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Abstract

In this paper we first show that call options, together with the market portfolio, are sufficient to obtain Pareto efficiency while put options are not. Next we investigate how investors' heterogeneous preferences and beliefs affect their investment strategies and who buys options from whom. We show that an investor buys options with strike prices below a threshold from investors who have lower cautiousness/optimism while selling options with strike prices above the threshold to investors who have higher cautiousness/optimism. We also show that the investor's threshold increases with increases in his cautiousness and optimism.

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Introduction

What is the role of options in an economy? Ross (1976), Breeden and Litzenberger, Hakansson (1978), Leland (1980) and Brennan and Solanki (1981) all searched for an answer. Leland (1980) presented an important result that shows explicitly how investors construct their optimal sharing rules using options. He suggested that a twice differentiable payoff function can be generated by a fraction of the reference portfolio and a further portfolio of options on the portfolio.[1] In a recent paper Carr and Madan (2001) have formalized this result and rigorously showed how a twice differentiable function is generated by a portfolio of call options and put options.

Leland (1980) and Brennan and Solanki (1981) also sought answers to two closely related questions: Who buys and who sells options? And how do investors' heterogeneous preferences and beliefs affect their investment strategies? Leland characterized investors who buy portfolio insurance (options) in terms of global convexity of their optimal sharing rules. He examined two cases. In the first case he assumed homogeneous beliefs and concluded that investors who have higher/lower cautiousness than the representative investor will have globally convex/concave optimal sharing rules, which means they buy/sell portfolio insurance.[2] In the second case he assumed identical cautiousness and concluded that investors who are more/less optimistic than the market will have globally convex/concave optimal sharing rules, which means they buy/sell portfolio insurance.

Brennan and Solanki (1981) also discussed these two cases and obtained similar results, although they focused on obtaining analytical expressions of investors' optimal sharing rules for HARA utility functions and lognormal beliefs.

Leland (1980) and Brennan and Solanki's (1981) work has important implications for portfolio management. However, they made an assumption in their models which requires that there exists a representative investor who has constant (positive) cautiousness. As is well known, this assumption is restrictive. In general there rarely exists such a representative investor since heterogeneity among investors has an impact on the equilibrium of the market.[3] Thus the case is not closed.

In this paper we begin by further investigating the role of options in an economy with heterogeneous preferences/beliefs. Assuming investors are heterogeneous, we show that if the value of the market portfolio is bounded below and unbounded above (as usually assumed), every investor's optimal sharing rule can be generated by a portfolio of call options on the market portfolio with all strike prices, plus (possibly) a fraction of the market portfolio; but it can not be done using put options in a similar way.[4] This implies that limited liability, which keeps the value of the market portfolio bounded below, tends to give call options and put options different roles in achieving Pareto efficiency. This result also has an implication for portfolio insurance. From the construction of investor i's optimal sharing rule we see clearly that his net position is just a series of call options (short or long). There is no floor in the investor's position at all. Thus it is inappropriate to state that the investor wants to buy or sell portfolio insurance. This implies that heterogeneity in preferences and beliefs cannot explain the demand for portfolio insurance.

Later we continue Leland (1980) and Brennan and Solanki's (1981) studies to investigate the impact on investors' investment strategies of heterogeneous preferences and beliefs and characterize option buyers and sellers. But we use a different approach. Instead of exogenously assuming the characteristics of the representative investor, we derive them endogenously.

By re-examining the two special cases in Leland's (1980) work, we show that a rational investor buys options with strike prices below a threshold from investors who have lower cautiousness/optimism while selling options with strike prices above the threshold to investors who have higher cautiousness/optimism. Moreover, the investor's threshold increases with increases in his cautiousness and optimism.

The difference between our results and those in Leland (1980) is worth noting. For example, we show that in the first case mentioned above, only the investor who has the lowest/highest cautiousness has a globally concave/convex optimal sharing rule. This is in contrast to Leland's conclusion that investors who have lower/higher cautiousness than the representative investor (assumed to have positive constant cautiousness) all have globally convex/concave optimal sharing rules. Similarly, we show that in the second case only the least/most optimistic investor's optimal sharing rule is globally concave/convex. This is in contrast to Leland's conclusion that investors who are more/less optimistic than the market all have globally convex/concave optimal sharing rules.

Because the scenario given in this paper is different from that described by Leland (1980) and Brennan and Solanki (1981), their suggestion for portfolio management needs to be adjusted accordingly. Leland suggested that investors who have average expectation but higher cautiousness than average and those who have average cautiousness but are more optimistic than the market would benefit from a 'run with your winners, cut your losers' kind of dynamic strategy. Investors with opposite characteristics would prefer the 'buy low, sell high' strategy, which is equivalent to writing a call (or selling insurance).

The results given in this paper suggest that the above strategy is too simple. Most portfolio managers (investors, in our model) invest in the market portfolio partially covered by options with low strike prices and write options with high strike prices. Thus the optimal strategy for them must be more complicated. The strategy "run with your winners, cut your losers" is not always optimal. We can conclude only that it is optimal for an investor when the value of the portfolio is below a threshold, but when the value of the portfolio is above the threshold, the strategy "buy low and sell high" is more appropriate. The threshold for the change of strategy increases with higher cautiousness and optimism.

Our work is related to Benninga and Blume (1985), Brennan and Cao (1996), and Franke, Stapleton and Subrahmanyam (hereafter FSS) (1998). Benninga and Blume (1985) investigated the optimality of a certain insurance strategy in which an investor buys a risky asset and a put on that asset. Brennan and Cao (1996) investigated the impact of asymmetric information on the demand for options in an economy with exponential utility and normally distributed returns. They developed a dynamic noisy rational expectation model in which investors can trade continuously and concluded that well informed investors tend to buy on good news and sell on bad news. FSS (1998) investigated the impact of background risk on investors' optimal sharing rules in an economy in which investors have power utility functions with the same power coefficient. They concluded that investors with background risk tend to buy options while investors without background risk tend to sell them.

In Section I we introduce a two-period economy in which we study investors' optimal sharing rules. Section II shows how options help to achieve Pareto efficiency in an economy with heterogeneity under general conditions. In Section III we investigate the impact of heterogeneity in preferences on investors' optimal sharing rules. Section IV shows the impact of heterogeneity in beliefs. In Section V we compare our model with Leland's (1980) model. Section VI concludes the paper. Detailed proofs are in the appendices.

I A Two-Period Economy

In this section we introduce a two-period economy. We assume there are N investors indexed by $i = 1, 2, ..., N$. Let X be the payoff of the market portfolio at the end of the two periods. Assume that there is a complete market for state-contingent claims on X.[5] Thus all investors can buy and sell state-contingent claims on X so that, as discussed in Leland (1980), any investor i can choose a payoff function $x_i(X)$. Let $u_i(x)$ be investor i's utility function. Let $f(X)$ be the objective probability density function and $f_i(X)$ investor i's subjective probability density function respectively. We assume that there exists a pricing kernel, $\phi(X)$, whose functional form will be determined in an equilibrium of the economy.

