# PARTICLE FILTERING WITH ALPHA-STABLE DISTRIBUTIONS

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**ABSTRACT**

In this paper we introduce a novel sequential Monte Carlo technique, which is based on the family of symmetric $\alpha$-stable ($S\alpha S$) distributions. Sequential Bayesian estimation generally involves recursive estimation of *filtering* and *predictive distributions* of unobserved signals from their noisy measurements. In our proposed algorithm, the relevant density functions are approximated by particles drawn from stable distributions. We call this novel technique $S\alpha S$ particle filtering ($S\alpha S$PF). We assess the performance of the $S\alpha S$PF in comparison with the Gaussian Sum particle filter (GSPF) [1] and a standard (non-parametric) particle filter (PF). Results obtained using highly nonlinear models with simulated data show that the $S\alpha S$PF outperforms the GSPF and compares very favorably with the PF.

## 1. INTRODUCTION

Recently, there has been considerable interest in solving the problem of sequentially estimating the state of a dynamic system based on sensor measurements [2]. Among the different solutions, the Kalman filter [3] remains probably the most well known together with its variants - the extended Kalman filter (EKF) [2] and the unscented Kalman filter (UKF) [4]. However, Kalman filters are generally limited by the ubiquitous nonlinearity and non-Gaussianity of the physical world. Particle filtering methods (sometimes also refered to as Monte Carlo techniques) have been thus proposed as powerful tools, able to handle multivariate data and nonlinear / non-Gaussian processes [5, 6]. Particle filtering relies on a sample-based reconstruction of probability density functions. Multiple particles (samples) of the variables of interest are generated, each one associated with a weight that characterises the quality of a specific particle. An estimate of the variable of interest is obtained by the weighted sum of particles. The different particle filters differ from each other in the way they propagate the probability density functions. Most sequential Monte Carlo algorithms are non-parametric techniques, directly propagating a sample based representation of the probability density function. However, some recently developed algorithms, such as the Gaussian particle filter (GPF) [7] offer a parameterised solution, i.e. they use a Gaussian assumption and propagate the first two moments (mean and covariance). They approximate posterior densities by single Gaussians like the EKF and its variants. The GPF has also been used as a building block for more complex filters, called the Gaussian sum particle filter (GSPF) [1] that approximate the posterior densities by mixtures of Gaussian components.

A more general distribution that includes the Gaussian density as a limiting case is the $\alpha$-stable distribution. It has been used to model many phenomena where the Gaussian distribution is not a reasonable choice, for instance, noises with an impulsive nature. Signals and noises of such class contain sharp spikes or occasional bursts [8]. Impulsive noises, which can be modelled with $\alpha$-stable distributions include atmospheric noise in radio links, ambient acoustic noise in underwater sonar and submarine communications, as well as lightening, switching transients and accidental hits in telephone lines. Different phenomena have also been modelled successfully with $\alpha$-stable distributions in economics [9], physics, biology, and electrical engineering [8], including communications and image processing [10, 11].

So far alpha-stable distributions have not been used within the Monte Carlo framework, apart from in [12, 13, 14, 15]. In [14] they have been applied to solve parameter estimation problems in time-series. A Rao-Blackwellised implementation was reported without a direct evaluation of the $\alpha$-stable probability density function, which in general is unavailable in closed form. In [12, 13] a batch-based Markov chain Monte Carlo (MCMC) method is proposed, whilst [14] develops a sequential Monte Carlo framework for on-line estimation with measurements corrupted with $\alpha$-stable noise. Finally, in [15] Monte Carlo techniques were employed in order to derive a maximum likelihood estimator for the parameters of an $S\alpha S$ distributed signal mixed with additive Gaussian noise.

In this paper we develop a parameterised particle filter, called $S\alpha S$ particle filter. It propagates the filtering density

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through sequential evaluation of the parameters of the stable density. Since $\alpha$-stable distributions incorporate the Gaussian distribution as a particular case, the $S\alpha S$ particle filter can be regarded as a natural generalisation of the GPF [7] and of the GSPF [1].

