# Factorising integrable operators beyond Airy

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Painlevé's first equation $-q'' = 6q^2 + 2x$ arises in string theory; let $q$ be the monotone solution for $x > 0$ as constructed by Slemrod. For suitable $2 \times 2$ real matrices with zero trace, we introduce the differential equation $\frac{d}{d\lambda}\Psi = (T_2\lambda^2 + T_1\lambda + T_0)\Psi$, which generalises Airy's equation, then form the Lax pair of this with $\frac{d}{dx}\Psi = \begin{bmatrix} 0 & 1 \\ \frac{1}{2}\lambda - q & 0 \end{bmatrix}\Psi$. By analogy with the Airy kernel, we introduce a kernel $K$ from the solutions of $\frac{d}{d\lambda}\Psi = (T_2\lambda^2 + T_1\lambda + T_0)\Psi$; so $K$ is an integrable operator of Tracy–Widom type. We introduce an energy matrix $E$ and suppose that there exists a solution $\Psi$ of bounded energy. We prove that $K$ can be factorised as $K(\xi, \eta) = \int_0^\infty \langle \Phi(\xi + w), \sigma\Phi(\eta + w)\rangle \, \mathrm{d}w$ for some constant signature matrix $\sigma$. We follow a similar procedure for a Lax pair associated with Painlevé's second transcendental equation.

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## 1. Introduction

The string equation arises in Hermitian matrix models of 2D quantum gravity, as considered by Douglas [6]. One can consider the Schrödinger equation

$$\frac{\partial^2}{\partial x^2} f(\xi, x) + 2q(x) f(\xi, x) = \lambda f(\xi, x) \tag{1.1}$$

where $\xi$ undergoes an evolution. With $g(\xi, x) = \frac{\partial}{\partial x} f(\xi, x)$, Tracy & Widom observed [14] that the kernel

$$K(\xi, \eta) = \frac{f(\xi, x) g(\eta, x) - f(\eta, x) g(\xi, x)}{\eta - \xi} \tag{1.2}$$

is analogous to kernels that arise in random matrix theory. For example, the differential equation

$$-\frac{\partial^2}{\partial x^2} f(s, x) + x f(s, x) = -s f(s, x) \tag{1.3}$$

arises when one considers the soft edge of the eigenvalue distribution of Hermitian matrices with Gaussian random entries.

The Airy function

$$\mathrm{Ai}(t) = \int_{-\infty}^{\infty} e^{it\xi + \xi^3/3} \frac{\mathrm{d}\xi}{\sqrt{2\pi}} \tag{1.4}$$

satisfies the equation $\mathrm{Ai}''(t) = t\mathrm{Ai}(t)$ and $\mathrm{Ai}(t) \asymp \frac{1}{2}\pi^{-1/2} t^{-1/4} \exp(-2/3 t^{3/2})$ as $t \longrightarrow \infty$ [9]. McLeod and Hastings ([9], theorem 1 (iii)) proved that there exists a unique solution $y(t)$ to the Painlevé II equation

$$y''(t) = 2y(t)^3 + t y(t) - \frac{1}{2} - \theta \tag{1.5}$$

such that

$$y(t) \asymp k \mathrm{Ai}(t) \tag{1.6}$$

for some $k > 0$ as $t \longrightarrow \infty$.

Tracy & Widom considered the integral operator on $L^2(0, \infty)$ that has solution $f(s, x) = \mathrm{Ai}(s + x)$ and analysed the kernel

$$\frac{\mathrm{Ai}(s + x)\mathrm{Ai}'(t + x) - \mathrm{Ai}(t + x)\mathrm{Ai}'(s + x)}{s - t}, \tag{1.7}$$

making essential use of the fundamental identity

$$\mathrm{Ai}(x + s, y + t) = \int_0^{\infty} \mathrm{Ai}(x + s + u)\mathrm{Ai}(y + t + u)\, \mathrm{d}u. \tag{1.8}$$

In this paper we prove an analogous result.

Let $L = \frac{\partial^2}{\partial x^2} + 2q(x)$ and $P = \frac{\partial^3}{\partial x^3} + 3q\frac{\partial}{\partial x} + \frac{3}{2}q$ be the linear differential operators that are associated with the Schrödinger and Korteweg-de Vries equations. Then the string equation

$$[L, P] = 1 \tag{1.9}$$

reduces to a form of the Painlevé I equation

$$-q'' = 6q^2 + 2x + c \tag{1.10}$$

(as in [13], p50) with $2q$ the potential of the Schrödinger equation (1.1) and $c$ a constant. We assume without loss that $c = 0$.