Let w_{i0} be investor i's initial endowment, expressed as a fraction of the spot value of the total wealth in the economy. Let x_{i0} be investor i's amount of wealth consumed in the first period and x_i in the second period respectively. Then the investor has the following utility maximization problem:

$$\max_{x_{i0},x_i} u_i(x_{i0}) + \rho_i E_i[u_i(x_i)], \tag{1}$$

subject to

$$x_{i0} + E(\phi x_i) = w_{i0}(X_0 + E(\phi X)), \tag{2}$$

where ρ_i is his time preference parameter, $E_i(.)$ denotes the expectation operator under the subjective probability measure with p.d.f.$f_i(X)$, and $E(.)$ denotes the expectation operator under the true probability measure with p.d.f.$f(X)$. In equilibrium, the market is cleared and we have

$$\sum_i x_{i0}(X_0) = X_0 \quad \text{and} \quad \sum_i x_i(X) = X. \tag{3}$$

Since negative consumption is not allowed, we require that for every i, $x_{i0} \geq 0$ and $x_i \geq 0$. We assume that all utility functions have infinite marginal utility of zero consumption.[6] This implies that the first order condition is an equality, as follows

$$f(X)\phi(X) = \rho_i f_i(X) u_i'(x_i)/u_i'(x_{i0}), \tag{4}$$

which can be rewritten as

$$u_i'(x_i) = (u_i'(x_{i0})/\rho_i)\phi(X)/g_i(X), \tag{5}$$

where

$$g_i(X) = f_i(X)/f(X).$$

To focus on the main issues addressed in this paper, we will not discuss the existence of an equilibrium.[7] Moreover, whenever needed, sufficient regularity conditions are assumed about all functions involved.

II The Role of Options

As noted by Leland (1980) "For analytical tractability, however, most of these models (including the CAPM) have assumed homogeneous expectations, and investors with linear risk tolerance utility functions with the same slope — precisely those assumptions that eliminate the demand for options!" As noted by Benninga and Mayshar (1997), "the empirical evidence seems to contradict this assumption." In this section we discuss the impact of heterogeneity of preferences on investors' optimal sharing rules.

Let $\gamma_i(x)$ be investor i's relative risk aversion, i.e., $\gamma_i(x) \equiv -xu_i''(x)/u_i'(x)$. Let

$$\epsilon_i(X) \equiv -X(\ln g_i(X))'.$$

We call it investor i's (coefficient of) optimism.[8] Let

$$\delta_i(X) \equiv \gamma_i(x_i(X)) - \epsilon_i(X).$$

We call it investor i's delta coefficient. We call $\delta_i(0)$ and

$$\hat{\delta}_i(\infty) \equiv \gamma_i(\infty) - \epsilon_i(\infty)$$

investor i's delta coefficient at zero and infinity respectively.[9]

We will see later that investors' delta coefficients determine the representative investor's coefficient of relative risk aversion.

Although an equilibrium is assumed in the economy, since investors have heterogeneous preferences, there does not exist an aggregate investor in Rubinstein's sense.[10] But we still have a representative investor in the sense of Benninga and Mayshar's (1997) "pricing representative" investor. He is so called because if the economy had only one investor, namely the "pricing representative" investor with the total endowment of the economy, then the equilibrium state prices in the economy would remain unchanged.

Differentiating both sides of (5) leads to the following result:

$$x_i'(X) = R_i^{-1}(x_i)(R_e(X) - g_i'(X)/g_i(X)), \tag{6}$$

where

$$R_i(x) = -u_i''/u_i'(x) \quad \text{and} \quad R_e(X) = -\phi'(X)/\phi(X)$$

are the coefficients of absolute risk aversion of investor i and the representative investor respectively. Equation (6) can be written as

$$Xx_i'(X)/x_i(X) = (\gamma_e(X) + \epsilon_i(X))/\gamma_i(x_i), \tag{7}$$

where

$$\gamma_i(x) = xR_i(x) \quad \text{and} \quad \gamma_e(X) = XR_e(X)$$

are the coefficients of relative risk aversion of investor i and the representative investor respectively. Noting that $\sum_i x_i' = 1$, from (6) we obtain

$$\gamma_e(X) = \sum_i s_i(\gamma_i(x_i) - \epsilon_i(X)), \tag{8}$$

where

$$s_i \equiv R_i^{-1}(x_i) / \sum_i R_i^{-1}(x_i).$$

From (8) we can see that the representative investor's coefficient of relative risk aversion is a weighted average of individual investors' delta coefficients with weights equal to their risk tolerance. Noting that an investor's delta coefficient is equal to his coefficient of relative risk aversion minus his coefficient of optimism, we can see that investors' (negative) coefficients of optimism have the same impact on the representative investor's coefficient of relative risk aversion as their coefficients of relative risk aversion.

As is well known, in an economy where investors are all homogeneous, investors all have linear sharing rules. In that case any investor's marginal optimal payoff, share of the economy and ratio of his risk tolerance to the aggregate risk tolerance are all constants across states. However, in an economy with heterogeneous investors, in general, these values are state dependent. This is shown in Lemma 1.

The following lemma tells us how an investor's marginal optimal payoff, share of the economy and ratio of his risk tolerance to the aggregate risk tolerance in the worst and best states are related to his delta coefficients at zero and infinity respectively.

Lemma 1 *The following two statements hold:*

1. *If investors are ordered such that* $\delta_1(0) > \delta_2(0) > ... > \delta_N(0)$*, then*

$$s_1(0) = x_1'(0) = \lim_{X\to 0} x_1/X = 1$$

and for every $i \neq 1$*,*

$$s_i(0) = x_i'(0) = \lim_{X\to 0} x_i/X = 0.$$

2. *If investors are ordered such that* $\delta_1(\infty) > \delta_2(\infty) > ... > \delta_N(\infty)$ *and* $\delta_N(\infty) > -\min_i\{\epsilon_i(\infty)\}$*, then*

$$s_N(\infty) = x_N'(\infty) = \lim_{X\to +\infty} x_N/X = 1$$

and for every $i \neq N$*,*

$$s_i(\infty) = x_i'(\infty) = \lim_{X\to +\infty} x_i/X = 0.$$

Proof: See Appendix A. Q.E.D.

Statement 1 in Lemma 1 tells us that in the worst state, any investor's marginal optimal payoff, share of the economy and ratio of his risk tolerance to the aggregate risk tolerance are zero, except the investor who has the highest delta coefficient at zero. Statement 2 tells us that in the best state, any investor's marginal optimal payoff, share of the economy and ratio of his risk tolerance to the aggregate risk tolerance are zero, except the investor who has the lowest delta coefficient at infinity. The results imply that in the worst state the economy is like one where the only investor has the highest delta coefficient at zero while in the best state it is like one where the only investor has the lowest delta coefficient at infinity.

Here we have assumed that investors are completely heterogeneous. However, first we may note that investors are not required to appear in the same position in the two statements. Second, this assumption is just for simplicity. It is not difficult to generalize the case to one where some investors have the same delta coefficients (at zero or infinity).