The structure of the paper is as follows. Section 2 recalls the main properties of the $\alpha$-stable distribution. Section 3 develops an $\alpha$-stable particle filter. Section 4 presents results with synthetic data examples and assesses the performance of the $S\alpha S$ particle filter in comparison with the standard PF and the GSPF. Finally, conclusions and open issues for future research are highlighted in Section 5.

## 2. BASIC PROPERTIES OF $S\alpha S$ DISTRIBUTIONS

The appeal of $S\alpha S$ distributions as a statistical model for signals derives from some important theoretical and empirical reasons. First, stable random variables satisfy the stability property which states that linear combinations of jointly stable variables are indeed stable. Second, stable processes arise as limiting processes of sums of independent identically distributed (i.i.d.) random variables via the generalised central limit theorem. Actually, the *only* possible non-trivial limit of normalised sums of i.i.d. terms is stable. On the other hand, strong empirical evidence suggests that many data sets in several physical and economic systems exhibit heavy tail features that justify the use of stable models [8].

Generally, there is no closed-form expression for the probability density function of $S\alpha S$ distributions. Consequently, the most convenient way to define them is by means of their characteristic function

$$\varphi(\omega) = \exp(j\delta\omega - \gamma|\omega|^{\alpha}) \tag{1}$$

where

- $\alpha$ is the *characteristic exponent* , with values $0 < \alpha \leq 2$. It is arguably the most important parameter as it determines the shape of the distribution. It controls the *heaviness* of the tails of the density function. A small positive value of $\alpha$ indicates severe impulsiveness, and thus tails are *heavier*, while a value of $\alpha$ close to 2 indicates more Gaussian type behaviour. A value of $\alpha = 1$ corresponds to Cauchy distribution.

- $\gamma$ is the *dispersion* parameter ($\gamma > 0$), which determines the spread of the density around the location parameter. It behaves in a similar way to the variance of the Gaussian density, and it is, in fact, equal to half the variance when $\alpha = 2$, for the Gaussian case.

- $\delta$ is the *location parameter* ($-\infty < \delta < \infty$). It corresponds to the *mean* for $1 < \alpha \leq 2$, and to the *median* for $0 < \alpha \leq 1$.

A $S\alpha S$ distribution characterised by the above three parameters is denoted as $\mathcal{S}(\alpha, \gamma, \delta)$. The case $\alpha = 2$ corresponds to the *Gaussian distribution*, while $\alpha = 1$ corresponds to the *Cauchy distribution*. The *density functions* in these two cases are given by

$$f_{\alpha=2}(\gamma, \delta; x) = \frac{1}{\sqrt{4\pi\gamma}} exp\left\{-\frac{(x-\delta)^2}{4\gamma}\right\}, \tag{2}$$

$$f_{\alpha=1}(\gamma, \delta; x) = \frac{\gamma}{\pi[\gamma^2 + (x-\delta)^2]} \,. \tag{3}$$

General $\alpha$-stable density members do not possess finite second or higher moments [16]. In particular, the variance of a stable distribution with $\alpha < 2$ does not exist, making the use of variance as a measure of dispersion meaningless. However, the *dispersion* of a stable random variable plays an analogous role to the variance. The larger the dispersion of an $\alpha$-stable variable is, the more spread it is around its location parameter [8].

## 3. AN ALPHA-STABLE PARTICLE FILTER

Consider a dynamic system described by the following discrete-time state-space model

$$\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k, \boldsymbol{v}_k), \tag{4}$$

$$\boldsymbol{y}_{k+1} = h(\boldsymbol{x}_{k+1}, \boldsymbol{w}_{k+1}), \tag{5}$$

where $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$ is the unobserved system state vector, $\boldsymbol{v}_k \in \mathbb{R}^{n_v}$ is the system noise assumed to belong to the class of $\alpha$-stable symmetric processes; $\boldsymbol{y}_k \in \mathbb{R}^{n_z}$ is the measurement vector, $\boldsymbol{w}_k \in \mathbb{R}^{n_w}$ is a white Gaussian noise, and $k$ is the discrete time. Functions $f(.)$ and $h(.)$ are nonlinear in general.