Let

$$T = T_2\lambda^2 + T_1\lambda + T_0, \tag{1.11}$$

where

$$T_0 = \begin{bmatrix} -z & 4q \\ q^2 - \frac{1}{2}x & z \end{bmatrix}, \tag{1.12}$$

$$T_1 = \begin{bmatrix} 0 & 4 \\ -q & 0 \end{bmatrix}, \tag{1.13}$$

$$T_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{1.14}$$

and $z(x)$ and $q(x)$ are real differentiable functions; note that all the matrices have trace zero.

Let $S = \begin{bmatrix} 0 & 2 \\ \frac{1}{2}\lambda - q & 0 \end{bmatrix}$ and $\Psi = \begin{bmatrix} f \\ g \end{bmatrix}$. We begin by introducing the Lax pair

$$\frac{\partial}{\partial x}\Psi = S\Psi \tag{1.15}$$

and

$$\frac{\partial}{\partial \lambda}\Psi = -T\Psi. \tag{1.16}$$

There are various linearizations of the Painlevé equations by Lax pairs [10, 11]. Here we use the Lax pair of Jimbo *et al* [10] which has the advantage of requiring only $2 \times 2$ matrices.

**Lemma 1.1.** *If $q$ satisfies $P_I$, then*

$$\frac{\partial T}{\partial x} + \frac{\partial S}{\partial \lambda} = [S, T] \tag{1.17}$$

*and the differential equations are compatible.*

**Lemma 1.2.** (Slemrod [13]) *There exists a solution of (1.10) such that*

1. $s \longmapsto q(-s)$ *is monotone decreasing*
2. $q(-s) \asymp -\frac{1}{\sqrt{3}}\sqrt{s}$ *as* $s \longrightarrow \infty$
3. $-\frac{1}{\sqrt{3}}\sqrt{s} - \frac{1}{2}s^{-3/4} < q(-s) < -\frac{1}{\sqrt{3}}\sqrt{s}$
4. $\lambda + \frac{2}{\sqrt{3}}\sqrt{s} < \lambda - 2q(s) < \lambda + \frac{2}{\sqrt{3}}\sqrt{s} + s^{-3/4}$.

*Proof.* See Slemrod's paper [13].

The choice of sign is due to scaling transformations carried out on the Painlevé I equation, made to enable us to work with solutions defined on the positive rather than the negative axis, as this is more natural.

We make further scaling transformations to allow us to use $P_I$ in the form of (1.10); we introduce

$$Q(s) = \frac{6}{12^{2/5}} q\Big( - \frac{s}{12^{1/5}}\Big), \tag{1.18}$$

which satisfies $P_I$ in the form

$$-Q'' = Q^2 - s \tag{1.19}$$

□

Setting $\lambda^2 = w$ gives

$$2\frac{d}{dw}\Psi = \Big(\sqrt{w}T_2 + T_1 + \frac{1}{\sqrt{w}}T_0\Big)\Psi. \tag{1.20}$$

Making the transformation of variables $\lambda^2 = w$ to get from (1.11) to (1.20) preserves operator monotonicity. Whereas $\lambda \longmapsto \lambda^2$ is not operator monotone increasing, $w \longmapsto \sqrt{w}$ and $w \longmapsto -\frac{1}{\sqrt{w}}$ are both operator monotone increasing on $(0, \infty)$; the relevance of this was observed in [2].

For a solution $\Psi$ of (1.20) take the energy

$$E = \Big\langle \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{\sqrt{w}-q} \end{bmatrix} \Psi(w), \Psi(w) \Big\rangle \ (w > 0) \tag{1.21}$$

and let

$$K(\xi, \eta) = \frac{\Big\langle J\Psi(\xi), \Psi(\eta)\Big\rangle}{\eta - \xi}, \quad (\xi \neq \eta) \tag{1.22}$$

for the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{1.23}$$

the kernel analogous to Tracy & Widom's kernel.

In many significant examples of kernels in random matrix theory one has a factorisation formula for $K$ as in (1.8) and (1.24) below. This enables one to use methods from the theory of linear systems to analyse $K$ as in [1, 2].

The main result of this paper is the factorisation theorem, analogous to (1.8).

