PHYSICAL REVIEW A VOLUME 35, NUMBER 1 JANUARY 1, 1987

Skewed probability densities in the ring-laser gyroscope: A colored noise effect

K. Vogel and H. Risken
Abteilung für Theoretische Physik, Universität Ulm, D-7900 Ulm, West Germany

W. Schleich
*Center for Theoretical Physics, Department of Physics and Astronomy, University of Texas at Austin, Austin, Texas 78712 and Center for Advanced Studies, Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131 and Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, West Germany**

Mark James and Frank Moss
Department of Physics, University of Missouri–St. Louis, St. Louis, Missouri 63121

P. V. E. McClintock
Department of Physics, University of Lancaster, Lancaster LA1 4YB, United Kingdom

(Received 11 September 1986)

The influence of noise color on the phase diffusion in a ring-laser gyroscope is studied by use of an (formally) exact solution of the time-independent Fokker-Planck equation. Novel asymmetries in the distribution which develop with increasing noise correlation time are predicted and verified with an electronic model of the corresponding Langevin equation.

Conventional, active ring-laser gyroscopes (RLG's)[1] are limited by the well-known lock-in effect.[2] At low rotation rates the backscattering phase locks the counterpropagating waves, and the beat note between the waves disappears. Information about the rotation rate can be obtained in this dead band from the phase difference between the two waves,[3] but since a beat note is easier to detect, many methods to avoid locking have been studied.[1,2] One approach consists of deliberately introducing additional noise, which can reduce the size of the dead band.[4-7] All previous studies have been restricted to the case of Gaussian white noise as, for example, due to spontaneous emission of the laser atoms.[8] In contrast, external noise necessarily has a correlation time $\tau_c > 0$, and the resulting Fokker-Planck (FP) equation is two dimensional.[9] Apart from numerical solutions, only two methods for studying this problem have been demonstrated:[10] One consists of solving the FP equation in terms of infinite matrix continued fractions, and the other is a modeling of the Langevin equation by an electronic circuit.[11] At present, the continued-fraction method represents the only exact solution of a higher-dimensional FP equation.

Here we present solutions of an FP equation for the RLG in the presence of colored noise and compare them to measured results obtained from an electronic circuit model. We focus on the steady-state statistical density P_{ss}, though a discussion of other quantities of interest together with the detailed calculations[12] will be presented elsewhere.[13] We emphasize that the Langevin equation describing the evolution of the phase difference between two counterpropagating waves also arises in a number of other physical situations; for example, radio physics,[14] Josephson junctions,[15] self-locking of a laser,[16] and charge-density waves.[17] Thus there is a great interest in this particular equation and its solutions in the presence of fluctuations.

The equation of motion for the phase difference ϕ is given by[1]

$$\dot{\phi}=a+b\sin\phi+\varepsilon(t)\ , \tag{1}$$

where a is the Sagnac frequency.[1,2] The coupling between the two waves, due to the mirror imperfections, is accounted for by $b\sin\phi$, where b is the backscattering coefficient.[2] We consider Gaussian noise $\varepsilon(t)$ of strength D defined by

$$\langle\varepsilon(t)\varepsilon(s)\rangle=(D/\tau_c)e^{-|t-s|/\tau_c}\ . \tag{2}$$

Introducing $\dot{\varepsilon}=-(1/\tau_c)\varepsilon+F(t)$, where $\langle F(t)F(s)\rangle=(2D/\tau_c^2)\delta(t-s)$, a two-dimensional FP equation can readily be obtained:[9]

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\phi}[(a+b\sin\phi+\varepsilon)P]+\frac{1}{\tau_c}\frac{\partial(\varepsilon P)}{\partial\varepsilon}+\frac{D}{\tau_c^2}\frac{\partial^2 P}{\partial\varepsilon^2}\ . \tag{3}$$

We impose periodic boundary conditions for ϕ and natural boundary conditions for ε.

