_i}(\infty) - \overline{\epsilon_i}(\infty)$ and $\underline{\delta}(\infty) = \min_i\{\underline{\delta_i}(\infty)\}$. We have the following lemma.

[2]Since investors have heterogeneous beliefs and preferences, we usually do not have an aggregation investor in Rubinstein's sense. It is shown by Rubinstein (1974) and Brennan and Kraus (1978) that there exists an aggregate investor in Rubinstein's sense if and only if either all investors have identical cautiousness and beliefs or all investors have exponential utility functions. But we still have a representative investor who was called a "pricing representative" investor by Benninga and Mayshar (1997). He/she was so called because if the economy had only one investor, namely the "pricing representative" investor with the total endowment of the economy, then the equilibrium state prices in the economy would remain unchanged. We still call him/her the representative investor.

Lemma 1 (i) *Assume for every i, $u_{i0}(x)$ is strictly increasing and differentiable and $\lim_{x\to 0} u'_{i0}(x) = +\infty$.* (ii) *Assume for every i, $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative, $\underline{\gamma_i}(0) > 0$, $\overline{\gamma_i}(0) < +\infty$, and $\underline{\gamma_i}(\infty) > 0$.* (iii) *Assume for every i, $f_i(X)$ and $f(X)$ are positive and differentiable in $X \in (0, +\infty)$ almost surely and there exist $\nu > 0$, $M > 0$ and $\varepsilon_0 > 0$ such that $E_{(X<\nu)}(X^{1-(\overline{\delta}(0)+\varepsilon_0)}) < +\infty$ and $E_{(X>M)}(X^{1-(\underline{\delta}(\infty)-\varepsilon_0)}) < +\infty$. Then given any $x_0 \in C$, $E(X\phi(X;x_0)) < \infty$ and given any $x_0 \in C^+$ for every i, $-\infty < E_i(u_i(x_i(X;x_0))) < +\infty$.*

Proof: (A) First we show that $E(X\phi(X)) < +\infty$.

When $X \to 0$, we have $x_i \to 0$, for every i. From Equation (15) given arbitrarily small $\varepsilon > 0$, we have for sufficiently small X

$$\limsup_{X\to 0} \gamma_e(X) < \overline{\delta}(0) + \varepsilon, \tag{16}$$

which implies

$$\phi(X) < A_0 X^{-(\overline{\delta}(0)+\varepsilon)}, \tag{17}$$

where A_{01} is a positive constant.

From the above equation and Condition (iii), we conclude that for sufficiently small ν

$$E_{(X<\nu)}(X\phi) < +\infty \tag{18}$$

On the other hand from (15) we conclude that given small $\varepsilon > 0$, for sufficiently large X

$$\gamma_e(X) > \underline{\delta}(\infty) - \varepsilon,$$

where $\underline{\delta}(\infty) = \min_i\{\underline{\delta_i}(\infty)\}$. This implies that for sufficiently large X

$$\phi(X) < A_0 X^{-(\underline{\delta}(\infty)-\varepsilon)} \tag{19}$$

From the above equation and Condition (iii), we conclude that for sufficiently large M

$$E_{(X>M)}(X\phi) < +\infty \tag{20}$$

Since $\phi(X)$ is continuous in X, from (18) and (20) we obtain $E(X\phi(X)) < +\infty$.

(B) Secondly we show that $E_i(u_i(x_i))$ is bounded above.

(Ba) Suppose $\underline{\gamma_i}(\infty) > 1$.

We can show that $u_i(x)$ is naturally bounded above. Since $\underline{\gamma_i}(\infty) > 1$, for sufficiently large x, we have

$$\gamma_i(x) > 1 + \varepsilon,$$

where $0 < \varepsilon < \gamma_i(\infty) - 1$. It follows that

$$u'_i(x) < Ax^{-(1+\varepsilon)}$$

where A is a positive constant. Hence for sufficiently large x° and $x > x^\circ$,

$$\begin{aligned}
u_i(x) &= u_i(x^\circ) + \int_{x^\circ}^{x} u_i'(x)dx \\
&\leq u_i(x^\circ) + \int_{x^\circ}^{x} Ax^{-(1+\varepsilon)}dx \\
&= u_i(x^\circ) - \frac{A}{\varepsilon}(x^{-\varepsilon} - x^{\circ-\varepsilon}) \\
&< u_i(x^\circ) + \frac{A}{\varepsilon}x^{\circ\varepsilon}
\end{aligned}$$

(Bb) Suppose $\underline{\gamma_i}(\infty) \leq 1$.
Given small $\varepsilon > 0$, for sufficiently large x, we have

$$\gamma_i(x) > \underline{\gamma_i}(\infty) - \varepsilon,$$

Using a similar argument as in (Ba), we obtain that given small $\varepsilon > 0$, for sufficiently large x,

$$u_i(x) < A_1 x^{1-(\underline{\gamma_i}(\infty)-\varepsilon)}, \tag{21}$$

where A_1 is a positive constant.

Since $\lim_{X\to\infty} X\frac{g_i'(X)}{g_i(X)} = \epsilon(\infty)$, given small $\varepsilon > 0$ for sufficiently large X we have

$$A_2 X^{-(\overline{\epsilon_i}(\infty)+\varepsilon)} < g_i(X) < A_2 X^{-(\underline{\epsilon_i}(\infty)-\varepsilon)}. \tag{22}$$

where A_2 is a positive constant.

