Observation of a Noise-Induced Phase Transition with an Analog Simulator

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The first measurements are presented of the phase diagram of a noise-induced phase transition for a model, nonlinear system with multiplicative noise proposed by Horsthemke and Lefever. Measurements of two of the critical exponents are also presented. The present results are in good agreement with theoretical predictions based on the Stratonovic treatment of the stochastic evolution equation.

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It has been observed that in certain first-order, nonlinear, differential equations a fluctuating parameter can cause the appearance of new, statistically favored states, which were unknown to the system in the absence of fluctuations. The new states appear when the fluctuation intensity exceeds some critical value, and hence have been called noise-induced phase transitions (NIPT's) by Horsthemke and Lefever, who first studied their properties.[1] The differential equations are phenomenological representations of some nonlinear processes subject to external, multiplicative noise with applications in physics, chemistry, and biology.[2] Currently, there is intense interest in a very similar problem in nonlinear stochastic dynamics with application to dye-laser statistics.[3,4] Previous attempts to observe NIPT's in natural systems[5,6] have not been completely satisfactory. While switching phenomena have been observed, no quantitative results which could be compared to the theory were possible.

In this Letter, we report the first measurements of the stationary statistical density made using an analog simulator. This technique has made it possible to (1) observe the NIPT directly as the appearance of double maxima in the density, (2) measure the critical noise intensity and hence obtain the phase diagram, and (3) measure the critical exponents. Our results are in quite good agreement with the theory obtained with use of the Stratonovic, rather than the Ito, stochastic calculus, and are, to our knowledge, the first quantitative results able to distinguish between these two theoretical approaches. An important motive for this work is to demonstrate the ease with which the detailed statistical properties of nonlinear, stochastic systems can be observed.

We have chosen to simulate the genetic model[2]

$$dX/dt = \tfrac{1}{2} - X + \lambda_t X(1-X), \qquad (1)$$

where the noisy parameter is $\lambda_t \to \lambda + \sigma\xi_t$. Here ξ_t is the derivative, in the sense of generalized functions, of the Wiener process W_t, and hence $\sigma\xi_t$ is a Gaussian, white noise of variance σ^2 and zero mean, so that $\langle\lambda_t\rangle = \lambda$. Equation (1) thus becomes

$$dX = f(X,\lambda)dt + \sigma g(X) dW_t, \qquad (2)$$

where $g(X,t) = X(1-X)$, and $f(X,\lambda) = \frac{1}{2} - X + \lambda X(1-X)$. Equations (1) and (2) are, in fact, not defined for white noise. (van Kampen has called them "pre-equations.")[7] Depending upon how one chooses to approximate the right-hand side of Eq. (2), one obtains the Fokker-Planck equation for the probability density function $\rho(X,t)$ in either the Ito version,

$$\partial_t \rho = -\partial_X f(X,\lambda)\rho(X,t) + \tfrac{1}{2}\sigma^2 \partial_X{}^2 g^2(X)\rho(X,t), \qquad (3a)$$

or the Stratonovic version,

$$\partial_t \rho = -\partial_X (f + \tfrac{1}{2}\sigma^2 g' g)\rho + \tfrac{1}{2}\sigma^2 \partial_X{}^2 g^2 \rho, \qquad (3b)$$

where $g' \equiv \partial_X g$. Equations (3) have the well known, stationary ($\partial_t \rho = 0$) solutions

$$\rho_s = N g^{-\nu} \exp\{(2/\sigma^2)\int^X [f(z)/g^2(z)]\,dz\}, \qquad (4)$$

where $\nu = 1$ or 2 in the Stratonovic or Ito versions, respectively, and N is some normalizing constant. In the case of our model, Eq. (4) results in

$$\rho_s = N[X(1-X)]^{-\nu} \exp\{-[X(1-X)\sigma^2]^{-1} - (2\lambda/\sigma^2)\ln[(1-X)/X]\}. \qquad (5)$$

This function has a single maximum for $\sigma < \sigma_c$, and double maxima for $\sigma > \sigma_c$. For the case $\lambda = 0$, when $\nu = 1$ (Stratonovic), $\sigma_c^{(0)} = 2$; and when $\nu = 2$ (Ito), $\sigma_c^{(0)} = \sqrt{2}$. For $|\lambda| > 0$, $\sigma_c > \sigma_c^{(0)}$, and the curve $\sigma_c(\lambda)$ constitutes the phase diagram.

Fokker-Planck systems with external noise have been discussed at length in the literature,[8] as have the Ito and Stratonovic formulations.[7,9] While the Ito approach is a mathematically correct treatment of the white-noise problem, it is not general. van Kampen[7] and West *et al.*[9] have pointed out that the Stratonovic version should be used to describe dynamical systems, since a theorem due to Wong and Zakai[10] obtains that result as the white-noise limit of a "real" (i.e., band-limited) noise problem. In the case of the model Eq. (1), the two approaches result in values for the critical noise variance, σ_c^2, which differ by a factor of 2 when $\lambda = 0$, so that the question can be easily resolved.