It has been shown[12,13] that

$$P_{ss}(\phi,\varepsilon)=(1/\sqrt{2\pi})\mathcal{H}_0(\varepsilon)\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\mathcal{S}_{m,n}\mathcal{H}_m(\varepsilon)e^{in\phi}\ ,$$

where $\mathcal{H}_m$ is given by

$$\mathcal{H}_m(\varepsilon)=\mathcal{N}_m\exp[-\varepsilon^2/(4D/\tau_c)]H_m(\varepsilon/\sqrt{2D/\tau_c})\ .$$

The normalization factors $\mathcal{N}_m$ are chosen to be $\mathcal{N}_m=(m!2^m\sqrt{2\pi D/\tau_c})^{-1/2}$, and the H_m are the familiar Hermite polynomials. Since P_{ss} is real and thus $\mathcal{S}_{m,-n}=\mathcal{S}^*_{m,n}$, only the coefficients $\mathcal{S}_{m,n}$ for $n\geq 0$ must be determined. They can be obtained by iteration from

$$\mathcal{S}_{m,n}=(\mathbf{S}_m)_n=(\underline{R}_m\mathbf{S}_{m-1})_n\ \text{ for } m\geq 1\ , \tag{4}$$

where $\underline{R}_m$ is the infinite matrix continued fraction

$$\underline{R}_m = -[\underline{A}_m + \underline{B}_m \underline{R}_{m+1}]^{-1}\underline{C}_m \ , \tag{5}$$

and where the matrices $\underline{A}_m$, $\underline{B}_m$, and $\underline{C}_m$ are defined by

$$(\underline{A}_m)_{n,n'} := -\left(ina + \frac{m}{\tau_c}\right)\delta_{n,n'} + \frac{nb}{2}(\delta_{n+1,n'} - \delta_{n-1,n'}) \ , \tag{6a}$$

$$(\underline{B}_m)_{n,n'} := -in\sqrt{D/\tau_c}\sqrt{m+1}\,\delta_{n,n'} \ , \tag{6b}$$

$$(\underline{C}_m)_{n,n'} := -in\sqrt{D/\tau_c}\sqrt{m}\,\delta_{n,n'} \ . \tag{6c}$$

The start vector $\mathbf{S}_0$ is determined by

$$(\underline{A}_0 + \underline{B}_0\underline{R}_1)\mathbf{S}_0 = 0 \ \text{ with } \mathcal{S}_{0,0} = 1/\sqrt{2\pi} \ . \tag{7}$$

We now briefly outline the procedure for calculating the coefficients $\mathcal{S}_{m,n}$. Substituting the matrices $\underline{A}_m$, $\underline{B}_m$, and $\underline{C}_m$ into Eq. (5) and using downward iteration[9] yields the matrices $\underline{R}_m$ and, in particular, $\underline{R}_1$. The start vector $\mathbf{S}_0$ is then determined from Eq. (7) by combining the results for $\underline{R}_1$ and Eqs. (6a) and (6b). Substituting the thus-calculated $\mathbf{S}_0$ together with $\underline{R}_1$ into Eq. (4) we arrive at $\mathbf{S}_2$. Continuing this iteration yields $\mathbf{S}_m$.

For the sake of clarity, the phase variable ϕ is displayed over two periods from -2π to 2π. For simplicity, we set $D = a = b = 1$. In the absence of noise, the rotation rate $a = b$ marks the boundary between running and locked solutions and is thus well suited for studies of the influence of τ_c.