Equations (21) and (22) imply that given small $\varepsilon > 0$ for sufficiently large X

$$\frac{u_i(x_i(X))}{g_i(X)} < \frac{u_i(X)}{g_i(X)} < AX^{1-(\underline{\delta}(\infty)-\varepsilon)}, \tag{23}$$

where A is a positive constant.

The above equation and Condition imply that there exists sufficiently large $M > 0$ such that

$$E_{i(X>M)}(u_i(x_i)) < +\infty. \tag{24}$$

Since x_i approaches zero when X approaches zero, $u_i(x_i)$ will be bounded above when X approaches zero. Thus there exists $\nu > 0$ such that

$$E_{i(X<\nu)}(u_i(x_i)) < +\infty. \tag{25}$$

Since $u_i(x_i)$ is continuous in X, from (24) and (25) we conclude that $E_i(u_i(x_i))$ is bounded above.

(C) Thirdly we show that for any $x_0 \in C^+$, $E_i(u_i(x_i(X; x_0)))$ is bounded below. This is proved in Appendix A. Q.E.D.

Apparently if $\overline{\delta}(0) < 1$, then it naturally holds that for sufficiently small $\varepsilon_0 > 0$, $E_{(X<\nu)}(X^{1-(\overline{\delta}(0)+\varepsilon_0)}) < +\infty$. A special example is for every i, $\underline{\epsilon_i}(0) = +\infty$ while $\overline{\gamma_i}(0) < +\infty$. In this case, $\overline{\delta}(0) = -\infty < 1$. We can also see that if $\underline{\delta}(\infty) > 1$, then it naturally holds that for sufficiently small $\varepsilon_0 > 0$, $E_{(X>M)}(X^{1-(\underline{\delta}(\infty)-\varepsilon_0)}) < +\infty$. A special example is for every i, $\overline{\epsilon_i}(\infty) = -\infty$ while $\underline{\gamma_i}(\infty) > -\infty$. In this cases $\underline{\delta}(\infty) = +\infty > 1$.

The conditions that $\underline{\gamma_i}(0) > 0$ and $\underline{\gamma_i}(\infty) > 0$ imply that $\lim_{x\to 0} u_i'(x) = +\infty$ and $\lim_{x\to\infty} u_i'(x) = 0$ respectively. The condition that $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative together with the conditions that $\underline{\gamma_i}(0) > 0$ and $\underline{\gamma_i}(\infty) > 0$ guarantees that $\underline{\gamma_i} = \inf_x\{\gamma_i(x)\} > 0$, a result used when we prove that for any $x_0 \in C^+$, $E_i(u_i(x_i(X;x_0)))$ is bounded below in Appendix A.

Now we are ready to present an important result.

Proposition 2 (i) *Assume for every i, $u_{i0}(x)$ is strictly increasing and differentiable and $\lim_{x\to 0} u_{i0}'(x) = +\infty$.* (ii) *Assume for every i, $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative, $\underline{\gamma_i}(0) > 0$, $\overline{\gamma_i}(0) < +\infty$, and $\underline{\gamma_i}(\infty) > 0$.* (iii) *Assume for every i, $f_i(X)$ and $f(X)$ are positive and differentiable in $X \in (0,+\infty)$ almost surely and there exist $\nu > 0$, $M > 0$ and $\varepsilon_0 > 0$ such that $E_{(X<\nu)}(X^{1-(\overline{\delta}(0)+\varepsilon_0)}) < +\infty$ and $E_{(X>M)}(X^{1-(\underline{\delta}(\infty)-\varepsilon_0)}) < +\infty$. Then there exists a pricing kernel under which every investor has an optimal portfolio, i.e., there is equilibrium. Furthermore, if for every i, $u_i(x)$ is $n+1$ $(n>1)$ times differentiable and $g_i(X)$ is n times differentiable, then $x_i(X)$ and $\phi(X)$ are n times differentiable in X.*

Proof: Since all the conditions in Lemma 1 are satisfied, applying this lemma, we conclude that given any $x_0 = (x_{10}, x_{20}, ..., x_{N0}) \in C$, it holds that $E(\phi(X;x_0)) < +\infty$ and given any $x_0 \in C^+$, for every i, $E(x_i(X;x_0)\phi(X;x_0)) < +\infty$ and $-\infty < E_i(u_i(x_i(X;x_0))) < +\infty$. These results together with $\lim_{x\to 0} u_{i0}'(x) = +\infty$, and equations $\lim_{x\to 0} u_i'(x) = +\infty$ and $\lim_{x\to\infty} u_i'(x) = 0$, which are implied by $\underline{\gamma_i}(0) > 0$ and $\underline{\gamma_i}(\infty) > 0$, meet the conditions in Proposition 1. Applying this lemma, we conclude that there exists a pricing kernel under which there is a solution to problem (1) subject to (2), (3) and (4) for every investor i, i.e., there is an optimal portfolio for every investor. Furthermore, if for every i, $u_i(x)$ is $n+1$ $(n > 1)$ times differentiable and $g_i(X)$ is n times differentiable, then $x_i(X)$ and $\phi(X)$ are n times differentiable in X. Q.E.D.

We give the following remarks on the above result.