In many statistical signal processing problems the primary objective is the reconstruction of the filtering probability density function $p(\boldsymbol{x}_k|\boldsymbol{y}_{1:k})$, with $\boldsymbol{y}_{0:k} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$ being the set of all measurements available up to the moment $k$. Denote with $N$ the number of particles.

The standard particle filtering technique is a *non- parametric* inference technique, whilst the $\alpha$-stable particle filter that we develop is a *parametric* technique, since it reduces the uncertainty to the calculation of the parameters of the $\alpha$-stable distribution. The $\alpha$-stable particle filter represents an extension to the GPF and of the GSPF. The GPF is based on the particle filtering concept, and it approximates the filtering and predictive state distributions by single Gaussians. The algorithm achieves this by the propagation of the first two moments of the Gaussian distribution, namely the mean and covariance through particles.

In a similar way, the $S\alpha S$PF approximates the filtering and predictive state densities by $\alpha$-stable densities using the Monte Carlo methodology. It propagates the parameters of the $S\alpha S$ distribution, $\alpha$, $\gamma$ and $\delta$ through particles.

The $S\alpha S$PF algorithm is described in *Table 1*.

**Table 1** The symmetric $\alpha$-stable particle filter

**Initialisation**

For $k = 0$ draw samples from $\mathcal{S}(\boldsymbol{\alpha}_0, \boldsymbol{\gamma}_0, \boldsymbol{\delta}_0)$ and denote them $\{\boldsymbol{x}_0^{(j)}\}_{j=1}^N$, for each state vector component $i = 1, \ldots, n_x$. In order to sample from a stable distribution we use the method proposed by Chambers et. al [17]. Set initial weights $W_0^{(j)} = 1/N$.

**Time update**

For $k = 0, 1, 2, \ldots$,

1. For $j = 1, \ldots, N$, sample $\boldsymbol{x}_{k+1}^{(j)} \sim p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k^{(j)})$, from the motion model (4).

**Measurement update**

2. For $j = 1, \ldots, N$, compute the weights

$$W_{k+1}^{(j)} = \frac{p(\boldsymbol{y}_{k+1}|\boldsymbol{x}_k^{(j)})\mathcal{S}(\boldsymbol{\alpha}_k, \boldsymbol{\delta}_k, \boldsymbol{\gamma}_k)}{q(\boldsymbol{x}_k^{(j)}|\boldsymbol{y}_{0:k+1})} \quad (6)$$

Under the assumption that the importance function $q(\boldsymbol{x}_k^{(j)}|\boldsymbol{y}_{0:k+1}) = \mathcal{S}(\boldsymbol{\alpha}_k, \boldsymbol{\delta}_k, \boldsymbol{\gamma}_k)$, then the weights are equal to:

$$W_{k+1}^{(j)} = p(\boldsymbol{y}_{k+1}|\boldsymbol{x}_{k+1}^{(j)}).$$

3. Normalise the weights $W_{k+1}^{(j)} = W_{k+1}^{(j)} / \sum_{j=1}^N W_{k+1}^{(j)}$.

4. Update the characteristic exponent $\boldsymbol{\alpha}_{k+1}$, and the other parameters $\boldsymbol{\gamma}_{k+1}$ and $\boldsymbol{\delta}_{k+1}$ from the shifted particles $W_{k+1}^{(j)}\boldsymbol{x}_{k+1}^{(j)}$ using the $\log|S\alpha S|$ method proposed in [18].