We now present the following proposition.

Proposition 1 *The following two statements hold:*

1. *Assume that investors are ordered such that* $+\infty > \delta_1(0) > \delta_2(0) > ... > \delta_N(0)$.

   (a) Then $\gamma_e(0) = \delta_1(0)$.

   (b) Moreover, every investor i's optimal sharing rule can be constructed as follows:

$$x_i(X) = x_i'(0)X + \int_0^{+\infty} x_i''(K)c[X;K]dK, \tag{9}$$

   where $x_1'(0) = 1$, $x_2'(0) = ... = x_N'(0) = 0$ *and* $c[X;K]$ *denotes the payoff of the call option on the market portfolio with strike price* K.

2. *Assume that investors are ordered such that*

   $+\infty > \delta_1(\infty) > \delta_2(\infty) > ... > \delta_N(\infty)$ *and* $\delta_N(\infty) > -\min_i\{\epsilon_i(\infty)\}$.

(a) Then $\gamma_e(\infty) = \delta_N(\infty)$.

(b) Moreover, no investor i's optimal sharing rule can be constructed as in (10)

$$x_i(X) = a_i X + \int_0^{+\infty} \beta(K) p[X;K] dK + b_i, \tag{10}$$

where a_i and b_i are constants, $\beta(K)$ is a function of K and $p[X;K]$ denotes the payoff of the put option on the market portfolio with strike price K.

Proof: See Appendix B.

Results 1(a) and 2(a) tell us that the representative investor's coefficient of relative risk aversion is state dependent. In the worst state, it equals the highest value of investors' delta coefficients at zero; in the best state, it equals the lowest value of investors' delta coefficients at infinity. This implies that in the worst state the representative investor acts like the investor who has the highest delta coefficient at zero while in the best state he acts like the investor who has the lowest delta coefficient at infinity.

Result 1(b) explains the composition of investors' optimal sharing rules. It shows that the investors who have the highest delta coefficients at zero hold the market portfolio plus a portfolio of short or long positions in call options on the market portfolio while all other investors just hold a portfolio of short or long positions in call options on the market portfolio. The result also tells us that additional to the market portfolio, call options on the market portfolio with a continuum set of strike prices are sufficient to obtain Pareto efficiency in an economy with heterogeneous investors. This result shows clearly the special role of call options in such an economy.

If there is a risk free asset, then we can use put options to replicate the call options. Thus investor i's optimal sharing rule can be constructed in the following alternative way:[11]

$$\begin{aligned} x_i(X) &= x_i(S_0) - x_i'(S_0)S_0 + x_i'(S_0)X + \int_0^{S_0} x_i''(K)p[X;K]dK \\ &\quad + \int_{S_0}^{+\infty} x_i''(K)c[X;K]dK, \end{aligned} \tag{11}$$

where $0 < S_0 < +\infty$ is arbitrary. Even if there is no risk free asset, as long as we can find $S_0 > 0$ such that $x_i(S_0) - x_i'(S_0)S_0 = 0$, then we can still replace some of the call options with put options as in (11) with no need to borrow or lend.[12] *This implies that heterogeneity among investors can explain demand for both call options and put options.*

Although we can replace some of the call options with put options, Result 2(b) tells us that we cannot replace all call options with put options. That is, no investor can construct his optimal sharing rule using put options on the market portfolio in a way similar to that in Result 1(b) using call options. The result highlights a difference between the role of call options and that of put options. The difference results from the fact that the value of the market portfolio is unbounded above while it is bounded below. Since the value of the market portfolio is unbounded above, $x_i(S_0) - x_i'(S_0)S_0$ is unbounded when $S_0 \to +\infty$, i.e., the amount of loans made to replicate call options becomes infinitely large. This result implies that limited liability, which keeps the value of the market portfolio bounded below, tends to cause call options and put options to have different roles in achieving Pareto efficiency.

Moreover, eq. (11) has implications for portfolio insurance. If an investor's optimal sharing rule is convex, one may conclude that he has a demand for portfolio insurance. However, this conclusion may be superficial. Note that although an investor can construct his optimal sharing rule by buying put options, he will simultaneously buy a fraction of the market portfolio and make some loans; i.e., he is just replicating call options using put options, the market portfolio and loans. Recall that eq. (9) shows clearly that the net position of investor i is just a series of call options. There is no floor in the investor's position at all. This implies that it is inappropriate to state that the investor has a demand for portfolio insurance. Heterogeneity in preferences and beliefs has been used to explain the demand for portfolio insurance using options (see, for example, Leland (1980)). However, the above result implies that this explanation is incomplete.

III Impact of Heterogeneous Preferences

Both heterogeneity in preferences and heterogeneity in beliefs may impact on the composition of investors' optimal sharing rules. To see their impact clearly, we can separate the two effects. In this section we assume that investors have homogeneous beliefs and discuss the impact on investors' optimal sharing rules and investment strategies of heterogeneity in their preferences.

The following assumptions are made in this section:

(i) Investors have homogeneous beliefs.

(ii) Investors are risk averse and have constant relative risk aversion.

(iii) Investors are ordered such that

$$C_1 < C_2 < ... < C_N, \quad \text{which is equivalent to} \quad \gamma_1 > \gamma_2 > ... > \gamma_N,$$

where $C_i = 1/\gamma_i$ and γ_i are investor i's cautiousness and relative risk aversion respectively.

Note that Assumption (iii) is purely for simplicity. As in the last section, we can easily generalize this case to one where some investors have the same coefficients of relative risk aversion.

Now since investors have homogeneous beliefs, letting $g_i(X) = 1$ in eq. (6), we obtain

$$x_i'(X) = R_e(X)/R_i(x_i), \tag{12}$$

where $R_e(X)$ is the representative investor's absolute risk aversion. Differentiating both sides of eq. (12), we obtain:

$$x_i''(X) = R_i^{-1}(x_i)R_e^2[C_i(x_i) - C_e(X)], \tag{13}$$

where $C_i(x_i)$ is investor i's cautiousness and $C_e(X)$ is that of the representative investor.

Two points can be made from eq. (13). First, if investor i has higher cautiousness at state $X = K$ than does investor j (along their optimal sharing rules), i.e., if $C(x_i(K)) > C(x_j(K))$, then investor i is more likely to buy options on the market portfolio with strike price K than is investor j. Secondly, an investor, say i, buys options on the market portfolio with strike price K if and only if his cautiousness is higher (along his optimal sharing rule) than the representative investor's at state $X = K$; i.e., $C_i(x_i(K)) > C_e(K)$. These two results were first given by Leland (1980) although he characterized option buyers in terms of global convexity of their optimal sharing rules.

Since $\sum_i x_i'' = 0$, from eq. (13) we obtain

$$C_e(X) = \sum_i x_i'(X) C_i(x_i). \tag{14}$$

From eq. (14) we can see that the representative investor's coefficient of cautiousness is a weighted average of the individual investors' coefficients with weight equal to their risk tolerance relative to the aggregate risk tolerance.