Remark 1, we stress that it is necessary to show $E_i(u_i(x_i))$ is bounded both from above and from below. Back and Dybvig (1993) ignored the fact that $E_i(u_i(x_i))$ can be minus infinity when they showed the existence of an optimal solution to the maximization problem for an individual investor with an exogenously given pricing kernel. But it is apparent that if for all $x_0 \in C^+$, $E_i(u_i(x_i(X;x_0))) = -\infty$, there will be no optimal solution to the maximization problem.

Remark 2, the condition that for every i, $\lim_{x\to 0} u_i'(x) = +\infty$, which is implied by the assumption that for every i, $+\infty > \gamma_i(0) > 0$, is very important. Now assume investors have homogeneous beliefs, i.e., for every i, $g_i = 1$. We show that it is necessary for the first order condition to be an equality if there exists equilibrium with interior solutions except that the investors are homogeneous. Suppose for some i, $\lim_{x\to 0} u_i'(x) < +\infty$ and for some j, $\lim_{x\to 0} u_j'(x) = +\infty$. When $X \to 0$, from (5) we can see that it cannot hold that $x_i \to 0$ and $x_j \to 0$ simultaneously. This implies that equilibrium with interior solutions cannot exist. Assume for every i, $\lim_{x\to 0} u_i'(x) < +\infty$. Suppose the first order conditions are equalities. Let $x_i \to 0$ for every i. From (5), we obtain $u_{i0}'(x_{i0}) = \phi(0)u_i'(0)$, for $i = 1, 2, ..., N$. Since $\sum_{i=1}^N x_{i0} = X_0$, we solve the equations and obtain x_0. Apparently the obtained x_{i0} has nothing to do with w_{i0}. Assume there is equilibrium. Given x_0, the pricing kernel can be obtained from (7) which has nothing to do with w_{i0}. Now x_{i0} and ϕ are both independent of w_{i0}, which is inconsistent with (2).

Remark 3, the proposition can be generalized to the case where investor i's utility function has a positive threshold parameter,[3] $a_i > 0$, i.e.,

$$\lim_{x\to a_i} u_i'(x) = +\infty.$$

In this case, the threshold parameter of the distribution of X will be $\sum_i a_i$, i.e., we must have $X > \sum_i a_i$. Then we have to change the definition of relative risk aversion and ϵ to $\hat{\gamma}_i(x) = -(x - a_i)\frac{u_i''(x)}{u_i'(x)}$ and $\hat{\epsilon}(X) = -(X - \sum_i a_i)\frac{g_i'(X)}{g_i(X)}$ respectively. The proposition will hold in this case if we use $\hat{\gamma}_i$ and $\hat{\epsilon}$ to replace γ_i and ϵ respectively.

Let $\overline{\gamma}(0) = \max_i\{\overline{\gamma_i}(0)\}$. In a special case when investors have homogeneous beliefs, we have the following result implied by the above proposition:

Corollary 1 (i) *Assume for every i, $u_{i0}(x)$ is strictly increasing and differentiable and* $\lim_{x\to 0} u_{i0}'(x) = +\infty$*.* (ii) *Assume for every i, $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative, $\underline{\gamma_i}(0) > 0$, $\overline{\gamma_i}(0) < +\infty$, and $\underline{\gamma_i}(\infty) > 0$.* (iii) *Assume that investors have homogeneous beliefs.* (iv) *Assume there exists $\nu > 0$, $M > 0$ and $\varepsilon_0 > 0$ such that $E_{(X<\nu)}(X^{1-(\overline{\gamma}(0)+\varepsilon_0)}) < +\infty$ and $E_{(X>M)}(X^{1-(\underline{\gamma}(\infty)-\varepsilon_0)}) < +\infty$. Then there exists a pricing kernel under which every investor has an optimal portfolio, i.e., there is equilibrium. Furthermore, if for every i, $u_i(x)$ is $n+1$ $(n > 1)$ times differentiable, then $x_i(X)$ and $\phi(X)$ are n times differentiable in X.*

Proof: Since investors have homogeneous beliefs, for every i, $\epsilon_i(X) = 0$. Thus the conclusion immediately follows Proposition 2.

4 The Case with Background Risk

In this section we discuss conditions for the existence of an optimal portfolio for every investor in an exchange economy when investors are exposed to back-

[3] See footnote 4.

ground risk. Assume investor i has a non-insurable background risk z_i. Assume the values of z_i are in domain $(\underline{z_i}, +\infty)$ or $[\underline{z_i}, +\infty)$, where $\underline{z_i} < 0$. Now assume that the investor's utility function in the second period is $u_i(x_i + z_i - \underline{z_i})$, where we have added a threshold parameter $-\underline{z_i}$ to the utility function. This threshold parameter will be explained in the end of this section. Let $e_i = z_i - \underline{z_i}$. With other things unchanged as in problem (1), the ith investor's utility maximization problem becomes

$$\max_{x_{i0}, x_i} u_{i0}(x_{i0}) + E_i[\tilde{u}_i(x_i)], \tag{26}$$

where $\tilde{u}_i(x) = E_{e_i}(u_i(x_i + e_i))$ is investor i's indirect utility function.