**Theorem 1.3.** *Suppose that $\Psi$ is a solution of (1.24) such that $E$ is bounded. Let $\sigma$ be a constant diagonal matrix with diagonal entries $\pm$ 1. Then there exists a Hilbert space $H$ and $\Phi : (0,\infty) \longrightarrow H$ such that*

$$K(\xi,\eta) = \int_0^\infty \langle \Phi(\xi+w), \sigma\Phi(\eta+w)\rangle \, dw. \tag{1.24}$$

The remainder of the paper is arranged as follows. In §2 we recall basic identities concerning Pick functions which we require in the proof of the factorisation theorem. In §3 we state bounds on solutions of (1.20). In §4 we prove the main result. Boutroux [3] identified a solution of $P_I$ which is asymptotic to the Weierstrass elliptic function. This requires a different kind of analysis to that contained within this paper.

In §5 we prove an analogue to theorem 1.3 but for kernels associated with Painlevé's second equation. The $P_{II}$ equation can be expressed as the consistency condition for the Lax pair

$$\frac{dW}{d\mu} = \left(\mu\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -y & -(z+2y^2+t) \\ \frac{1}{2} & y \end{bmatrix} + \frac{1}{2\mu}\begin{bmatrix} \theta & 0 \\ z & -\theta \end{bmatrix}\right)W \tag{1.25}$$

$$\frac{dW}{dt} = -\left(\mu\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -y & 0 \\ \frac{1}{2} & y \end{bmatrix}\right)W \tag{1.26}$$

as considered by Harnad *et al.* [8] and stated in [11]. We obtain a similar factorisation to theorem 1.3.

Whereas Jimbo *et al.* [11] compute $2\times 2$ Lax pairs for all the Painlevé transcendental equations $P_I - P_{VI}$, we have not yet obtained a factorisation theorem for $P_{III}$ since this involves both $\lambda$ and $\lambda^{-2}$.

## 2. Matrices in the differential equation

In this section we make some basic definitions. Let $\sqrt{w} = \exp\left(\frac{1}{2}\log w\right)$ where we take the logarithm that has the principal branch of the argument. We consider

$$T(w) = \frac{1}{2}\left(\sqrt{w}T_2 + T_1 + \frac{1}{\sqrt{w}}T_0\right). \tag{2.1}$$

From the theory of Pick functions, we know

$$\sqrt{w} = \int_{-\infty}^0 \frac{\sqrt{|t|}}{\pi}\left(\frac{1}{t-w} - \frac{t}{t^2+1}\right) dt \tag{2.2}$$

for Re $w > 0$. See [5] for a general theory of such representations.

The following lemmas are elementary.

**Lemma 2.1.** *For any $w$ such that $Re\ w > 0$,*

1.

$$\frac{1}{\sqrt{w}} = \frac{1}{\pi}\int_0^\infty \frac{1}{w+u}\frac{\mathrm{d}u}{\sqrt{u}}, \tag{2.3}$$

2.

$$\int_0^\infty \frac{\sqrt{w}}{(w+\xi)(w+\eta)}\mathrm{d}w = \frac{\pi}{\sqrt{\xi}+\sqrt{\eta}}. \tag{2.4}$$

For the remainder of the paper we work with functions and vectors that have real values.

**Definition 1.** The rank of a finite $n \times m$ matrix $B$ is the dimension of the image of $B$ on $\mathbb{R}^m$. The signature of an $n \times n$ real symmetric matrix is the number of positive eigenvalues minus the number of negative eigenvalues.

**Lemma 2.2.** 1. *The matrices $JT_0$, $JT_1$ and $JT_2$ are all real symmetric.*

2. *For large positive $s$, and $q(-s)$ as in lemma 1.2(2)*
   *(i) $JT_0$ has rank 2 and signature 0,*
   *(ii) $JT_1$ has rank 2 and signature 0,*
   *(iii) $JT_2$ has rank 1 and signature -1.*

*Proof.* The proof of lemma 2.2 is by direct calculation from lemma 1.2(3). □

**Remark 2.3.** If we introduce a new variable $w$ and calculate $\det(wI - T)$ we obtain the equation

$$\det(wI - T) = w^2 - z^2 - 4\lambda^3 - 4q^3 + 2\lambda x + 2qx. \tag{2.5}$$

The characteristic equation $\det(wI - T) = 0$ reduces to

$$w^2 = z^2 + 4\lambda^3 + 4q^3 - 2\lambda x - 2qx. \tag{2.6}$$

The spectral curve given by (2.6) defines an elliptic curve in $(w, \lambda)$.

## 3. Transforming The Differential Equation

First we observe that for suitable choices of the constants, (2.1) reduces to a form of Bessel's equation.