We now present the following proposition.

Proposition 2 *In the economy the following statements hold:*

1. $C_e(\infty) = C_N$ *and* $C_e(0) = C_1$.

2. $C_e'(X) > 0$.

3. *There exist*

$$+\infty = X_1^\circ > X_2^\circ > ... > X_N^\circ = 0$$

such that

$$\textit{when}\ \ X > X_i^\circ,\ x_i''(X) > 0\ \ \textit{and when}\ \ X < X_i^\circ,\ x_i''(X) < 0.$$

Proof: See Appendix C.

Statements 1 and 2 tell us that when investors have different constant cautiousness the representative investor has increasing cautiousness and that his cautiousness increases from the lowest cautiousness among investors to the highest.

Statement 3 explains how the convexity of an investor's optimal sharing rule depends on his cautiousness relative to the others. If an investor has the lowest cautiousness in the economy, his optimal sharing rule is globally concave. If an investor has the highest cautiousness, his optimal sharing rule is globally convex. All other investors' optimal sharing rules are convex at low market portfolio values (bad states) and concave at high market portfolio values (good states). The threshold value at which an investor's optimal sharing rule turns from convex to concave varies with his cautiousness. More precisely, it increases with increases in cautiousness.

From Statement 3 and eq. (9) we can see that if an investor has the lowest cautiousness in the economy, he takes only short positions in options. If an investor has the highest cautiousness in the economy, he takes only long positions in options. All other investors buy options with low strike prices from those who have lower cautiousness and sell options with high strike prices to those with higher cautiousness. Moreover, the threshold strike price at which an investor turns from buying options to selling options increases with increases in his cautiousness.

Intuitively, risk-averse investors tend to hedge downside uncertainty (bad states) and capitalize on upside uncertainty (good states). This results in optimal sharing rules, which are convex at bad states and concave at good states. The fact that an investor with higher cautiousness has a higher threshold value at which he turns from buying options to selling them implies that his standard of good states is higher than that of an investor with lower cautiousness.

Moreover, from eq. (11) every investor $i \neq N$, can construct his optimal sharing rule by replicating call options with strike prices under X_i°. That is,

$$\begin{aligned} x_i(X) &= x_i(X_i^\circ) - x_i'(X_i^\circ)X_i^\circ + x_i'(X_i^\circ)X + \int_0^{X_i^\circ} x_i''(K)p[X;K]dK \\ &+ \int_{X_i^\circ}^{+\infty} x_i''(K)c[X;K]dK, \end{aligned} \tag{15}$$

where X_i° is given in Statement 3. We can verify that

$$x_i(X_i^\circ) - x_i'(X_i^\circ)X_i^\circ = \inf_X \{x_i(X) - x_i'(X)X\} \leq 0, \tag{16}$$

i.e., the amount of loans he makes reaches its maximum; and

$$x_i'(X_i^\circ) = \sup_X \{x_i'(X)\}, \tag{17}$$

i.e., the fraction of the market portfolio the investor holds reaches its maximum. This apparently results from the investor's replicating strategy.

IV Impact of Heterogeneous Beliefs

Assume investors have identical constant relative risk aversion. This implies that every investor i's utility function can be written as[13]

$$u_i(x) = x^{1-\gamma}/(1-\gamma), \tag{18}$$

where $\gamma > 0$.

We assume that all investors' beliefs are lognormal. Let $f(X)$ denote the objective probability density function and $f_i(X)$ investor i's subjective probability density function respectively. We can write

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}X} e^{\frac{-(\ln x-\mu)^2}{(2\sigma^2)}} \quad \text{and} \quad f_i(X) = \frac{1}{\sigma_i\sqrt{2\pi}X} e^{\frac{-(\ln x-\mu_i)^2}{(2\sigma_i^2)}}. \tag{19}$$

We further assume that investors agree on the variance of the growth rate of the market portfolio value.[14] That is,

$$\text{for every } i, \quad \sigma_i = \sigma.$$

However, investors disagree on the mean μ_i, which reflects their divergent opinions about the economy. We assume that investors are ordered such that

$$\mu_1 < \mu_2 < \ldots < \mu_N.$$

Note that this assumption is purely for simplicity. As in the last section, we can easily generalize the case to one where some investors have the same mean parameters.

Under the above assumption it can be verified that

$$\epsilon_i \equiv -X(\ln g_i(X))' = (\mu_i - \mu)/\sigma^2 \tag{20}$$

is a constant and

$$\epsilon_1 < \epsilon_2 < \ldots < \epsilon_N.$$

We now present the following result.

Proposition 3 *In the economy assume $\gamma \geq 3(\epsilon_N - \epsilon_1)$. Then there exist*

$$0 = X_1^\circ > X_2^\circ > \ldots > X_N^\circ = +\infty$$

such that

$$\textit{when } \; X < X_i^\circ, \; x_i''(X) > 0 \;\; \textit{and when } \; X > X_i^\circ, \; x_i''(X) < 0.$$

Proof: See Appendix D.

Proposition 3 gives a picture of the economy analogous to that given by Proposition 2. It tells us that the most optimistic investor has a globally convex optimal sharing rule and the most pessimistic investor has a globally concave optimal sharing rule. All other investors have optimal sharing rules that are convex in low market portfolio values and concave in high market portfolio values. The threshold market portfolio value at which an investor's optimal sharing rule turns from convex to concave varies with his optimism. More precisely, it increases with increases in the investor's mean parameter μ_i, which reflects his optimism.

Analogous to the first case studied in the last section, we can see that if an investor is the least optimistic in the economy, he takes only short positions in options. If an investor is the most optimistic, he takes only long positions in options. All other investors buy options with low strike prices written by less optimistic investors and sell options with high strike prices to more optimistic investors. Moreover, the threshold strike price at which an investor turns from buying options to selling options increases with increases in his optimism.

Intuitively, a more optimistic investor will think a call option has a higher probability of exercise than a less optimistic investor. Thus the more optimistic an investor, the more likely he buys a call option. This intuition is consistent with the result derived above.

V Comparison with Leland's (1980) Results

In this section we compare our model (in which we investigate the impact on investors' investment strategies in the special cases) with Leland's (1980) model. Recall that in the first case we assume investors have homogeneous beliefs and constant relative risk aversion, while in the second case we assume investors have identical constant relative risk aversion and heterogeneous lognormal beliefs. In the following comparison we use (L) to denote Leland's model and (H) to denote our model.

(a) On the representative investor:

- (L): The representative investor is assumed to have constant relative risk aversion. Apparently, when investors are heterogeneous this is possible only if the system is not closed, i.e., $\sum_i x_i(X) \neq X$.

- (H): We assume that the economy is closed i.e., $\sum_i x_i(X) = X$. This enables us to derive the characteristics of the representative investor endogenously. We find that the derived representative investor has declining relative risk aversion.