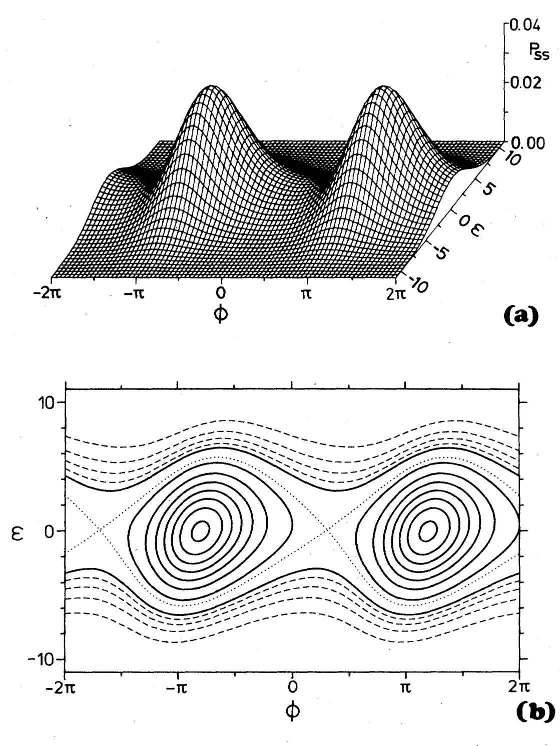

FIG. 1. The theoretical statistical density $P_{ss}(\phi,\varepsilon)$ for $D = 1$, $\tau_c = 0.10$ (nearly white noise), and $a = b = 1$. (a) The three-dimensional plot, and (b) contours of constant probability for equally spaced probabilities 0.005, 0.01 . . . (solid lines) and 0.001, 0.002 . . . (dashed lines). The separatrix (dotted line) is at 0.007 847.

An example solution is shown in Fig. 1 for small τ_c (approximately white noise), where we have plotted $P_{ss}(\phi,\varepsilon)$ in (a) and the corresponding contours of constant probability in (b). The effect of τ_c is illustrated by the contour plot shown in Fig. 2 for $\tau_c = 1$, but with all other parameters unchanged. A noticable effect of increasing τ_c is to markedly increase the ratio of peak height to saddle-point probability densities, as was observed[11,18,19] previously in the case of bistable systems. In addition, noise color has a profound influence on the shape of the contours. Whereas for small τ_c, $P_{ss}(\phi,\varepsilon)$ is more nearly symmetric about the $\varepsilon = 0$ axis, as shown in Fig. 1(b), increasing τ_c destroys this symmetry as is evident in Fig. 2. Moreover, the contours are skewed toward an axis running from lower left toward upper right for $a > 0$. (The reverse is true for $a < 0$.) Note that in Fig. 1(b), the scale of ε is ± 10, so that there is small skewing, whereas in Fig. 2 the scale of ε is ± 3.3, showing considerable skewing. These properties have a strong influence on the locking characteristic of the RLG, since the mean beat frequency $\langle\langle\dot{\phi}\rangle_\varepsilon\rangle_t$ is obtained by averaging over the noise using $P_{ss}(\phi,\varepsilon)$. Therefore the noise color induced skewing and asymmetry are of technological importance as well as of academic interest.

In order to test these predictions we have constructed the electronic circuit model shown in Fig. 3, which mimics Eq. (1) to a steady-state accuracy of a few percent. Apart from the hybrid analog-digital system providing the $\sin\phi$ forcing (shown within the broken line box in Fig. 3) the basic principle of operation is identical to that of a bistable system discussed previously.[11] We measure and display ϕ over two periods from -2π to 2π. Periodic boundary conditions (PBC) are achieved by a special circuit (see Fig. 3) which resets the integrator so that $\phi_b \to -\phi_b$ (with $\dot{\phi}$ held constant) each time the trajectory $\phi(t)$ crosses $\phi_b = \pm 2\pi$. The noise generator and filter provide a time correlated noise voltage ε defined by Eq. (2), with dimensionless correlation time $\tau_c = \tau_n/\tau_i$, where τ_n is the actual correlation time and τ_i is the integrator time constant. In operation, $\phi(t)$ and ε are digitized in a