**Output**

5. Estimate the overall state

$$\hat{\boldsymbol{x}}_{k+1} = \sum_{j=1}^N W_{k+1}^{(j)}\boldsymbol{x}_{k+1}^{(j)}, \quad (7)$$

6. Set $k = k + 1$ and return to step 2.

An advantage of the $S\alpha S$PF compared to the PF and the GSPF is that the $S\alpha S$PF does not require a resampling procedure. Note that the resampling step in the PF is with respect to the particles, whilst in the GSPF it is applied to the mixing components. The $S\alpha S$PF relies on a parametric representation of the filtering and predictive state densities, whilst the PF representations of the state densities is non-parametric, and in this sense the PF is more general than the $S\alpha S$PF. On the other hand, the $S\alpha S$PF represents a generalisation of the GPF and GSPF.

## 4. PERFORMANCE EVALUATION AND RESULTS

The developed $S\alpha S$PF is compared to a standard PF and a GSPF.

***Example 1.*** Consider the system model given in [7, 1]

$$x_{k+1} = 0.5x_k + 25\frac{x_k}{1 + x_k^2} + 8cos(1.2k) + v_k, \quad (8)$$

with a measurement equation

$$y_{k+1} = x_{k+1} + w_{k+1}, \quad (9)$$

where $v_k \sim \mathcal{N}(0, Q)$, $w_k \sim \mathcal{N}(0, R)$, with $Q = 1$, $R = 2.5$. The $S\alpha S$PF and the PF are run with $N = 1000$ particles, whilst the GSPF with $N = 1000$ and 10 mixing components which means that the last has 10000 particles. The actual state and the state estimates for a single realisation for all filters are given in Figure 1, the mean errors are shown in Figure 2 over 100 Monte Carlo runs.

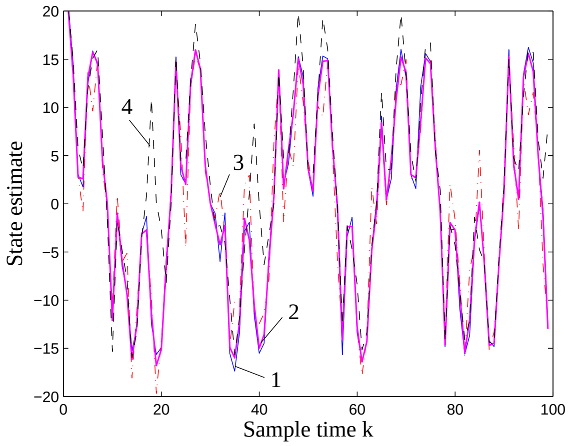


**Fig. 1**. State estimate from a single realisation: 1 - actual state, 2 - standard PF, 3 - $S\alpha S$PF, 4 - GSPF

***Example 2.*** This example considers a scalar system model with an additive $\alpha$-stable noise,

$$x_{k+1} = -2x_k + 1.5x_k^2 + v_k, \quad (10)$$

and measurement equation

$$y_{k+1} = x_{k+1} + w_{k+1}, \quad (11)$$

where $v_k \sim \mathcal{S}(\alpha_{noise}, \delta_{noise}, \gamma_{noise})$. The measurement noise is Gaussian $w_k \sim \mathcal{N}(0, R)$, with the following parameters: $\alpha_{noise} = 1.5$, $\gamma_{noise} = 0.5$, $\delta_{noise} = 0$, $R = 1$.