(b) On options' role:

- (L): Together with the market portfolio, call options cannot achieve Pareto efficiency. Let investor i be one of those who have lower cautiousness than the representative investor in the first case. According to Leland's model, we have

$$Xx_i'(X)/x_i(X) = \gamma_e/\gamma_i.$$

Since $C_e = 1/\gamma_e > C_i = 1/\gamma_i$, we have $\lim_{X\to 0} x_i'(0) = +\infty$. This implies that $x_i(X)$ cannot be written as (9). Similarly, we can show in the second case that the optimal sharing rule of an investor who is less optimistic than the market cannot be written as in (9).

- (H): Together with the market portfolio, call options with all strike prices can achieve Pareto efficiency. That is, every investor's optimal sharing rule can be written as in (9).

(c) On investors' optimal sharing rules:

- (L): In the first case investors who have higher/lower cautiousness than the representative investor have globally convex/concave optimal sharing rules. In the second case investors who are more/less optimistic than the market have globally convex/concave optimal sharing rules.

- (H): In the first case only the investor who has the highest/lowest cautiousness has globally convex/concave optimal sharing rules. All other investors' optimal sharing rules are convex at low market portfolio values and concave at high market portfolio values. In the second case only the most/least optimistic has globally convex/concave optimal sharing rules. All other investors' optimal sharing rules are convex at low market portfolio values and concave at high market portfolio values.

(d) On who buys and who sells options and on demand for portfolio insurance:

- (L): Investors who have average expectations, but have higher cautiousness than average, will wish to buy portfolio insurance (options); investors who have average cautiousness, but whose expectations of returns are more optimistic than average, will wish to buy portfolio insurance (options).
- (H): Most investors buy options with low strike prices and sell options with high strike prices. Heterogeneity in preferences and beliefs cannot explain demand for portfolio insurance.

(e) On investment strategy:

Let class (1) be the investors who have average expectation but higher cautiousness than the representative investor and class (2) be the investors who have average cautiousness but more optimistic expectations than the market.

- (L): "Since the dynamic trading strategy which yields call option returns (or insured returns) involves buying into the portfolio as its value goes up, but selling out as its value goes down, our results also suggest that investors in class (1) and (2) would benefit from a 'run with your winners, cut your losers' kind of dynamic strategy rather than a simply 'buy and hold' policy. Investors with opposite characteristics would prefer the 'buy low, sell high' strategy which is equivalent to writing a call (or selling insurance)."
- (H): Most investors invest in the market portfolio partially covered by op-

tions with low strike prices and write options with high strike prices. Thus the optimal strategy for them must be more complicated. The strategy "run with your winners, cut your losers" is not always optimal for investors in class (1) and (2). We can conclude only that it is the optimal strategy for an investor when the value of the market portfolio is below his threshold, but when the value of the market portfolio is beyond the threshold, the strategy "buy low and sell high" is more appropriate. The threshold for the change of strategy increases with increases in cautiousness and optimism.

VI Conclusions

This paper first shows that, in an economy with heterogeneous preferences and beliefs, if the value of the market portfolio is bounded below but unbounded above, additional to the market portfolio, call options with all strike prices can achieve Pareto efficiency while put options cannot. This implies that limited liability, which keeps the value of the market portfolio bounded below, tends to cause call options and put options to have different roles in achieving Pareto efficiency. Another implication of the result is that although heterogeneity in preferences and beliefs can explain the demand for options, they cannot explain demand for portfolio insurance. We also show the impact on investors' optimal sharing rules of heterogeneity among investors. The endogenously derived characteristics of the representative investor in the paper are different from those exogenously assumed by Leland (1980) and Brennan and Solanki (1981) and lead to different optimal sharing rules. This gives different implications for portfolio management. Generally speaking, the optimal sharing rule of a typical investor will be convex in low market portfolio values and concave in high market portfolio values. Optimally, a typical investor should follow the strategy "run with your winners and cut your losers" when the value of the portfolio is below a threshold, while he should change to "buy low and sell high" when the value of the portfolio is above the threshold. The threshold increases with increases in an investor's cautiousness and optimism.

Appendix A Proof of Lemma 1

We show the proof of the second half. The first half can be analogously proved.

Given that for every i, $\gamma_i(\infty) > 0$, we have for every i, $\lim_{x\to+\infty} R_i(x) = 0$. Now consider the situation when X approaches infinity. In that case if x_i is bounded, then $s_i = R_i^{-1}(x_i)/(\sum_i R_i^{-1}(x_i))$ will approach zero. From (8), this implies that the value of $\gamma_e(X)$ when X approaches infinity is determined by the values of $\gamma_i(x)$ and $\epsilon_i(x)$. Hence we have

$$\liminf_{X\to+\infty} \gamma_e(X) \geq \underline{\delta}(\infty). \tag{21}$$

From (7) and (21), we conclude that for every i,

$$\liminf_{X\to+\infty} X\frac{x_i'(X)}{x_i(X)} \geq 1.$$

It follows that

$$\text{for every } i, \quad x_i(\infty) = +\infty. \tag{22}$$

On the other hand, from (7) we have

$$\frac{d\ln w_i(X)}{d\ln X} = \frac{\gamma_e(X) - \gamma_i(x_i) + \epsilon_i(X)}{\gamma_i(x_i)}. \tag{23}$$

This together with (7), (21) and the result that $x_i(\infty) = +\infty$ implies that for every $i \neq N$,

$$\liminf_{X\to+\infty} \frac{d\ln w_i(X)}{d\ln X} = \liminf_{X\to+\infty} \frac{\gamma_e(X) - \gamma_i(x_i) + \epsilon_i(X)}{\gamma_i(x_i)} < 0,$$

where $w_i(X) = x_i(X)/X$. It follows that for every $i \neq N$, for sufficiently large X,

$$\frac{d\ln w_i(X)}{d\ln X} = \frac{\gamma_e(X) - \gamma_i(x_i) + \epsilon_i(X)}{\gamma_i(x_i)} < 0,$$

which implies that

$$\text{for any} \quad i \neq N, \quad \lim_{X \to +\infty} w_i(X) = 0. \tag{24}$$

Since $s_i(X) = R_i^{-1}(x_i)/\sum_i R_i^{-1}(x_i)$, we have

$$s_i(X) = w_i(X) \frac{\gamma_i^{-1}(x_i)}{\sum_i w_i(X)\gamma_i^{-1}(x_i)}. \tag{25}$$

But we have

$$0 < \inf_X \gamma_i(x_i) \leq \sup_X \gamma_i(x_i) < +\infty, \tag{26}$$

which is implied by the conditions that for every i, $0 < \gamma_i(\infty) < 0$ and $0 < \gamma_i(0) < +\infty$ and the fact that for every i, $\gamma_i(x_i)$ is differentiable in X. It follows that

$$0 < \inf_X \frac{\gamma_i^{-1}(x_i)}{\sum_i w_i(X)\gamma_i^{-1}(x_i)} < +\infty.$$

From this and (25) we conclude that

$$\lim_{X \to +\infty} w_i(X) = 0 \quad \text{is equivalent to} \quad \lim_{X \to +\infty} s_i(X) = 0. \tag{27}$$

This and (24) imply that

$$\text{for any} \quad i \neq N, \quad \limsup_{X \to +\infty} s_i(X) = 0.$$

Now rewrite (7) as

$$x_i'(X) = s_i(X) - w_i(X) \frac{\epsilon_i(X)}{\gamma_i(x_i)}. \tag{28}$$

Since for any $i \neq N$, $w_i(\infty) = 0$, $s_i(\infty) = 0$ and $\epsilon_i(X)/\gamma_i(x_i)$ is bounded from above and below, it follows that for any $i \neq N$, $x_i'(\infty) = 0$.