The simulator of Eq. (1) is shown in Fig. 1. All sum and difference operations and the two integrations are accomplished with standard operational-amplifier circuits. Multiplications are performed by commercially available analog multipliers.[11] The simulator was designed with a scale factor of unity, so that its output, $X(t)$, in volts is numerically comparable to the response of Eq. (1). In the steady state ($dX/dt = 0$) and for $\sigma = 0$, Eq. (1) has the (stable) solution

$$X_s = (1/2\lambda)[\lambda - 1 + (1 + \lambda^2)^{1/2}], \tag{6}$$

and we were able to adjust the simulator to agree with this to within ±4% over the range $-7 \leq \lambda \leq +7$. The slow integrator, with a time constant of 3 sec, functions primarily to stabilize the mean response so that $\langle X(t)\rangle = X_s$ at its output. The bandwidth of the slow integrator is so narrow that the noise response at its output is reduced by a factor of $\sim 2 \times 10^{-4}$. The fast integrator takes advantage of the fact that $\langle f(X, t)\rangle \cong 0$, so that the noise on $X(t)$ can be integrated separately and subsequently added to $\langle X(t)\rangle$.

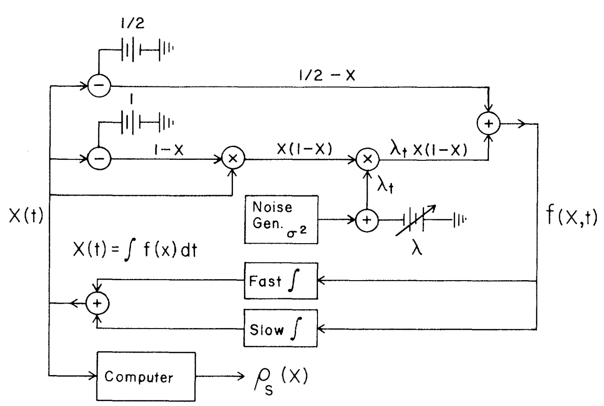

FIG. 1. The analog simulator.

Most of the data were obtained with the bandwidth of the noise generator limited to an upper cutoff frequency of 300 Hz, though we have also made measurements for a number of cutoff frequencies between 60 and 1500 Hz, with no significant effect on the results. Since even at 60 Hz the noise correlation time is much shorter than the dynamical response time of Eq. (1), $X(dX/dt)^{-1}$, for $|\lambda| \leq 7$, the simulator should approximate the predictions of the white-noise theories. Our results do not depend significantly on the time constant of the fast integrator. Increasing this time constant beyond the noise correlation time simply has the effect of reducing the noise amplitude at the integrator output, but measurements, for example, of σ_c are not affected.

Stationary density functions $\rho_s(X)$ are measured with the computer shown in Fig. 1, according to the following algorithm: (1) a time series of typically 4096 digitized points of $X(t)$ is obtained from the simulator; (2) the occurrence frequency of X between X and $X + \Delta X$ is computed for $0 \leq X < 1$ (volt) typically with a resolution of 1

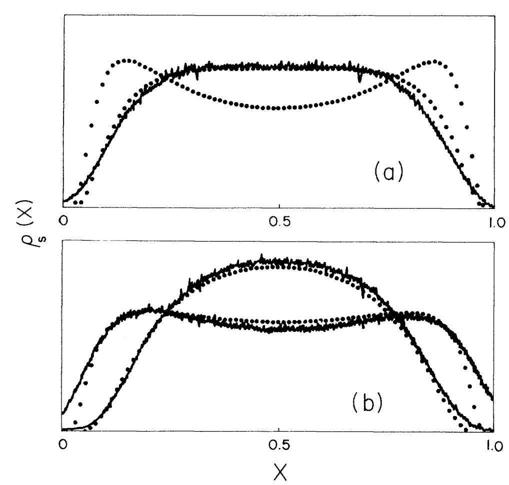

FIG. 2. (a) A measured density function for $\sigma = 2.0$ (continuous curve) compared to Eq. (5) for $\sigma = 2$ (solid circles); Ito, $\nu = 2$ (double maxima), and Stratonovic, $\nu = 1$ ($\simeq$ zero slope). (b) Measured density functions (continuous curves) for $\sigma = 1.5$ (single maximum) and $\sigma = 2.5$ (double maximum) compared to the Stratonovic version of Eq. (5) for the same values of σ (solid circles).

part in 256 and stored as $\rho_1(X)$; (3) the first time series is erased and a second obtained with the new frequency added to ρ_1 to form ρ_2; (4) the steps (1)–(3) are repeated n times (typically $n = 2048$) until the resulting $\rho_n \simeq \rho_s$ is suitably averaged. Some examples of ρ_s for $\lambda = 0$ are shown in Fig. 2 (continuous curves) compared to the calculated results from Eq. (5) (solid circles). Figure 2(a) shows a measured result for $\sigma = 2$ along with the two ($\nu = 1, 2$) calculated densities also for $\sigma = 2$. The curve with the double maxima (Ito) clearly does not describe the measured density well. Figure 2(b) shows results for $\sigma = 1.5$ (single maximum) and $\sigma = 2.5$ (double maximum) compared to the Stratonovic densities.

