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Swept-parameter-induced postponements and noise on the Hopf bifurcation

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The postponement of Hopf bifurcations driven by a swept bifurcation parameter is demonstrated with an electronic Brusselator. Noise on the swept parameter destroys the postponement and the bifurcation as it is usually defined.

We have recently studied two examples of dynamical systems with swept bifurcation parameters in the presence of noise.[1,2] In both of these systems, the bifurcation was of the simplest "pitchfork" type, i.e., it represented the transition from a monostable to a bistable state. In the first[1] work the statistics of branch selection were studied as the bifurcation parameter was swept into the region of bistability. In the second[2] the magnitude of the postponement of the bifurcation was studied as a function of sweep velocity, noise intensity, and noise correlation times. In both cases the noise was additive. Our essentially experimental studies were made on analog simulators of the appropriate Langevin equation, and were motivated by prior theoretical works: in the first instance, a proposed mechanism for the selection of molecular chirality,[3] and in the second, a dynamical switching process in a ring-laser model.[4]

In this Brief Report we consider the simplest Hopf bifurcation: the transition from a fixed point to a limit cycle, driven by a swept and noisy bifurcation parameter. We observe that the bifurcation is postponed, and that both the postponement and the Hopf bifurcation itself are destroyed by the noise. A deterministic theory of the swept-parameter-induced postponement of the Hopf bifurcation in the Brusselator has been reported, but only in abstract form.[5] The origin of the postponements is undoubtedly a critical slowing down at the bifurcation, but, to our knowledge, no theory has yet been published in complete form. There also exists a recent theory of the purely noise-induced postponements for this bifurcation.[6]

Our measurements were made on an electronic model of the Brusselator:

$$\dot{x}=A-[1+B(t)]x+x^2y \ , \tag{1}$$

$$\dot{y}=B(t)x-x^2y \ , \tag{2}$$

with a time-dependent bifurcation parameter

$$B(t)=vt+V_n(t,\tau) \ . \tag{3}$$

In practice, $B(t)$ was driven by a triangular wave of slope v (V/s), to which was added a colored noise voltage V_n of intensity D, and correlation time τ, defined by

$$\langle V_n \rangle = 0 \ , \tag{4a}$$

$$\langle v_n(t)V_n(s)\rangle=(D/\tau)e^{-|t-s|/\tau} \ . \tag{4b}$$

The electronic model and its operation have been described previously[7] and will not be further detailed here. We need only recall that the Brusselator limit cycle frequency for $B>B_c=1+A^2$ is ω_B, and that its inverse scales time in our circuit. The dimensionless correlation time and sweep velocity,

$$\tau=\tau_n\omega_B \ , \tag{5a}$$

$$V=v\omega_B^{-1} \ , \tag{5b}$$

where τ_n and v are the corresponding unscaled quantities, can be conveniently defined. In the course of a sweep, V measures the change in $\langle B(t)\rangle$ during one period of the limit cycle. During this experiment we set $\omega_B=10^4$ s^{-1}. The actual noise correlation time τ_n varied from 10^{-5} s ($\tau=0.1$, or quasiwhite noise) to 10^{-3} s ($\tau=10$, or very colored noise).

The deterministic ($V_n=0$) and static ($v=0$) steady states of Eq. (1) (with $A=1$) are, for $B<B_c$, given by $x_s=1$ and $y_s=B$. The deterministic and static bifurcation point is at $B_c=2$. For $B>B_c$ (but not too close to B_c) the limit cycle grows in amplitude as $(B-B_c)^{1/2}$. When $B(t)$ is swept in time ($v>0$) the bifurcation point is postponed to a new value $B^*>B_c$.

Example measured trajectories of $x(t)$ and $y(t)$ for $V_n=0$ are shown in Fig. 1 for $V=13.3$ mV ($v=0.133$ V/ms). It is evident that the onset of at least the large amplitude oscillations is delayed, but it is not immediately obvious how the postponed bifurcation point can be defined.