On the other hand since

$$\sum_i w_i(X) = \sum_i x_i'(X) = \sum_i s_i(X) = 1,$$

it follows that

$$w_N(\infty) = x_N'(\infty) = s_N(\infty) = 1.$$

Q.E.D.

Appendix B Proof of Proposition 1

The Results 1(a) and 2(a) immediately Lemma 1 and Equation (8). We need only to prove the other results.

We first prove 1(b). Applying Lemma 1, we have $x_1'(0) = 1$, $x_2'(0) = ... = x_N'(0) = 0$. Now rewrite the right side of (9) as

$$x_i'(0)X + \int_0^X x_i''(K)[X-K]dK,$$

which can be written as

$$x_i'(0)X + X\int_0^X x_i''(K)dK - \int_0^X Kx_i''(K)dK,$$

or

$$x_i'(0)X + X(x_i'(X) - x_i'(0)) - \int_0^X Kdx_i'(K).$$

This is equivalent to $x_i(X) - x_i(0)$. But from Lemma 1 $x_i(0) = 0$. Hence (9) is proved.

We now show 2(b). Equation (10) can be written as

$$x_i(X) = a_i X + \int_X^{+\infty} \alpha(K)[K - X]dK + b_i. \tag{29}$$

Differentiating both sides of the above equation twice, we have

$$\alpha(K) = x_i''(K).$$

Substituting this into (29), we have

$$x_i(X) = a_i X + \int_X^{+\infty} x_i''(K)[K - X]dK + b_i. \tag{30}$$

The second term in the right side of (30) can be written as

$$\begin{aligned}
&\int_X^{+\infty} x_i''(K)[K - X]dK \qquad (31)\\
&= \lim_{Y\to+\infty}(\int_X^Y K dx_i'(K) - X\int_X^Y dx_i'(K))\\
&= \lim_{Y\to+\infty}(Yx_i'(Y) - x_i(Y) + x_i(X) - Xx_i'(Y))
\end{aligned}$$

Applying Lemma 1 we conclude that for every i, $x_i'(\infty)$ exists. Hence the right side of (31) is equivalent to

$$x_i(X) - Xx_i'(\infty) + \lim_{Y\to+\infty}(Y\frac{x_i'(Y)}{x_i(Y)} - 1)x_i(Y).$$

From (7) it can be rewritten as

$$x_i(X) - Xx_i'(\infty) + \lim_{Y\to+\infty}(\frac{\gamma_e(Y) + \epsilon_i(Y)}{\gamma_i(x_i(Y))} - 1)x_i(Y). \tag{32}$$

On the other hand, since for every i, $-\epsilon_i(\infty) < \underline{\delta}(\infty)$ and $\gamma_e(\infty) = \underline{\delta}(\infty)$ we have for every i

$$\lim_{Y\to+\infty}\frac{\gamma_e(Y) + \epsilon_i(Y)}{\gamma_i(x_i(Y))} > 0. \tag{33}$$

Again since for every i, $-\epsilon_i(\infty) < \underline{\delta}(\infty)$ and $\gamma_e(\infty) = \underline{\delta}(\infty)$, using (7) we derive that for every i

$$\lim_{Y \to +\infty} X \frac{x_i'(Y)}{x_i(Y)} > 0,$$

which implies that for every i

$$\lim_{Y \to +\infty} x_i(Y) = +\infty. \tag{34}$$

Again since for every i, $-\epsilon_i(\infty) < \underline{\delta}(\infty)$ and $\gamma_e(\infty) = \underline{\delta}(\infty)$, we have for every $i \neq N$,

$$\lim_{Y \to +\infty} \left(\frac{\gamma_e(Y) + \epsilon_i(Y)}{\gamma_i(x_i(Y))} - 1\right) < 0. \tag{35}$$

From the above two equations we conclude that for every $i \neq N$

$$\lim_{Y \to +\infty} \left(\frac{\gamma_e(Y) + \epsilon_i(Y)}{\gamma_i(x_i(Y))} - 1\right) x_i(Y) = -\infty. \tag{36}$$

Hence for every $i \neq N$, the right side of (10) is not defined.

Moreover, we have

$$\sum_i \left(Y \frac{x_i'(Y)}{x_i(Y)} - 1\right) x_i(Y) = Y \sum_i x_i'(Y) - \sum_i x_i(Y) = 0.$$

This and (36) imply that

$$\lim_{Y \to +\infty} \left(Y \frac{x_N'(Y)}{x_N(Y)} - 1\right) x_N(Y) = +\infty.$$

Hence for $i = N$, the right side of (10) is not defined either. Q.E.D.

Appendix C Proof of Proposition 2

Statement 1 immediately follows Lemma 1 and (14). Thus we need only to prove Statements 2 and 3.

(a) We first show the proof of Statement 2.

From (13) we have

$$x_i''(X) = R^2(X)R_i^{-1}(x_i)(C_i(x_i) - C(X)),$$

where we have omitted subscript "e" in $R_e(X)$ and $C_e(X)$. Differentiating both sides of the above equation, we have

$$\frac{x_i'''(X)}{x_i''(X)} = -2\frac{R'(X)}{R(X)} - \frac{R_i'(x_i)}{R_i(x_i)}x_i'(X) + \frac{C_i'(x_i)x_i'(X) - C'(X)}{C_i(x_i) - C(X)},$$

which can be rewritten as:

$$x_i'''/x_i''(= 2(P - R) + (P_i - R_i)x_i' + (C_i'x_i' - C')/(C_i - C),$$

where we have omitted the arguments of the functions. Applying (12) and (13) and rearranging the terms, we obtain

$$x_i'''(X) = \frac{R^2}{R_i}(C_i - C)(-2(P - R) + (P_i - R_i)\frac{R}{R_i}) - \frac{R^2}{R_i}C' + C_i'x_i'\frac{R^2}{R_i}.$$

Since $\sum_i x_i''' = 0$ we have

$$R\sum_i R_i^{-1}(C_i - C)(-2(P - R) + (P_i - R_i)\frac{R}{R_i}) + \sum_i C_i'x_i'^2 - C' = 0.$$

Since $C_i' = 0$, we have

$$C' = -2R(P - R)\sum_i (C_i - C)/R_i + R^2\sum_i (C_i - C)C_i/R_i.$$

From (14) we obtain $\sum_i (C_i - C)/R_i = 0$. Thus we have

$$C' = R^2 \sum_i C_i(C_i - C)/R_i,$$

which can be rewritten as:

$$C' = R^2(\sum_i C_i^2/R_i - R(\sum_i C_i/R_i)^2).$$

Rearranging the terms, we obtain

$$C' = R^3(\sum_i R_i^{-1} \sum_i C_i^2/R_i - (\sum_i C_i/R_i)^2).$$

Applying Cauchy's inequality, we obtain $C' > 0$.