**Definition 2.** The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ (see [7], 8.494 (10), p 922) are Bessel functions of the third kind defined by

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i}J_\nu(z)}{i\sin(\nu\pi)} \tag{3.1}$$

and

$$H_\nu^{(2)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i}J_\nu(z)}{-i\sin(\nu\pi)}, \tag{3.2}$$

where

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu} \tag{3.3}$$

is the Bessel function of the first kind of order $\nu$.

In the case given by $q = 0$ we can write down an explicit solution given by Bessel's equation, demonstrating that for certain choices of $q$ there exist bounded solutions to (2.1).

**Remark 3.1.** 1. *The differential equation* $\frac{d}{dw}\Psi = \begin{bmatrix} 0 & 2 \\ -\frac{1}{2}\sqrt{w} & 0 \end{bmatrix} \Psi$ *has general solution* $\Psi = \begin{bmatrix} y \\ y' \end{bmatrix}$ *where*

$$y(w) = w^{1/2} \left( a_2 H_\nu^{(2)}\left( -\frac{\sqrt{8}}{5} w^{5/4} \right) + a_1 H_\nu^{(1)}\left( -\frac{\sqrt{8}}{5} w^{5/4} \right) \right) \tag{3.4}$$

*for* $\nu = \frac{2}{5}$ *and constants* $a_1$ *and* $a_2$.

2. *For some particular values of the constants,*

$$y(w) \asymp w^{-1/8} \cos \frac{8}{5} w^{5/4} (1 + o(1)) \quad \text{as } w \longrightarrow \infty. \tag{3.5}$$

3. *For any solution of* $y'' = -\frac{1}{2}\sqrt{w}y$,

$$E = \frac{(y')^2}{\sqrt{w}} + \frac{1}{2} y^2 \tag{3.6}$$

*is decreasing with increasing* $w$, *so* $y$ *is bounded.*

Now we return to the general case of (2.1).

**Lemma 3.2.** 1. *Let* $\Psi$ *be a solution of*

$$\frac{d}{dw}\Psi(w) = \frac{1}{2}\left( \sqrt{w} T_2 + T_1 + \frac{T_0}{\sqrt{w}} \right) \Psi(w) \tag{3.7}$$

*for* $w > 1$ *and let*

$$E = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{\sqrt{w-q}} \end{bmatrix} \Psi(w), \Psi(w) \right\rangle. \tag{3.8}$$

*Then for* $w > 1$ *there exists* $C(q, z, s)$ *such that*

$$\log E(w) \leq \frac{2}{3} C(q, z, s) w^{3/4} + C_1(q, z, s) \tag{3.9}$$

*where* $C_1(q, z, s) = -\frac{2}{3} C(q, z, s) w_0^{3/4} + \log E(w_0)$.

2. *For suitable $q, z$ and $s$, (3.9) holds with $C(q,z,s)=0$, so $E$ is bounded.*

*Proof.* After some reduction of $\frac{dE}{dw}$ we find, using (1.20), that

$$\begin{aligned}\frac{dE}{dw} &\leq \left\langle \frac{2}{(\sqrt{w}-q)\sqrt{w}} \begin{bmatrix} -z(\sqrt{w}-q) & 4q\sqrt{w}-2s \\ 4q\sqrt{w}-2s & 8z \end{bmatrix} \Psi(w), \Psi(w) \right\rangle \\ &\leq \frac{C(q,z,s)}{w^{1/4}} \left\langle \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{\sqrt{w}-q} \end{bmatrix} \Psi(w), \Psi(w) \right\rangle \end{aligned} \tag{3.10}$$

for some constant $C(q,z,s)$. We obtain a suitable $C$ by simultaneously reducing the quadratic forms to diagonal forms by Lagrange's method. □

## 4. Factorisation Theorem

Let $\Psi$ be a solution as in lemma 3.2(2) and

$$K(\xi,\eta) = \frac{\left\langle J\Psi(\xi), \Psi(\eta) \right\rangle}{\eta-\xi}. \tag{4.1}$$

As $\Psi(\xi)$ and $\Psi(\eta)$ both have real entries we use a real inner product. Moreover, we have $\left\langle J\Psi(\xi), \Psi(\xi) \right\rangle = 0$, so $K$ is an integrable operator in the style of [14].