In Fig. 2 we have shown two measured trajectories of the function $[x(t)-1]^2$ which is approximately zero for $B<B_c$. The measurement of this function rather than $x(t)$ or $y(t)$ alone has two advantages: first, since $x_s=$const for $B<B_c$, there is no time delay inherent in the system's approach to B_c so that the observed postponements are associated only with the bifurcation process; and second, since $[x(t)-1]=0$ for $B<B_c$, we are able to substantially increase the sensitivity of the measuring apparatus in order to look more closely at the bifurcation point. The upper trace (sensitivity $\times 1$) shows that the quadratic behavior of the limit cycle amplitude with B

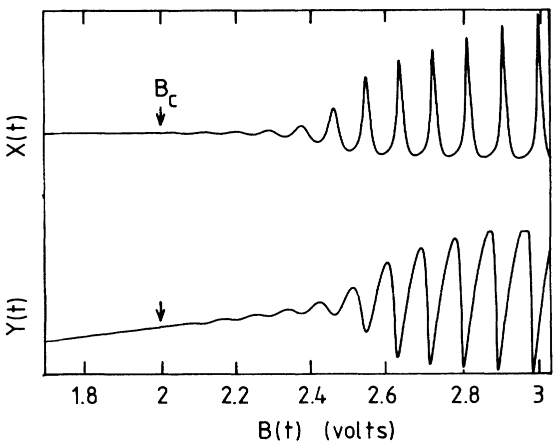

FIG. 1. Example trajectories of $x(t)$ and $y(t)$ plotted on an arbitrary vertical scale in volts for $V_n=0$ and for $V=13.3$ mV ($v=0.133$ V/ms).

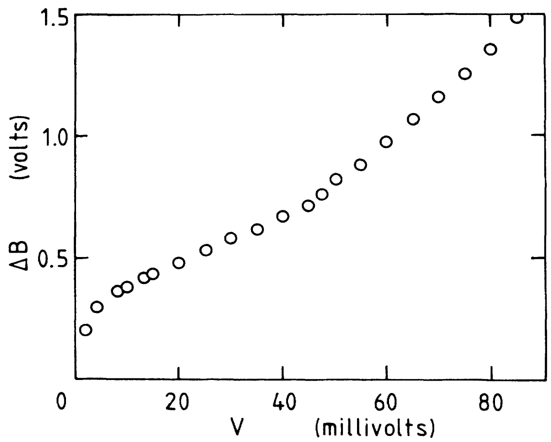

FIG. 3. The postponements vs sweep velocity for $V_n=0$.

for $B>B_c$, true for the static bifurcation, is also preserved in the dynamic case. The postponed bifurcation parameter B^*, may then be defined as the point at which the amplitude of $[x(t)-1]^2$ extrapolates to zero as shown by the dashed line.

The lower trace shows the same trajectory with amplitude multiplied by a factor of 64, where the small precursor oscillations in the range $B_c<B<B^*$ anticipate the onset of the quadratic amplitude dependence for B well above B^*. Both traces in Fig. 2 are the result of signal averaging 200 individual sweeps of $B(t)$. It is remarkable that the limit-cycle oscillations are phase coherent with the $B(t)$ sweep and that the coherence is robust enough to survive the 200-sample signal average. As we show below even very small external noise deliberately added to $B(t)$

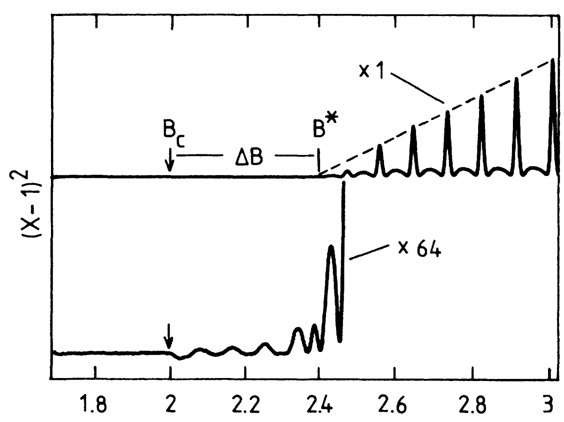

FIG. 2. The function $(x-1)^2$, plotted on an arbitrary vertical scale in V^2, as measured with 200 samples of signal averaging for $V=13.3$ mV and for $V_n=0$. The linear relation between squared limit-cycle amplitude and B, shown by the dashed line, defines the postponed bifurcation point B^*. The lower trace is the same trajectory but plotted on a vertical scale 64 times more sensitive than the top trace.