(b) We now prove Statement 3. From (12) and (13) we obtain

$$X\frac{x_i''}{x_i'} = \gamma_e(C_i - C_e(X)).$$

But from Proposition 1 we have $\gamma_e(0) = \gamma_1$ and $\gamma_e(\infty) = \gamma_N$. Hence we conclude that around zero and infinity the sign of Xx_i''/x_i' is the same as that of $C_i - C_e(\infty)$. Noting that for every i, $x_i'(X)$ is always positive, we conclude that around zero and infinity

$$Xx_i''/x_i' \text{ and } C_i - C_e(\infty) \text{ have the same sign.} \tag{37}$$

On the other hand, from Statement 1 and Statement 2, $C_e(X)$ is increasing in X from $C_e(0) = C_N$ to $C_e(\infty) = C_1$. This and (37) imply that for any i, there exists $X_i^\circ \in [0, +\infty]$ such that $x_i''(X) > (<)0$ when $X > (<)X_i^\circ$ and $C_e(X_i^\circ) = C_i$. Apparently $X_1^\circ = +\infty$, $X_N^\circ = 0$ and for any i, $X_i^\circ < X_{i+1}^\circ$. Q.E.D.

Appendix D Proof of Proposition 3

We first prove the following lemma.

Lemma 2 *In the economy we have the following result:*

1. *For any* $i \neq 1$, $\lim_{X\to 0} X^2 x_i''(X)/x_i > 0$*; if* $\gamma \geq \epsilon_N - \epsilon_1$*, then* $x_N''(X) > 0$.

2. *If* $\gamma \geq \epsilon_N - \epsilon_1$*, then for any* $i \neq N$, $\lim_{X\to+\infty} X^2 x_i''(X)/x_i < 0$*; if* $\gamma \geq 2(\epsilon_N - \epsilon_1)$*, then* $x_1''(X) < 0$.

3. *Assume* $\gamma \geq 2(\epsilon_N - \epsilon_1)$*. Then for any* i, $x_{i+1}''(X)/x_{i+1} > x_i''(X)/x_i$.

4. *Assume* $\gamma \geq 3(\epsilon_N - \epsilon_1)$*. Then for any* i, $(X^2 x_i''(X)/x_i)' < 0$.

Proof: From (7) we have

$$x_i'(X) = (x_i/X)(\gamma_e + \epsilon_i)/\gamma. \tag{38}$$

From (8) we have

$$\gamma_e(X) = \gamma - \sum_i w_i \epsilon_i, \tag{39}$$

where

$$w_i = x_i/X.$$

Differentiating both sides of (38), we obtain

$$x_i''(X) = x_i[(\gamma_e + \epsilon_i)^2/\gamma - C_e\gamma_e^2 - \epsilon_i]/(\gamma X^2). \tag{40}$$

Noting that $\sum_i x_i''(X) = 0$, from the above equation we obtain

$$C_e\gamma_e^2 = \sum_i w_i[(\gamma_e + \epsilon_i)^2/\gamma - \epsilon_i]. \tag{41}$$

(a) We first prove Statements 1 and 2. Substitute (41) into (40) we have

$$\gamma X^2 x_i''(X)/x_i = (\gamma_e + \epsilon_i)^2/\gamma - \sum_i w_i((\gamma_e + \epsilon_i)^2/\gamma - \epsilon_i) - \epsilon_i, \tag{42}$$

where $w_i = x_i/X$. Applying Equation (39), we can rewrite Equation (42) as

$$\gamma X^2 x_i''(X)/x_i = (\gamma - \sum_i w_i\epsilon_i + \epsilon_i)^2/\gamma - \sum_i w_i((\gamma - \sum_i w_i\epsilon_i + \epsilon_i)^2/\gamma - \epsilon_i) - \epsilon_i,$$

which can be simplified as

$$\gamma^2 X^2 x_i''(X)/x_i = \gamma(-\sum_i w_i\epsilon_i + \epsilon_i) + (-\sum_i w_i\epsilon_i + \epsilon_i)^2 - \sum_i w_i(\epsilon_i - \sum_i w_i\epsilon_i)^2. \tag{43}$$

Letting $i = N$ in (43), we obtain

$$\gamma^2 X^2 x_N''(X)/x_N = \gamma(-\sum_i w_i\epsilon_i + \epsilon_N) + (-\sum_i w_i\epsilon_i + \epsilon_N)^2 - \sum_i w_i(\epsilon_i - \sum_i w_i\epsilon_i)^2,$$

which we can rewrite as

$$\gamma^2 X^2 x_N''/x_N = \sum_i w_i(-\epsilon_i + \epsilon_N)(\gamma + \epsilon_N + \epsilon_i - 2\sum_i w_i\epsilon_i).$$

It follows that

$$\gamma^2 X^2 x_N''/x_N > \sum_i w_i(-\epsilon_i + \epsilon_N)(\gamma - (\epsilon_N - \epsilon_1)) \geq 0.$$

Hence $x_N'' > 0$.

Similarly we can shown that given that $\gamma \geq 2(\epsilon_N - \epsilon_1)$, $x_1'' < 0$.

From Lemma 1 and (43) we obtain

$$\gamma^2 \lim_{X^2 \to +\infty} \frac{X^2 x_i''(X)}{x_i} = (-\epsilon_N + \epsilon_i)(-\epsilon_N + \epsilon_i + \gamma), \tag{44}$$

and

$$\gamma^2 \lim_{X\to 0^+} \frac{X^2 x_i''(X)}{x_i} = (-\epsilon_1 + \epsilon_i)(-\epsilon_1 + \epsilon_i + \gamma). \tag{45}$$

Hence Statements 1 and 2 are proved.

(b) We now prove Statement 3. We have

$$\begin{aligned}&\gamma(-\sum_i w_i\epsilon_i + \epsilon_i) + (\sum_i w_i\epsilon_i - \epsilon_i)^2 - [\gamma(-\sum_i w_i\epsilon_i + \epsilon_j) + (\sum_i w_i\epsilon_i - \epsilon_j)^2] \\ &= (\epsilon_i - \epsilon_j)(\gamma + \epsilon_i + \epsilon_j - 2\sum_i w_i\epsilon_i).\end{aligned}$$

From (43) and the above equation, we have

$$\gamma^2 X^2 (\frac{x_i''(X)}{x_i} - \frac{x_j''(X)}{x_j}) = (\epsilon_i - \epsilon_j)(\gamma + \epsilon_i + \epsilon_j - 2\sum_i w_i\epsilon_i). \tag{46}$$

Since $\gamma \geq 2(\epsilon_N - \epsilon_1)$, if $i \neq j$, we have

$$\epsilon_i + \epsilon_j - 2\sum_i w_i\epsilon_i + \gamma > \gamma - 2(\epsilon_N - \epsilon_1) \geq 0.$$

This implies that the right side of Equation (46) is positive if and only if $\epsilon_i > \epsilon_j$.