**Theorem 4.1.** *Suppose that $\Psi$ is a solution of (1.20) such that $E$ is bounded. Then there exists a Hilbert space $H$ and $\Phi : (0,\infty) \longrightarrow H$ such that*

$$K(\xi,\eta) = \lim_{\zeta\to\infty} \int_0^{\zeta} \langle \Phi(\xi+w), \sigma\Phi(\eta+w) \rangle \, dw. \tag{4.2}$$

*Proof.* By lemma 3.2 a solution

$$\Psi = \begin{bmatrix} f \\ g \end{bmatrix} \tag{4.3}$$

exists, where $f$ is bounded and $g$ is oscillating with amplitude that grows slowly.

Let

$$\sigma = \begin{bmatrix} -I_n & 0 \\ 0 & I_m \end{bmatrix}. \tag{4.4}$$

Consider

$$
\begin{aligned}
\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\frac{\left\langle J\Psi(\xi),\Psi(\eta)\right\rangle}{\eta-\xi} &= \frac{\left\langle J\Psi'(\xi),\Psi(\eta)\right\rangle+\left\langle J\Psi(\xi),\Psi'(\eta)\right\rangle}{\eta-\xi} \\
&= \frac{\left\langle JT(\xi)\Psi(\xi),\Psi(\eta)\right\rangle+\left\langle J\Psi(\xi),T(\eta)\Psi(\eta)\right\rangle}{\eta-\xi} \\
&= \frac{\left\langle (JT(\xi)+T(\eta)^tJ)\Psi(\xi),\Psi(\eta)\right\rangle}{\eta-\xi} \\
&= \left\langle \frac{\sqrt{\xi}JT_2-\sqrt{\eta}JT_2}{\eta-\xi}\Psi(\xi),\Psi(\eta)\right\rangle \\
&\quad + \left\langle \frac{JT_1-JT_1}{\eta-\xi}\Psi(\xi),\Psi(\eta)\right\rangle \\
&\quad + \left\langle \frac{\frac{JT_0}{\sqrt{\xi}}-\frac{JT_0}{\sqrt{\eta}}}{\eta-\xi}\Psi(\xi),\Psi(\eta)\right\rangle,
\end{aligned}
\tag{4.5}
$$

where we have used lemma 2.2(1).

The first term involves

$$
\begin{aligned}
\frac{JT_2\sqrt{\xi}+T_2^tJ\sqrt{\eta}}{2(\eta-\xi)} &= JT_2\left(\frac{\sqrt{\xi}-\sqrt{\eta}}{2(\eta-\xi)}\right) \\
&= -\frac{JT_2}{2\pi}\int_0^\infty \frac{\sqrt{w}}{(\xi+w)(\eta+w)}\mathrm{d}w,
\end{aligned}
\tag{4.6}
$$

further the second term vanishes

$$
JT_1+T_1^tJ=0 \tag{4.7}
$$

and by lemma 2.1(2), the final term gives

$$
\begin{aligned}
\frac{\frac{JT_0}{\sqrt{\xi}}+\frac{T_0^tJ}{\sqrt{\eta}}}{2(\eta-\xi)} &= \frac{JT_0}{2}\left(\frac{\frac{1}{\sqrt{\xi}}-\frac{1}{\sqrt{\eta}}}{\eta-\xi}\right) \\
&= \frac{JT_0}{2\pi}\int_0^\infty \frac{\mathrm{d}w}{(\xi+w)(\eta+w)\sqrt{w}}.
\end{aligned}
\tag{4.8}
$$

Thus

$$\begin{aligned}
&\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)K(\xi,\eta)\\
&\qquad=\left\langle\frac{JT_0}{2\pi}\int_0^\infty\frac{\mathrm{d}w}{(\xi+w)(\eta+w)\sqrt{w}}\Psi(\xi),\Psi(\eta)\right\rangle\\
&\qquad\quad+\left\langle-\frac{JT_2}{2\pi}\int_0^\infty\frac{\sqrt{w}\,\mathrm{d}w}{(\xi+w)(\eta+w)}\Psi(\xi),\Psi(\eta)\right\rangle\\
&\qquad=\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{d}w}{(\xi+w)(\eta+w)\sqrt{w}}\left\langle JT_0\Psi(\xi),\Psi(\eta)\right\rangle\\
&\qquad\quad-\frac{1}{2\pi}\int_0^\infty\frac{\sqrt{w}\,\mathrm{d}w}{(\xi+w)(\eta+w)}\left\langle JT_2\Psi(\xi),\Psi(\eta)\right\rangle
\end{aligned}\tag{4.9}$$

where $-JT_0\geq 0$ and $JT_2\geq 0$.