Apparently the only difference between this maximization problem and problem (1) is that the direct utility function $u_i(x_i)$ in the later is replaced by the indirect utility function $\tilde{u}_i(x_i)$ in the former. Since we assume that for every i, $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative, it is apparent that for every i, $\tilde{u}_i(x)$ is also strictly increasing and strictly concave and has continuous second derivative. Before we proceed, we first clarify notation. To reduce unnecessary complexity, we will use the notation already introduced and just add an $\sim$ to the heads of all notation related with the indirect utility function $\tilde{u}_i(x_i)$. According to Proposition 2, to check the existence of equilibrium, we need first to check if it holds that for every i, $\underline{\tilde{\gamma}_i}(0) > 0$, $\overline{\tilde{\gamma}_i}(0) < +\infty$ and $\underline{\tilde{\gamma}_i}(\infty) > 0$. Secondly we need to check if there exists $\nu > 0$, $M > 0$ and $\varepsilon_0 > 0$ such that $E_{(X<\nu)}(X^{1-(\overline{\tilde{\delta}}(0)+\varepsilon_0)}) < +\infty$ and $E_{((X>M))}(X^{1-(\underline{\tilde{\delta}}(\infty)-\varepsilon_0)}) < +\infty$. Apparently this is not an easy task; However, we have the following result.

Proposition 3 (i) *Assume for every i, $u_{i0}(x)$ is strictly increasing and differentiable and $\lim_{x\to 0} u'_{i0}(x) = +\infty$.* (ii) *Assume for every i, $u_i(x)$ is strictly increasing and strictly concave and has continuous second derivative, $\underline{\gamma_i}(\infty) > 0$, $\lim_{x\to 0} u'_i(x) = +\infty$, and there exists $\omega_i > 1$ such that $0 < \lim_{x\to 0} \frac{u''_i(x)}{x^{-\omega_i}} < +\infty$.* (iii) *Assume for every i, $f_i(X)$ and $f(X)$ are positive and differentiable in $X \in (0, +\infty)$ almost surely and there exists $\nu > 0$, $M > 0$ and $\varepsilon_0 > 0$ such that $E_{(X<\nu)}(X^{1-(\overline{\theta}+\varepsilon_0)}) < +\infty$ and $E_{(X>M)}(X^{1-(\underline{\delta}(\infty)-\varepsilon_0)}) < +\infty$, where $\overline{\theta} = \max_i\{\omega_i - 1 - \underline{\epsilon_i}(0)\}$.* (iv) *Assume that there exist $\nu > 0$ and $M > 0$ such that for $0 < x < \nu$, $E_{(e_i>M)}(-u''_i(x + e_i)) < A_0$, where A_0 is a positive constant.* (v) *Assume that either of the following conditions is satisfied:* (v(a)) *The background risk is discretely distributed at $e_i = 0$ with probability $P_i = Prob(e_i = 0) > 0$;* (v(b))) *The background risk is continuously distributed at $e_i = 0$ with probability density function $h_i(x)$ and there exists $\beta_i \in (-1, \omega_i - 2)$ such that $\liminf_{x\to 0} \frac{h_i(x)}{x^{\beta_i}} > 0$ and $\limsup_{x\to 0} \frac{h_i(x)}{x^{\beta_i}} < +\infty$. Then there exists a pricing kernel under which every investor has an optimal portfolio, i.e., there is equilibrium. Furthermore, if for every i, $u_i(x)$ is $n+1$ ($n > 1$) times differentiable and $g_i(X)$ is n times differentiable, then $x_i(X)$ and $\phi(X)$ are n times differentiable in X.*

Proof:

(A) We first show that

$$\underline{\tilde{\gamma}_i}(\infty) \geq \underline{\gamma_i}(\infty) \tag{27}$$

We have

$$\begin{aligned} \tilde{\gamma}_i(x) &= -x\frac{E_{e_i}(u_i''(x+e_i))}{E_{e_i}(u_i'(x+e_i))} \\ &= \frac{E_{e_i}(\frac{x}{x+e_i}\gamma_i(x+e_i)u_i'(x+e_i))}{E_{e_i}(u_i'(x+e_i))} \end{aligned} \tag{28}$$

Given arbitrarily small $\varepsilon > 0$, for sufficiently large $M > 0$ we have

$$\begin{aligned} \tilde{\gamma}_i(x) &\geq (\underline{\gamma_i}(\infty) - \varepsilon)\frac{E_{e_i > M}(u_i'(x+e_i))}{E_{e_i}(u_i'(x+e_i))} \\ &\geq \underline{\gamma_i}(\infty) - \varepsilon. \end{aligned}$$

Let $x \to +\infty$. We obtain

$$\underline{\tilde{\gamma}_i}(\infty) \geq \underline{\gamma_i}(\infty) - \varepsilon.$$

Let $\varepsilon \to 0$. We obtain (27).

(B) Now we show that $\lim_{x\to 0}\gamma_i(x)$ exists and

$$\gamma_i(0) = \omega_i - 1 \tag{29}$$

Since $\lim_{x\to 0} u_i'(x) = +\infty$ and there exists $\omega_i > 1$ such that $0 < \lim_{x\to 0}\frac{u_i''(x)}{x^{-\omega_i}} < +\infty$, we conclude that $\lim_{x\to 0}\frac{u_i'(x)}{x^{1-\omega_i}}$ exists and

$$\lim_{x\to 0}\frac{u_i'(x)}{x^{1-\omega_i}} = \frac{1}{1-\omega_i}\lim_{x\to 0}\frac{u_i''(x)}{x^{-\omega_i}}.$$