Thus we conclude that $x_i''(X)/x_i > x_j''(X)/x_j$, if and only if $\epsilon_i > \epsilon_j$; i.e., if investor i is more optimistic than investor j.

(c) Finally we prove Statement 4. From (43), we obtain

$$\frac{\gamma^2 X^2 x_i''(X)}{x_i} = -(\gamma + 2\epsilon_i)\sum_i w_i\epsilon_i + 2(\sum_i w_i\epsilon_i)^2 - \sum_i w_i\epsilon_i^2 + \gamma\epsilon_i + \epsilon_i^2.$$

It follows that

$$(\frac{\gamma^2 X^2 x_i''(X)}{x_i})' = -((\gamma + 2\epsilon_i) - 4\sum_i w_i\epsilon_i)\sum_i w_i'\epsilon_i - \sum_i w_i'\epsilon_i^2. \tag{47}$$

From (38) and (39), we obtain

$$w_i' = \frac{w_i}{\gamma X}(-\sum_i w_i \epsilon_i + \epsilon_i).$$

Substituting for w_i' in (47), we obtain

$$\begin{aligned}\gamma X(\frac{X^2 x_i''(X)}{x_i})' \qquad &= (\gamma + 2\epsilon_i - 4\sum_i w_i\epsilon_i)\sum_i w_i\epsilon_i(\sum_i w_i\epsilon_i \\ &-\epsilon_i) + \sum_i w_i\epsilon_i^2(\sum_i w_i\epsilon_i - \epsilon_i).\end{aligned}$$

We write it as

$$\begin{aligned}\gamma X(\frac{X^2 x_i''(X)}{x_i})' \qquad &= -(\gamma + 2\epsilon_i - 4\sum_i w_i\epsilon_i)\sum_i w_i(\sum_i w_i\epsilon_i - \epsilon_i)^2 \\ &- \sum_i w_i(\epsilon_i - \sum_i w_i\epsilon_i)^2(\epsilon_i + \sum_i w_i\epsilon_i).\end{aligned}$$

It follows that

$$\gamma X(\frac{X^2 x_i''(X)}{x_i})' < -(\gamma - 3(\epsilon_N - \epsilon_1))\sum_i w_i(\epsilon_i - \sum_i w_i\epsilon_i)^2 \leq 0.$$

Q.E.D.

We are now ready to prove the proposition. Applying Lemma 2, we obtain Statements 1, 2, 3 and 4. From Statements 1 and 2, we conclude that for any i, $1 < i < N$, when X is sufficiently large, $x_i'' < 0$ and when $X > 0$ is sufficiently small $x_i'' > 0$. This and Statement 4 imply that for any i, $1 < i < N$, there exists $X_i^\circ \in (0, +\infty)$ such that $x_i''(X) > (<)0$, when $X < (>)X_i^\circ$. From Statements 1 and 2, we also obtain that there exists $X_1^\circ = 0$ such that $x_1''(X) > (<)0$, when $X < (>)X_1^\circ$ and there exists $X_N^\circ = +\infty$ such that $x_N''(X) > (<)0$, when $X < (>)X_N^\circ$. From Statement 3 we conclude that for any i, $X_{i+1}^\circ > X_i^\circ$. Q.E.D.

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Notes

[1]See Section III in Leland (1980).

[2]An investor's cautiousness is defined by Wilson (1968) as the first derivative of his risk tolerance. It can be written as the ratio of the investor's absolute prudence to his absolute risk aversion minus one. A utility function has positive constant cautiousness if and only if it is a logarithmic utility function ($\ln(x+a)$) or a risk-averse power utility function with power coefficient smaller than one ($(x+a)^{1-\gamma}/(1-\gamma)$, $\gamma > 0$).

[3]For example, in the first case Benninga and Mayshar (2000) showed that heterogeneity among investors results in a representative investor with declining relative risk aversion.

[4]Although a call option can be always replicated by using a put option, we are not necessarily able to replicate all call options by using put options. If we do so, since the value of the market portfolio is unbounded above we can show that the amount of loans we need to make will approach infinity. See also Statement 2b in Proposition 1.

[5]Later we will show that a market of call options with all strike prices is sufficient.

[6]This will be implied by the condition that investors' coefficients of relative risk aversion are positive when their consumption approaches zero, which will be assumed through out the paper.

[7]It is understood that under sufficient regularity conditions, there always exists an equilibrium. Interested readers may refer to Borch (1962) and Mas-Colell (1985).

[8]It is so called since if for all X, $\epsilon_i(X) = -X(\ln g_i(X))' > 0$ then $E_i(X) > E(X)$. This can be shown as follows. It is apparent that $f_i(X)$ and $f(X)$ intersect at least once. But since $\epsilon_i(X) = -X(\ln g_i(X))' > 0$, they can intersect only once, say at X_0. Moreover $f(X)$ must intersect $f_i(X)$ from below. Thus we conclude that $X - X_0$ and $f_i(X) - f(X)$ always have the same sign. It follows that

$$E_i(X)-E(X) = E_i(X-X_0)-E(X-X_0) = \int_0^{+\infty} (X-X_0)(f_i(X)-f(X))dX > 0.$$

[9]In this paper given any function $h(x)$, $h(0)$ means $\lim_{x\to 0^+} h(x)$ and $h(\infty)$ means $\lim_{x\to+\infty} h(x)$. Note that $\hat{\delta}_i(\infty) \equiv \gamma_i(\infty) - \epsilon_i(\infty)$ is not necessarily equal to $\delta_i(\infty)$. They are equivalent only if $x_i(\infty) = +\infty$, which is indeed satisfied in this paper by assuming $\underline{\delta}(\infty) > -\min_i\{\epsilon_i(\infty)\}$. See Equation (22) in Appendix A).

[10]There exists an aggregate investor in Rubinstein's sense if security prices are independent of the allocation of wealth across investors. It is shown by Rubinstein (1974) and Brennan and Kraus (1978) that there exists an aggregate investor in Rubinstein's sense if and only if either all investors have identical cautiousness and beliefs or all investors have exponential utility functions.

[11]The result is an application of Theorem 1 in Carr and Madan (2001) which

states that every twice differentiable function $x_i(X)$ can be written as (11).

[12]In most cases, there exists $S_0 > 0$ such that $x_i(S_0) - x_i'(S_0)S_0 = 0$. For example, in the special case examined in the following section for investor i, S_0 is the point at which $\gamma_e(X)$ intersects $\gamma_i(x_i(X))$.

[13]Let investor i's utility function be $u_i(x) = (x+a_i)^{1-\gamma}/(1-\gamma)$. To guarantee that the first order conditions are equalities, every investor's marginal utility of zero consumption must be infinity. This requires that $a_i = 0$.

[14]Expressed as the continuously compounded growth rate, $\ln(X/X_0)$.