Now

$$\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{d}w}{(\xi+w)(\eta+w)\sqrt{w}}\left\langle JT_0\Psi(\xi),\Psi(\eta)\right\rangle\tag{4.10}$$

is bounded. Indeed, by lemma 3.2(2)

$$\left|\left\langle JT_2\Psi(\xi),\Psi(\eta)\right\rangle\right|\leq(c+\xi^{1/8})(c+\eta^{1/8}).\tag{4.11}$$

As

$$\int_0^\infty\frac{\sqrt{w}\,\mathrm{d}w}{(\xi+w)(\eta+w)}=\frac{\pi}{\sqrt{\xi}+\sqrt{\eta}}\tag{4.12}$$

by lemma 2.2, we therefore have

$$\begin{aligned}
&\left|-\frac{1}{2\pi}\int_0^\infty\frac{\sqrt{w}\,\mathrm{d}w}{(\xi+w)(\eta+w)}\left\langle JT_2\Psi(\xi),\Psi(\eta)\right\rangle\right|\\
&\qquad\qquad\leq\frac{1}{\sqrt{\xi}+\sqrt{\eta}}(c+\xi^{1/8})(c+\eta^{1/8}).
\end{aligned}\tag{4.13}$$

Thus

$$|K(\xi,\eta)|\leq\frac{|\xi|^{1/8}|\eta|^{1/8}+C}{\sqrt{\xi}+\sqrt{\eta}}\tag{4.14}$$

and hence

$$K(\xi,\eta)\longrightarrow 0\quad\text{as}\quad\xi,\eta\longrightarrow\infty.\tag{4.15}$$

Define the Hilbert spaces

$$H_0=L^2\left((0,\infty);\frac{dw}{\sqrt{w}}\right)\tag{4.16}$$

and

$$H_2=L^2\big((0,\infty);\sqrt{w}\,dw\big);\tag{4.17}$$

we consider $w$ as the variable of integration and $\xi$ and $\eta$ as parameters. For each $\xi$ we introduce $\phi_{0,\xi}(w) \in H_0$ and $\phi_{2,\xi}(w) \in H_2$ by the common formulae

$$\phi_{0,\xi}(w) = \frac{1}{\xi + w} \tag{4.18}$$

$$\phi_{2,\xi}(w) = \frac{1}{\xi + w}. \tag{4.19}$$

Then $\phi_{0,\xi}(w) \in H_0$ for each $\xi$ and $\phi_{0,\xi} : (0,\infty) \longrightarrow H_0$. Similarly for $\phi_{2,\xi}$; we have $\phi_{2,\xi}(w) \in H_2$ for each $\xi$ and $\phi_{2,\xi} : (0,\infty) \longrightarrow H_2$. Then

$$\begin{aligned} \|\phi_{0,\xi}(w)\|_{H_0} &= \left( \int_0^\infty \frac{\mathrm{d}w}{(\xi + w)^2 \sqrt{w}} \right)^{1/2} \\ &\leq \frac{C}{\xi^{3/4}} \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} \|\phi_{2,\xi}(w)\|_{H_2} &= \left( \int_0^\infty \frac{\sqrt{w}}{(\xi + w)^2} \mathrm{d}w \right)^{1/2} \\ &\leq \frac{C}{\xi^{1/4}}. \end{aligned} \tag{4.21}$$

As $JT_0$ has rank 2 and signature 0, we can write

$$-JT_0 = B_0^\dagger \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B_0 \tag{4.22}$$

for some $2 \times 2$ real symmetric matrix $B_0$. Let $H = (H_0 \otimes \mathbb{R}^2) \oplus (H_2 \otimes \mathbb{R}^2)$ and $\Phi : (0,\infty) \longrightarrow H$ be

$$\Phi(\xi) = \begin{bmatrix} \phi_{0,\xi}(w) \otimes B_0 \Psi(\xi) \\ \phi_{2,\xi}(w) \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Psi(\xi) \end{bmatrix}, \tag{4.23}$$

all components of which have real entries. Thus

$$\begin{aligned} &\left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) K(\xi, \eta) \\ &\quad = -\langle \phi_{0,\xi}, \phi_{0,\eta} \rangle \langle -\sigma B_0 \Psi(\xi), B_0 \Psi(\eta) \rangle \\ &\qquad - \langle \phi_{2,\xi}, \phi_{2,\eta} \rangle \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Psi(\xi), \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Psi(\eta) \right\rangle \\ &\quad = -\left\langle \Phi(\xi), \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \Phi(\eta) \right\rangle. \end{aligned} \tag{4.24}$$