From this we immediately obtain that

$$\lim_{x\to 0} -x\frac{u_i''(x)}{u_i'(x)} = \omega - 1.$$

(C) We show that

$$\overline{\tilde{\gamma}_i}(0) < \gamma_i(0). \tag{30}$$

From (28), given arbitrarily small $\varepsilon > 0$, for sufficiently small $\nu > 0$ when $x < \nu$

$$\begin{aligned} \tilde{\gamma}_i(x) &< \frac{E_{(e_i<\nu-x)}(\gamma_i(x+e_i)u_i'(x+e_i)) - xE_{(e_i\geq\nu-x)}(u_i''(x+e_i))}{E_{e_i}(u_i'(x+e_i))} \\ &< \frac{(\gamma_i(0)+\varepsilon)E_{(e_i<\nu-x)}(u_i'(x+e_i)) - xE_{(e_i\geq\nu-x)}(u_i''(x+e_i))}{E_{e_i}(u_i'(x+e_i))}. \end{aligned}$$

According to Condition iv), there exists $M > 0$ such that $-E_{(e_i \geq \nu - x)}(u_i''(x + e_i)) < A_0$, where A_0 is a positive constant. Now let $x \to 0$. We obtain

$$\overline{\tilde{\gamma}_i}(0) \leq \gamma_i(0) + \varepsilon.$$

Let $\varepsilon \to 0$. We immediately obtain (30).

(D) We show that if the background risk is discretely distributed at $e_i = 0$ with probability $P_i = Prob(e_i = 0) > 0$ then

$$\underline{\tilde{\gamma}_i}(0) \geq P_i \gamma_i(0). \tag{31}$$

We have

$$\begin{aligned} \tilde{\gamma}_i(x) &= -x \frac{E_{e_i}(u_i''(x + e_i))}{E_{e_i}(u_i'(x + e_i))} \\ &\geq -Prob(e_i = 0) x \frac{u_i''(x)}{u_i'(x)} \\ &= P_i \gamma_i(x). \end{aligned}$$

This implies (31).

(E) We show that if investor i's background risk is continuously distributed and satisfies condition (v(b)), then

$$\underline{\tilde{\gamma}_i}(0) \geq A_i \gamma_i(0), \tag{32}$$

where $A_i = \frac{\underline{b_i}\Gamma(\beta_i, -\omega_i)}{\overline{b_i}\Gamma(\beta_i, -(\omega_i - 1))}$, $\underline{b_i} = \liminf_{x\to 0} \frac{h_i(x)}{x^{\beta_i}}$, $\overline{b_i} = \limsup_{x\to 0} \frac{h_i(x)}{x^{\beta_i}}$ and $\Gamma(p, q) = \int_0^{+\infty} y^p (1 + y)^q dy$, for $-1 < p < -q - 1$. This is proved in Appendix B.

(F) From (27) in (A) and condition (ii), we obtain for every i, $\underline{\tilde{\gamma}_i}(\infty) > 0$. From (29) in (B) and (30) in (C), we obtain that for every i, $\overline{\tilde{\gamma}_i}(0) < +\infty$. From (31) in (D) and (32) in (E) we obtain that for every i, $\underline{\tilde{\gamma}_i}(0) > 0$. From (27) in (A) we obtain for every i

$$\underline{\tilde{\delta}_i}(\infty) \geq \underline{\delta_i}(\infty),$$

which implies

$$\underline{\tilde{\delta}}(\infty) \geq \underline{\delta}(\infty). \tag{33}$$

From (29) in (B) and (30) in (C), we obtain for every i

$$\overline{\tilde{\delta}_i}(0) \leq \omega_i - 1 - \underline{\epsilon_i}(0), \tag{34}$$

which implies

$$\overline{\tilde{\delta}}(0) \leq \overline{\theta},$$

where $\overline{\theta} = \max_i\{\omega_i - 1 - \epsilon_i(0)\}$. The above results together with assumptions (i) and (iii) made in the proposition, meet the conditions in Proposition 2. Applying this proposition, we conclude that there exists equilibrium. Q.E.D

We give the following remarks on the conditions. First any linear combination of risk-averse power utility functions will satisfy Condition (ii). Secondly, Condition (iv) is to guarantee that $\overline{\tilde{\gamma}_i}(0) \leq \gamma_i(0)$. It is apparent that if there exists $M > 0$ such that the probability $Prob(e_i > M) = 0$, then (iv) is satisfied. Or if for every i, $-u_i''(x)$ is bounded above for sufficiently large $x > 0$, then (iv) is also satisfied. For example, if $u''' \geq 0$, then (iv) is satisfied. Or if $\underline{\gamma_i}(\infty) > 0$ and there exist $M > 0$ and $a_0 < \underline{\gamma_i}(\infty)$ such that $\frac{\gamma(x)}{x^{1+a_0}}$ is bounded for $x > M$, then (iv) is satisfied.[4]

Thirdly, Condition (v) is to guarantee that $\underline{\tilde{\gamma}_i}(0) > 0$. In fact if Condition (v) is not met, for example, if the background risk is discretely distributed at $e_i = 0$ with probability $P_i = Prob(e_i = 0) = 0$, then we have $\lim_{x\to 0} \tilde{u}_i'(x) < +\infty$. If the background risk is continuously distributed and β_i and ω_i exist, but $\beta_i > \omega_i - 2$, we can also show that $\lim_{x\to 0} \tilde{u}_i'(x) < +\infty$.[5] According to Remark 2 on Proposition 2, if $\lim_{x\to 0} \tilde{u}_i'(x) < +\infty$, we will not have equilibrium with interior solutions (except that investors are homogeneous).