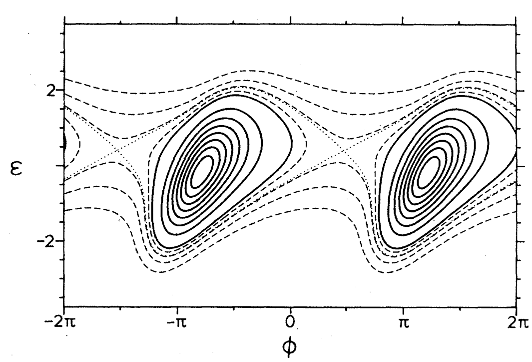

FIG. 2. Contour plot of $P_{ss}(\phi,\varepsilon)$ for $D = 1$, $\tau_c = 1.0$ (colored noise), and $a = b = 1$. The contours are drawn for equally spaced probabilities at 0.025, 0.05, . . . (solid lines) and 0.005, 0.010, . . . (dashed lines) with the separatrix at 0.015 76.

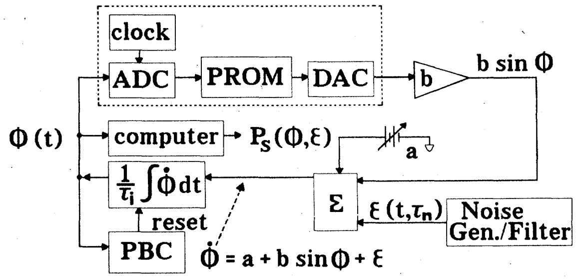

FIG. 3. The electronic model. The circuit is a closed-loop system which dynamically approximates Eq. (1). For this experiment the integrator time constant $\tau_i = 100$ μs, and the noise correlation time $\tau_n = 100$ μs (resulting in $\tau_c = 1$). The circuits which provide $\sin\phi$ forcing and periodic boundary conditions are discussed in the text.

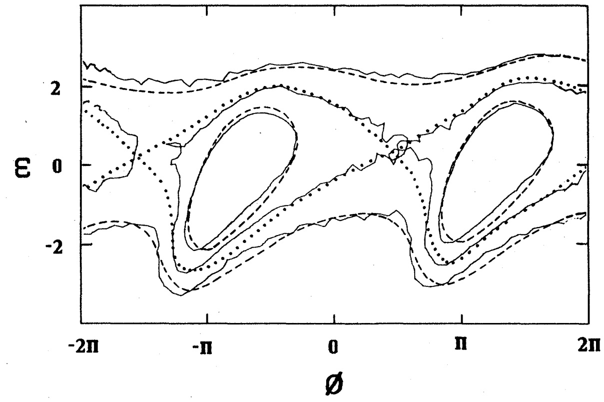

FIG. 4. A contour plot of $P_{ss}(\phi,\varepsilon)$ for $\tau_c = 1.0$ and $a = b = 1.0V/S$. Theoretical results (dashed lines) are replotted from Fig. 2 for $D = 1$ at $P_{ss} = 0.01576$ (dotted line separatrix), 0.05 (closed contour), and 0.005 (open contour). The measured results (solid lines) for the same densities were obtained for $D = 0.92 \pm 0.06V^2$.

time series of typically 5 million points, each from which the computer obtains $P_{ss}(\phi,\varepsilon)$.

In Fig. 4 we show a contour plot comparison of our measured results (solid lines) with the theoretical predictions (broken and dotted lines) for $\tau_c = 1.0$. In this example both the normalization and noise intensity D of the experimental results were adjusted (D only slightly) to achieve the fit to the theoretical separatrix as shown (dotted line). The open and closed contours then provide quantitative comparisons between theory and simulation. Though there are some discrepancies located where the density is most rapidly changing, we regard the overall agreement between theory and simulation as satisfactory, the errors being attributable to inaccuracies in the circuit.