For each $\zeta > 0$, we have

$$K(\xi, \eta) - K(\xi + \zeta, \eta + \zeta) = \int_0^\zeta \langle \Phi(\xi + w), \sigma \Phi(\eta + w) \rangle \, \mathrm{d}w. \tag{4.25}$$

Indeed

$$\begin{aligned}\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)&\big(K(\xi,\eta)-K(\xi+\zeta,\eta+\zeta)\big)\\ &=-\langle\Phi(\xi),\sigma\Phi(\eta)\rangle+\langle\Phi(\xi+\zeta),\sigma\Phi(\eta+\zeta)\rangle\\ &=\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\int_0^\zeta\langle\Phi(\xi+w),\sigma\Phi(\eta+w)\rangle\,\mathrm{d}w \end{aligned}\tag{4.26}$$

so

$$\begin{aligned}K(\xi,\eta)&-K(\xi+\zeta,\eta+\zeta)\\ &=\int_0^\zeta\langle\Phi(\xi+w),\sigma\Phi(\eta+w)\rangle\,\mathrm{d}w+R_\zeta(\xi-\eta)\end{aligned}\tag{4.27}$$

where $K(\xi,\eta)\longrightarrow 0$ as $\xi,\eta\longrightarrow\infty$ and $\int_0^\zeta\langle\Phi(\xi+w),\sigma\Phi(\eta+w)\rangle\,\mathrm{d}w\longrightarrow 0$ as $\xi,\eta\longrightarrow\infty$; hence $R_\zeta(\xi-\eta)=0$ and

$$K(\xi,\eta)-K(\xi+\zeta,\eta+\zeta)=\int_0^\zeta\langle\Phi(\xi+w),\sigma\Phi(\eta+w)\rangle\,\mathrm{d}w.\tag{4.28}$$

Since $K(\xi+\zeta,\eta+\zeta)\longrightarrow 0$ as $\zeta\longrightarrow\infty$, we deduce that

$$K(\xi,\eta)=\lim_{\zeta\to\infty}\int_0^\zeta\langle\Phi(\xi+w),\sigma\Phi(\eta+w)\rangle\,\mathrm{d}w.\tag{4.29}$$

□

**Remark 4.2.** Peller [12] has considered the singular numbers of Hankel operators with kernel $\Phi(x+y)$.

## 5. Linearization of the Painlevé II equation

In this section we repeat the method of proof of §4 for the Painlevé II equation. Here the analysis is simpler, as there are no square roots involved.

We introduce the matrices

$$A_2=\begin{bmatrix}0&1\\0&0\end{bmatrix},\tag{5.1}$$

$$A_1=\begin{bmatrix}-y&-(z+2y^2+t)\\ \frac{1}{2}&y\end{bmatrix},\tag{5.2}$$

$$A_0=\begin{bmatrix}\theta&0\\ z&-\theta\end{bmatrix},\tag{5.3}$$

which have zero trace, and let

$$A(\mu)=\mu A_2+A_1+\frac{1}{2\mu}A_0\tag{5.4}$$

(analogous to (1.20). Suppose that $y$ satisfies the $P_{II}$ equation

$$y'' = 2y^3 + ty - \frac{1}{2} - \theta. \tag{5.5}$$

Then we have a Lax pair

$$\frac{dW}{d\mu} = \Big(\mu A_2 + A_1 + \frac{1}{2\mu}A_0\Big)W \tag{5.6}$$

$$\frac{dW}{dt} = -\left(\mu\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -y & 0 \\ \frac{1}{2} & y \end{bmatrix}\right)W. \tag{5.7}$$

Direct calculation shows that the compatibility condition holds, as in lemma 1.1, when we take

$$z = -y' - y^2 - \frac{1}{2}t. \tag{5.8}$$

Observe that when $z = \theta = y = t = 0$, the equation (5.5) reduces to Airy's equation

$$\frac{dW}{d\mu} = \begin{bmatrix} 0 & \mu \\ \frac{1}{2} & 0 \end{bmatrix}W. \tag{5.9}$$

**Remark 5.1.** If $y$ is the solution (1.6) of $P_{II}$ then $z < 0$.

**Lemma 5.2.** *Let $\alpha \in \mathbb{N}$. Then there exists a unique rational solution $y$ of $P_{II}$ and $z < 0$ for all sufficiently large $t$.*