Now we explain why we should add a threshold parameter to the utility function and why it should be the lower bound of the domain in which the

[4]Since $\underline{\gamma_i}(\infty) > 0$, given arbitrary small $\varepsilon > 0$, for sufficiently large $x > 0$, we have

$$u_i'(x) < Ax^{-\underline{\gamma_i}(\infty)+\varepsilon},$$

where A is a positive constant. It follows that given arbitrary small $\varepsilon > 0$, for sufficiently large $x > 0$,

$$-u_i''(x) = \frac{\gamma_i(x)}{x} u_i'(x) < Ax^{a_0 - \underline{\gamma_i}(\infty)+\varepsilon}.$$

This together with the assumption that $u''(x)$ is continuous implies that for $e_i \geq \nu - x > 0$, $-u_i''(x + e_i) < A_0$, where A_0 is a positive constant. Thus $-E_{(e_i \geq \nu - x)}(u_i''(x + e_i)) < A_0$.

[5]From (43) and (45) in Appendix B, given arbitrarily small $\varepsilon > 0$, there exists $\nu > 0$ such that for $0 < x < \nu$

$$\begin{aligned}
E_{e_i}(u_i'(x + e_i)) &= \int_0^{+\infty} h(e_i) u_i'(x + e_i) de_i \\
&< A_0 \int_0^{\nu - x} e_i^{\beta_i} (x_i + e_i)^{-(\omega_i - 1)} de_i \\
&= A_0 \int_0^{\frac{\nu}{x} - 1} x^{\beta_i - \omega_i + 2} y^{\beta_i} (1 + y)^{-(\omega_i - 1)} dy \\
&< A_0 \int_0^{\frac{\nu}{x}} x^{\beta_i - \omega_i + 2} y^{\beta_i - \omega_i + 1} dy \\
&= \frac{A_0 \nu}{\beta_i - \omega_i + 2},
\end{aligned}$$

where A_0 is a positive constant independent of x.

background risk is distributed. We have assumed that the marginal (direct) utility is infinity when consumption approaches zero, i.e., $u_i'(0) = +\infty$. This implies that for utility function $u_i(x)$, it is naturally required that $x \geq 0$. Now given background risk z_i distributed in $(\underline{z_i}, +\infty)$, since $\underline{z_i} < 0$, we have to add a threshold parameter to the utility function; otherwise the indirect utility function will be $\tilde{u}_i(x_i) = E_{z_i}(u_i(x_i + z_i))$. Then the indirect utility function is not well defined for $0 < x_i < -\underline{z_i}$. In order to keep the indirect utility function well defined for any $x_i > 0$, the threshold parameter a_i should be no less than $-\underline{z_i}$. But if $a_i > -\underline{z_i}$, we will have $\tilde{u}_i'(0) = E_{z_i}(u_i'(a_i + z_i)) < u_i'(a_i + \underline{z_i}) < +\infty$. According to remark 2 on Proposition 2, there will be no equilibrium with interior solutions (except that investors are homogeneous). Thus the threshold parameter has to be $-\underline{z_i}$.

5 Conclusion

When the first order conditions for investors' utility maximization problems are all equalities, the pricing kernel is determined in equilibrium of the economy. In this case it would be inappropriate for us to give conditions for the existence of optimal portfolios involving the to-be-derived pricing kernel. In this paper we give conditions directly on investors' preferences, their beliefs about the market portfolio and the distributions of their background risk which guarantee that there is a pricing kernel under which every investor has an optimal portfolio i.e., there exists equilibrium.

Appendix A Proof of (C) in Lemma 1

We show that under the conditions given in Lemma 1, $E(u_i(x_i))$ is bounded below.

(Ca) Suppose $\overline{\gamma_i}(0) < 1$.

We can show that $u_i(x)$ is naturally bounded below. Since $\overline{\gamma_i}(0) < 1$, for sufficiently small x, we have

$$\gamma_i(x) < 1 - \varepsilon,$$

where $0 < \varepsilon < 1 - \gamma_i(0)$. It follows that

$$u_i'(x) < Ax^{-(1-\varepsilon)}$$

where A is a positive constant. Hence for sufficiently small x° and $x < x^\circ$

$$\begin{aligned} u_i(x) &= u_i(x^\circ) - \int_x^{x^\circ} u_i'(x)dx \\ &\geq u_i(x^\circ) - \int_x^{x^\circ} Ax^{-(1-\varepsilon)}dx \\ &= u_i(x^\circ) - \frac{A}{\varepsilon}(x^{\circ\varepsilon} - x^\epsilon) \\ &> u_i(x^\circ) - \frac{A}{\varepsilon}x^{\circ\varepsilon} \end{aligned}$$

Since for any $x_0 \in C^+$, $x_i(X; x_0) > 0$, we have $x_i(X^\circ; x_0) > 0$. From the above equation we conclude that $u_i(x_i)$ is bounded below.

(Cb) Suppose $\overline{\gamma_i}(0) \geq 1$.

From (5) and (13), we have

$$u_i'(x_i)x_i'(X) = \lambda_i g_i \phi \frac{x_i}{X} \frac{\gamma_e(X) + \epsilon_i(X)}{\gamma_i(x_i)} \tag{35}$$

Given arbitrarily small $\varepsilon > 0$, for sufficiently small $X > 0$, we have

$$\frac{\gamma_e(X) + \epsilon_i(X)}{\gamma_i(x_i)} < \frac{\overline{\delta}(0) + \overline{\epsilon_i}(0)}{\underline{\gamma_i}} + \varepsilon, \tag{36}$$

where $\overline{\delta}(0) + \epsilon_i(0) \geq 0$ and $\underline{\gamma_i} = inf_x\{\gamma_i(x)\} > 0$.