We conclude that the matrix continued-fraction theory offers an accurate method for evaluating the effects of colored noise on systems with periodic potentials. In addition to providing an independent method for obtaining $P_{ss}(\phi,\varepsilon)$ rapidly and simply, the electronic circuit offers a straightforward model for similar measurements which could be made on a RLG.

The authors consider it their pleasant duty to thank D. Hammons, P. Hanggi, V. Sanders, M. O. Scully, and H. D. Vollmer for fruitful and stimulating discussions. One of us (W.S.) would like to thank J. A. Wheeler for his continuous encouragement. We are grateful to Mr. W. Garver for the design and construction of the $\sin\phi$ forcing circuit. This work was partially supported by the National Science Foundation Grant No. PHY 8503890, by the British Science and Engineering Research Council, and by NATO Grant No. RG.85/0770.

*Present and permanent address.

[1]For a review on ring-laser gyroscopes, see W. W. Chow, J. Gea-Banacloche, L. Pedrotti, V. Sanders, W. Schleich, and M. O. Scully, Rev. Mod. Phys. **57**, 61 (1985).

[2]F. Aronowitz, in *Laser Applications,* edited by M. Ross (Academic, New York, 1977), Vol. 1.

[3]M. O. Scully (unpublished).

[4]J. D. Cresser, W. H. Louisell, P. Meystre, W. Schleich, and M. O. Scully, Phys. Rev. A **25**, 2214 (1982).

[5]J. D. Cresser, D. Hammons, W. H. Louisell, P. Meystre, and H. Risken, Phys. Rev. A **25**, 2226 (1982).

[6]J. D. Cresser, Phys. Rev. A **26**, 398 (1982).

[7]For noise in dithered ring-laser gyroscopes, see W. Schleich, C.-S. Cha, and J. D. Cresser, Phys. Rev. A **29**, 230 (1984); W. Schleich and P. Dobiasch, Opt. Commun. **52**, 63 (1984).

[8]M. Sargent, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, 1974).

[9]H. Risken, in *The Fokker-Planck Equation, Methods of Solutions and Applications,* Springer Series in Synergetics, Vol. 18, edited by H. Haken (Springer, New York, 1984).

[10]However, some approximate analytical results in the limit of small correlation times have been given by A. A. Kutsak and O. M. Lyashko, Zh. Prikl. Spektrosk. **43**, 188 (1985) [J. Appl. Spectros. **43**, 834 (1985)].

[11]See, for example, F. Moss and P. V. E. McClintock, Z. Phys. B **61**, 381 (1985), and references therein.

[12]K. Vogel, diploma thesis, University of Ulm, Ulm, West Germany, 1985 (unpublished).

[13]K. Vogel, H. Risken, and W. Schleich (unpublished).

[14]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vols. I and II.

[15]See, for example, Yu. M. Ivanchenko and L. A. Zil'berman, Zh. Eksp. Teor. Fiz. **55**, 2395 (1968) [Sov. Phys. JETP **28**, 1272 (1969)]; M. Buttiker, E. P. Harris, and R. Landauer, Phys. Rev. B **28**, 1268 (1983); E. G. Gwinn and R. M. Westervelt, Phys. Rev. Lett. **54**, 1613 (1985); T. Bohr, P. Bak, and M. H. Jensen, Phys. Rev. A **30**, 1970 (1984).

[16]H. Haken, H. Sauermann, Ch. Schmid, and H. D. Vollmer, Z. Phys. **206**, 369 (1967).

[17]See, for example, G. Gruner and A. Zettl, Phys. Rep. **119**, 117 (1985).

[18]P. Jung and H. Risken, Z. Phys. B **61**, 367 (1985).

[19]P. Hanggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985); L. Fronzoni, P. Grigolini, P. Hanggi, F. Moss, R. Mannella, and P. V. E. McClintock, *ibid.* **33**, 3320 (1986); F. Moss, P. Hanggi, R. Mannella, and P. V. E. McClintock, *ibid.* **33**, 4459 (1986).