*Proof.* By theorem 5.2 ([4], p348), there exist monic polynomials $Q_n$ of degree $\frac{1}{2}n(n+1)$ such that $y_n$ gives a solution of (5.5):

$$\begin{aligned} y_n &= \frac{d}{dz}\left\{\ln\left[\frac{Q_{n-1}(z)}{Q_n(z)}\right]\right\} \\ &= \frac{Q'_{n-1}(z)}{Q_{n-1}(z)} - \frac{Q'_n(z)}{Q_n(z)}. \end{aligned} \tag{5.10}$$

Let $\alpha_n$ be the largest real root of $Q_{n-1}Q_n$. Then for $t > \alpha_n$,

$$y_n = \mathrm{O}\Big(\frac{1}{t}\Big) \quad \text{as } t \longrightarrow \infty. \tag{5.11}$$

Now

$$\begin{aligned} y'_n &= \frac{Q''_{n-1}(z)}{Q_{n-1}(z)} - \frac{Q''_n(z)}{Q_n(z)} - \left(\frac{Q'_{n-1}(z)}{Q_{n-1}}\right)^2 + \left(\frac{Q'_n(z)}{Q_n}\right)^2 \\ &= \mathrm{O}\Big(\frac{1}{t^2}\Big) \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{5.12}$$

It follows that

$$\begin{aligned} z &= -y' - y^2 - \frac{1}{2}t \\ &\leq 0 \end{aligned} \tag{5.13}$$

for suitably large $t$. $\square$

We now prove an analogue of theorem 4.1. Let

$$K(\xi,\eta) = \frac{\Big\langle JW(\xi), W(\eta)\Big\rangle}{\xi-\eta}. \tag{5.14}$$

**Theorem 5.3.** *Suppose that there exists an $L^2$ solution $W$ of (5.6) and let $K$ be the corresponding kernel. Let $y$ be either*

1. *the solution (1.6) of $P_{II}$ such that $y(t) \asymp Ai(t)$ as $t \longrightarrow \infty$ or*
2. *a rational solution of $P_{II}$.*

*Then*

$$\begin{aligned} K(\xi,\eta) = &\int_0^\infty \Big\langle JA_2W(\xi+s), W(\eta+s)\Big\rangle \,\mathrm{d}s \\ &- \int_0^\infty \Big\langle JA_0\frac{W(\xi+s)}{\xi+s}, \frac{W(\eta+s)}{\eta+s}\Big\rangle \,\mathrm{d}s. \end{aligned} \tag{5.15}$$

*Proof.* Consider

$$\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\frac{\Big\langle JW(\xi), W(\eta)\Big\rangle}{\xi-\eta} = \frac{\Big\langle (JA(\xi)+A(\eta)^tJ)W(\xi), W(\eta)\Big\rangle}{\xi-\eta}. \tag{5.16}$$

As in the proof of theorem 4.1, we can write

$$\frac{JA_2\xi + A_2^tJ\eta}{\xi-\eta} = JA_2 \tag{5.17}$$

$$\frac{JA_1 + A_1^tJ}{\xi-\eta} = 0 \tag{5.18}$$

$$\frac{\frac{1}{2\xi}JA_0 + A_0^tJ\frac{1}{2\eta}}{\xi-\eta} = -\frac{JA_0}{\xi\eta}. \tag{5.19}$$

So for large $\xi$ and $\eta$

$$\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)K(\xi,\eta) = \Big\langle JA_2W(\xi), W(\eta)\Big\rangle - \Big\langle \frac{JA_0}{\xi\eta}W(\xi), W(\eta)\Big\rangle. \tag{5.20}$$

We assume that $W$ is square integrable; thus we can consider

$$\begin{aligned} L(\xi,\eta) = &\int_0^\infty \Big\langle JA_2W(\xi+s), W(\eta+s)\Big\rangle \,\mathrm{d}s \\ &- \int_0^\infty \Big\langle JA_0\frac{W(\xi+s)}{\xi+s}, \frac{W(\eta+s)}{\eta+s}\Big\rangle \,\mathrm{d}s. \end{aligned} \tag{5.21}$$

Then

$$\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\big(K(\xi,\eta)-L(\xi,\eta)\big) = 0. \tag{5.22}$$

Now

$$K(\xi, \eta) - L(\xi, \eta) = R(\xi - \eta) \tag{5.23}$$

for some function $R$.

But $K(\xi, \eta) \longrightarrow 0$ and $L(\xi, \eta) \longrightarrow 0$ as $\xi, \eta \longrightarrow \infty$. Thus

$$R(\xi - \eta) = 0. \tag{5.24}$$

The statement follows.

□

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