On the other hand, given arbitrarily small $\varepsilon > 0$ for sufficiently small $X > 0$ we have

$$A_0 X^{-(\underline{\epsilon_i}(0) - \varepsilon)} < g_i(X) < A_0 X^{-(\overline{\epsilon_i}(0) + \varepsilon)}, \tag{37}$$

where A_0 is a positive constant.

From (17), (35), (36) and (37), we conclude that given arbitrarily small $\varepsilon > 0$, for sufficiently small $X > 0$,

$$u'(x_i)x_i'(X) < A_1 X^{-(\overline{\epsilon_i}(0) + \overline{\delta}(0) + \varepsilon)}$$

where $A_1 = -A_0^2 \lambda_i(\frac{\overline{\delta}(0)+\overline{\epsilon_i}(0)}{\underline{\gamma_i}} + \varepsilon)$ is a positive constant. It follows that for sufficiently small $X^\circ > 0$, when $X < X^\circ$,

$$\begin{aligned} u_i(x_i(X)) &= u_i(x_i(X^\circ)) - \int_X^{X^\circ} u_i'(x_i(X)) x_i'(X) dX \\ &> u_i(x_i(X^\circ)) - \int_X^{X^\circ} A_1 X^{-(\overline{\epsilon_i}(0)+\overline{\delta}(0)+\varepsilon)} dX \\ &= u_i(x_i(X^\circ)) - \frac{A_1}{\alpha}(X^{\circ\alpha} - X^\alpha), \end{aligned}$$

where $\alpha = 1 - (\overline{\epsilon_i}(0) + \overline{\delta}(0) + \varepsilon)$. Since $\overline{\epsilon_i}(0) + \overline{\delta}(0) \geq \overline{\gamma_i}(0) \geq 1$ and $\varepsilon > 0$, we must have $\alpha < 0$. Since for any $x_0 \in C^+$, $x_i(X; x_0) > 0$, we have $x_i(X^\circ) = x_i(X^\circ, x_0) > 0$. Thus $u_i(x_i(X^\circ)) > -\infty$. It follows that given arbitrarily small ε for sufficiently small $X > 0$

$$u_i(x_i(X)) > -A_3 X^{1-(\overline{\epsilon_i}(0)+\overline{\delta}(0)+\varepsilon)}.$$

where A_3 is a positive constant. Combining the above equation and (37), we obtain that given arbitrarily small ε for sufficiently small $X > 0$

$$\frac{u_i(x_i(X))}{g_i(X)} > -AX^{1-(\overline{\delta}(0)+\varepsilon)}, \tag{38}$$

where A is a positive constant. Now applying Condition (iv) we conclude that there exists sufficiently small $\nu > 0$ such that

$$E_{i(X<\nu)}(u_i(x_i)) > -\infty. \tag{39}$$

We now show that there exists $M > 0$ such that

$$E_{i(X>M)}(u_i(x_i)) > -\infty. \tag{40}$$

From (15) given arbitrarily small $\varepsilon > 0$, we have for sufficiently large X,

$$\gamma_e(X) > \underline{\delta}(\infty) - \varepsilon. \tag{41}$$

If $\underline{\delta}(\infty) + \underline{\epsilon_i}(\infty) > 0$, from (13) and (41), we conclude that $x_i(X)$ will be increasing in X when X approaches infinity. This implies that for sufficiently large X, $u_i(x_i)$ is bounded below, thus (40) immediately follows. Otherwise, suppose $\underline{\delta}(\infty) + \underline{\epsilon_i}(\infty) \leq 0$.

For sufficiently large X, we must have

$$\frac{\gamma_e(X) + \epsilon_i(X)}{\gamma_i(x_i)} > \frac{\underline{\delta}(\infty) + \underline{\epsilon_i}(\infty)}{\underline{\gamma_i}} - \varepsilon, \tag{42}$$

where $\underline{\delta}(\infty) + \underline{\epsilon_i}(\infty) \leq 0$ and $\underline{\gamma_i} = inf_x\{\gamma_i(x)\} > 0$.

From (19), (22), (35), and (42) we conclude that given small $\varepsilon > 0$, for sufficiently large X,

$$u'(x_i)x_i'(X) > -A_1 X^{-(\underline{\epsilon_i}(\infty)+\underline{\delta}(\infty)-\varepsilon)}$$

where A_1 is a positive constant. It follows that for sufficiently large $X^\circ > 0$, when $X > X^\circ$,

$$\begin{aligned}
u_i(x_i(X)) &= u_i(x_i(X^\circ)) + \int_{X^\circ}^{X} u_i'(x_i(X))x_i'(X)dX \\
&> u_i(x_i(X^\circ)) - \int_{X^\circ}^{X} A_1 X^{-(\underline{\epsilon_i}(\infty)+\underline{\delta}(\infty)-\varepsilon)}dX \\
&= u_i(x_i(X^\circ)) - \frac{A_1}{\alpha}(X^\alpha - X^{\circ\alpha}),
\end{aligned}$$

where $\alpha = 1-(\underline{\epsilon_i}(\infty)+\underline{\delta}(\infty)-\varepsilon) > 0$. Since for any $x_0 \in C^+$, $x_i(X;x_0) > 0$, we have $x_i(X^\circ) = x_i(X^\circ, x_0) > 0$. Thus $u_i(x_i(X^\circ)) > -\infty$. It follows that there exists a positive constant A_2 such that for sufficiently large X

$$u_i(x_i(X)) > -A_2 X^{1-(\underline{\epsilon_i}(\infty)+\underline{\delta}(\infty)-\varepsilon)}$$