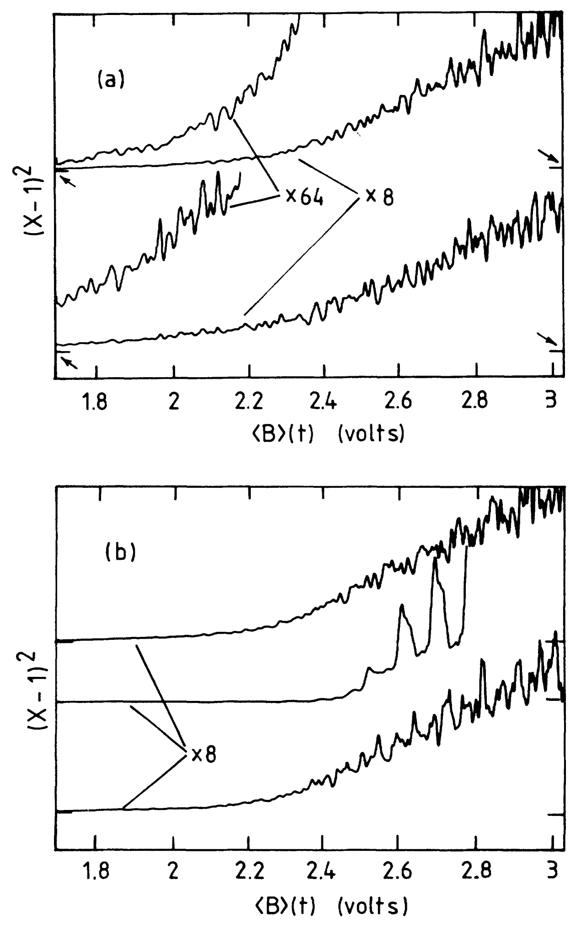

FIG. 4. The function $(x-1)^2$ measured with a 200-sample signal average, for (a) $\tau=1$ and $\langle V_n^2\rangle=0.01$, upper traces; $\langle V_n^2\rangle=0.04$, lower traces; (b) $\tau=0.1$, top trace; $\tau=10$, bottom two traces; with D held constant, middle trace; with $\langle V_n^2\rangle$ held constant, lower trace. See text for values.

destroys this coherence, and the signal-averaged trajectories become stochastic. Figure 2, for which the external noise is zero, therefore also indicates that the internal noise of the circuit has a negligible effect on the results. The postponed bifurcation is, no doubt, finally triggered by the internal noise, so that B^* is actually a statistical quantity. The persistent coherence, however, indicates that it must have a distribution much narrower than the width of a single limit cycle, and hence results in no measurable effect in this experiment. The postponed bifurcation can therefore be considered essentially as the result of a deterministic mechanism.

We have measured the magnitude of the postponement $\Delta B = B^* - B_c$, as defined in Fig. 2, for zero external noise as a function of V. The results are shown in Fig. 3. Notably large postponements (order 1 V) are induced by small (order 100 mV) sweep velocities.

It is not possible to make similar measurements when $V_n \neq 0$, because significant limit-cycle amplitude is evident even for $B < B_c$. This is shown in Fig. 4(a) where a small amount of colored noise ($\tau = 1$) has been added to the sweep. The upper pair of traces are for $\langle V_n^2 \rangle = 0.01$ V^2 (with $D = \tau \langle V_n^2 \rangle = 0.01$ V^2). The lower pair show the effect of increasing the noise intensity to $\langle V_n^2 \rangle = 0.04$ V^2. In both cases considerable offsets from zero, marked by the left and right tick marks indicated by arrows, are evident, so that it is not possible to identify any single bifurcation point. This difficulty has been discussed previously in terms of the power spectra,[8,9] and the two-dimensional statistical density of the limit cycle.[7]

Figure 4(b) shows the result of changing τ. The upper trace is for $\tau = 0.1$ and $\langle V_n^2 \rangle = 0.025$ V^2 (with $D = 0.0025$ V^2). The middle trace shows the effect of increasing τ to $\tau = 10$ while keeping $D = 0.0025$ V^2 constant ($\langle V_n^2 \rangle = 2.5 \times 10^{-4}$ V^2). The bottom trace is for $\tau = 10$ but for constant $\langle V_n^2 \rangle = 0.025$ V^2 ($D = 0.25$ V^2). It is evident that increasing τ to large values while keeping D constant moves the system toward the deterministic response, while if $\langle V_n^2 \rangle$ is held constant instead, increasing τ has little effect. We conclude that deterministic Hopf bifurcations can be sustantially postponed by sweeping the bifurcation parameter in time, and that noise on this parameter destroys the bifurcation when its definition is based on the mean limit-cycle amplitude.

Note added in proof. We have recently been informed of theoretical work on related problems by Wallet (Ref. 10).

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