The phase diagram is obtained by finding σ_c for $|\lambda| \geq 0$. We have obtained these data by first choosing a value for λ, then measuring ρ_s for several values of σ near σ_c. A density with a region of nearly zero slope identified $\sigma \simeq \sigma_c$. The results are shown in Fig. 3, where the measured data are represented by the open circles, while the predictions of the theory are shown by the solid curves. The random errors in these measurements are approximately equivalent to the size of the symbols. The single error bar shown represents a systematic error related to the accuracy of the steady-state response of the simulator compared to Eq. (6). Specifically, we have found that measured values for σ_c are quite sensitive to the slope of the steady-state response $(dX_s/d\lambda)_{\lambda\to 0}$. From Eq. (6), $\lim_{\lambda\to 0}(dX_s/d\lambda) = \frac{1}{4}$. The data shown in Fig. 3 were obtained for adjustments of the simulator such that $(dX_s/d\lambda)_{\lambda\to 0}$ was always within about 2% of $\frac{1}{4}$. However, if this slope was decreased (increased) by about 6%, the entire data set was shifted upward (downward) to near the top (bottom) of the systematic error bar. The Stratonovic prediction is therefore nicely bracketed by these data.[12]

We have also made preliminary measurements of two of the critical exponents for this transition. Following Lefever and Horsthemke,[13] we identify the peak separation m for $\sigma > \sigma_c$ as an order parameter. Then we would expect that, in the neighborhood of the transition,

$$m = \tfrac{1}{2}[(\sigma^2 - \sigma_c^2)/\sigma_c^2]^{\beta} \quad (7)$$

for $\lambda = 0$, and

$$m \simeq (\lambda)^{1/\delta} \quad (8)$$

for $\sigma = \sigma_c$. Measurements of m vs σ and λ were carried out in a straightforward way, and the re-

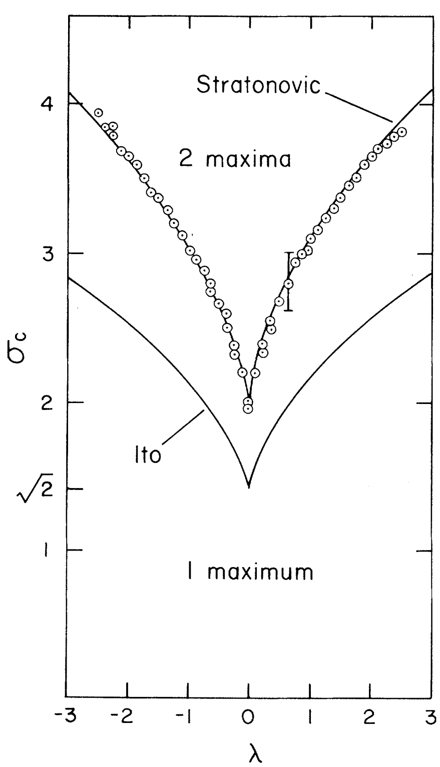

FIG. 3. The phase diagram. The measured data (open circles) are compared to the two theories (solid curves).

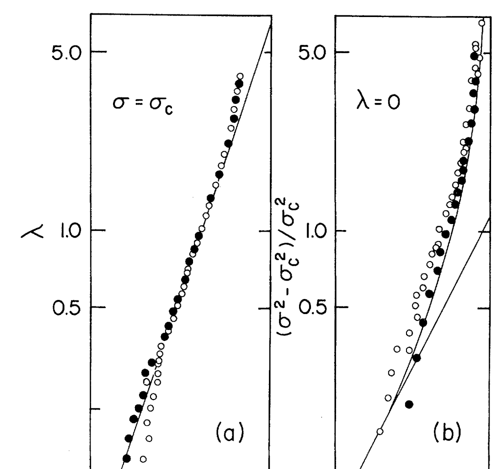

FIG. 4. (a) The peak separations vs λ for $\sigma = \sigma_c$. The open (solid) circles represent $\lambda > 0$ ($\lambda < 0$). The straight line has slope 3 ($\simeq \delta$). (b) The peak separation vs the reduced variance for $\lambda = 0$ and $\sigma > \sigma_c$. The solid (open) circles assume $\sigma_c = 2$ ($\sigma_c = 1.9$) in computing the ordinate. The curve is proportional to Eq. (7) with $\beta = \frac{1}{2}$. The straight line has slope 2 ($\simeq 1/\beta$).

sults are shown in Fig. 4 compared to the solid lines which represent the classical values $\beta = \frac{1}{2}$ and $\delta = 3$. The susceptibility $\chi = \lim_{\lambda\to 0}(\partial m/\partial\lambda)$ as a function of σ^2 is more difficult to measure with accuracy, so that we cannot provide an estimate for γ at present.

To summarize, we have provided the first measurements of the phase diagram and two of the critical exponents for a model noise-induced phase transition, and our results are in substantial agreement with theory based on the Stratonovic stochastic calculus.

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