From (22) and the above equation we obtain for sufficiently large X

$$\frac{u_i(x_i(X))}{g_i} > -A_3 X^{1-(\underline{\delta}(\infty)-\varepsilon)},$$

where A_3 is a positive constant. Now applying Condition (iv), we obtain (40).

Since $u_i(x_i)$ is continuous in X, from (39) and (40) we conclude that $E_i(u_i(x_i))$ is bounded below. Q.E.D.

Appendix B Proof of (E) in Proposition 3

Now assume e_i is continuously distributed at $e_i = 0$. From (B), we know that $\gamma_i(0) = \omega_i - 1$ and $\lim_{x\to 0} \frac{u_i'(x)}{x^{-\gamma_i(0)}}$ exists. Let $\alpha_i = \lim_{x\to 0} \frac{u_i'(x)}{x^{-\gamma_i(0)}}$. Given arbitrarily small $\varepsilon > 0$, for sufficiently small $\nu > 0$ when $x < \nu$

$$(\alpha_i - \varepsilon)x^{-\gamma_i(0)} < u_i'(x) < (\alpha_i + \varepsilon)x^{-\gamma_i(0)}. \tag{43}$$

From (28) and the above equation given arbitrarily small $\varepsilon > 0$, for sufficiently small $\nu > 0$ when $x < \nu$

$$\begin{aligned} \tilde{\gamma}_i(x) &> \frac{E_{e_i<\nu-x}(\frac{x}{x+e_i}(\gamma_i(0)-\varepsilon)u_i'(x+e_i))}{E_{e_i}(u_i'(x+e_i))} \\ &> (\gamma_i(0)-\varepsilon)\ \frac{\alpha_i-\varepsilon}{\alpha_i+\varepsilon}\ \frac{E_{e_i<\nu-x}(\frac{x}{x+e_i}(x+e_i)^{-\gamma_i(0)})}{E_{e_i<\nu-x}((x+e_i)^{-\gamma_i(0)}) + \frac{u_i'(\nu)}{\alpha_i+\varepsilon}}. \end{aligned} \tag{44}$$

Let $\underline{b_i} = \liminf_{x\to 0} \frac{h_i(x)}{x^{\beta_i}}$ and $\overline{b_i} = \limsup_{x\to 0} \frac{h_i(x)}{x^{\beta_i}}$, where $h_i(x)$ denotes the probability density function of e_i. Since $0 < \underline{b_i} \le \overline{b_i} < +\infty$, given arbitrarily small $\varepsilon > 0$, we have for sufficiently small $\nu > 0$ when $x < \nu$, for $e_i < \nu - x$

$$(\underline{b_i} - \varepsilon)e_i^{\beta_i} < h_i(e_i) < (\overline{b_i} + \varepsilon)e_i^{\beta_i}. \tag{45}$$

Substituting this equation into (44), given arbitrarily small $\varepsilon > 0$, we have for sufficiently small $\nu > 0$ when $x < \nu$

$$\begin{aligned} \tilde{\gamma}_i(x) &> A_\varepsilon\ \frac{\int_0^{\nu-x} \frac{x}{x+e_i} e_i^{\beta_i}(x+e_i)^{-\gamma_i(0)}de_i}{\int_0^{\nu-x} e_i^{\beta_i}(x+e_i)^{-\gamma_i(0)}de_i + \frac{u_i'(\nu)}{(\alpha_i+\varepsilon)(\overline{b_i}+\varepsilon)}} \\ &= A_\varepsilon\ \frac{\int_0^{\frac{\nu}{x}-1} y^{\beta_i}(1+y)^{-\gamma_i(0)-1}dy}{\int_0^{\frac{\nu}{x}-1} y^{\beta_i}(1+y)^{-\gamma_i(0)}dy + x^{(\gamma_i(0)-\beta_i-1)}\frac{u_i'(\nu)}{(\alpha_i+\varepsilon)(\overline{b_i}+\varepsilon)}} \end{aligned} \tag{46}$$

where $A_\varepsilon = (\gamma_i(0)-\varepsilon)\ \frac{\alpha_i-\varepsilon}{\alpha_i+\varepsilon}\ \frac{\underline{b_i}-\varepsilon}{\overline{b_i}+\varepsilon}$. Since $-1 < \beta_i < \gamma_i(0) - 1$ the integrals are well defined. Now let $x \to 0$. We obtain

$$\liminf_{x\to 0} \tilde{\gamma}_i(x) \ge A_\varepsilon \frac{\Gamma(\beta_i, -(\gamma_i(0)+1))}{\Gamma(\beta_i, -\gamma_i(0))},$$

where $\Gamma(p,q) = \int_0^{+\infty} y^p(1+y)^q dy$, for $-1 < p < -q-1$, and $\gamma_i(0) = \omega_i - 1$. Let $\varepsilon \to 0$. We obtain (32). Q.E.D.

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