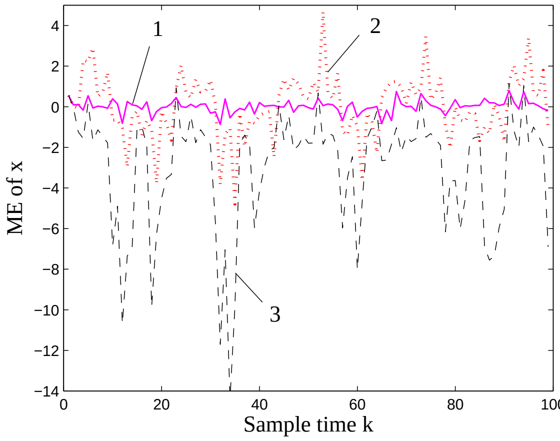


**Fig. 2**. Mean error from 100 independent Monte Carlo runs: 1 - standard PF, 2 - $S\alpha S$PF, 3 - GSPF

As in *Example 1*, the $S\alpha S$PF and the PF are run with $N = 1000$ particles, whilst the GSPF with $N = 1000$ and 10 mixing components which means that it has 10000 particles. Since the model noise is $\alpha$-stable, we draw samples from the $v_k \sim \mathcal{S}(\alpha_{noise}, \delta_{noise}, \gamma_{noise})$ and samples from $x_k \sim \mathcal{S}(\alpha_x, \delta_x, \gamma_x)$.

The state estimate from a single realisation is presented in Figure 3, the corresponding mean error from 100 independent Monte Carlo runs is shown in Figure 4. As it can be seen from the figures, the standard PF and the $S\alpha S$PF offer comparable results, while clearly outperforming the GSPF.

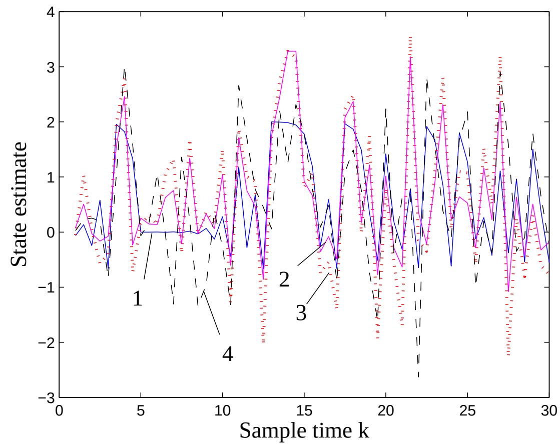


**Fig. 3**. State estimate from a single realisation: 1 - actual state, 2 - standard PF, 3 - $S\alpha S$PF, 4 - GSPF

The reason for this is that a Gaussian mixture distribution with a small number of components can not accurately model heavy-tail distributions whilst $S\alpha S$ distributions can more accurately model these heavy tail signals. Regarding the computational complexity the $S\alpha S$PF is more efficient because it estimates only the two parameters of the $\alpha$-stable distribution, whilst the GSPF approximates two moments per mixing component.

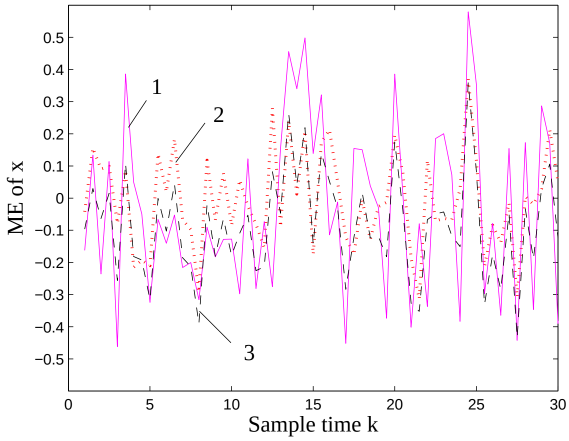


**Fig. 4**. Mean error from 100 independent Monte Carlo runs: 1 - standard PF, 2 - $S\alpha S$PF, 3 - GSPF

## 5. CONCLUSIONS AND FUTURE WORK

In this paper we have proposed an alpha-stable particle filter framework and investigated its performance. It is more general than the Gaussian sum particle filter and propagates the filtering and predictive state distributions through an update of the parameters of the $\alpha$-stable distribution. Results over synthetic data examples are presented.

So far the iterative calculation of the parameters of the $\alpha$-stable distribution were performed only for first-order systems. Since the multivariate $\alpha$-stable distribution has no covariance, methods to estimate its *spectral measure* should be devised. This is an open issue for future research, as well as the validation of the $S\alpha S$PF framework over applications with real data. The developed $S\alpha S$PF can be applied to different signal processing problems such as in queueing networks and systems with impulsive noises.

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