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Relaxation near a noise-induced transition point

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The transient behavior of a quadratic model system perturbed by a multiplicative white noise has been investigated. The relaxation time of the system, as a function of the noise intensity D, has been determined by analog experiment and by digital simulation. The results obtained are mutually consistent, but contradict a recent theoretical prediction by H. K. Leung [Phys. Rev. A **37**, 1341 (1988)] that there should be a critical slowing down of the system near the value of D for which a noise-induced transition occurs in the probability distribution. The discrepancy is resolved by deriving a new analytic result for the relaxation time, applicable to a range of systems described by separable stochastic differential equations.

Noise-induced transitions[1] are conventionally defined in terms of changes in the number of extrema in the probability distribution of a system variable. In at least some cases, for example the cubic bistable system,[2] it is known that a marked increase in the relaxation time T occurs in the vicinity of the transition, an effect that has naturally been described as "critical slowing down" in analogy with the well-known phenomenon associated with equilibrium phase transitions. In general, for a given system, it is only possible to obtain approximate results for T, based on moment expansion or on simulations. These techniques can produce tantalizingly different results and therefore, whenever possible, it is of crucial importance to obtain exact theoretical predictions.

In this paper, we report an exact analytic result for the relaxation time of a class of stochastic systems and apply it to a quadratic model discussed recently by Leung.[3] The model in question, introduced[4] by Eigen and Schuster to describe macromolecular self-replication under constraint and also relevant to a wide range of other applications (including, arguably, the spread of viral epidemics and the productivity of individual scientists, writers, and composers[5]), may be written

$$\dot{x} = W(x - x^2/\Omega) . \tag{1}$$

In the particular case of macromolecular self-organization, x represents the number of molecules which duplicate themselves precisely with a net replication rate W, and Ω relates to the size of the system. Leung[3] has considered the consequences of white noise fluctuations in W such that

$$W = W_0 + \xi(t) , \tag{2}$$

where W_0 is a constant, $\xi(t)$ is a Gaussian fluctuation with $\langle \xi(t) \rangle = 0$, and

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t') . \tag{3}$$

To investigate the stochastic transient behavior of the system, he employs a moment expansion approximation and a linear stability analysis, leading to the conclusion that the relaxation time T should diverge as the critical noise intensity D_c of the survival/extinction noise-induced transition is approached. Using the Stratonovich stochastic calculus, which is known[6] to be applicable to real physical systems, he finds that the critical value D_c/W is (with our above definition of D and for $\Omega = 1$) equal to 0.375.

Our motivation for deriving an analytic result for T arose after performing both digital and analog simulations to check this prediction. We first give a detailed comparison between the simulation results and the prediction of Leung and highlight a discrepancy between the two. Then, for a class of stochastic systems of which Eq. (1) is an example, by obtaining the Borel sum[7] of an asymptotic expansion for $\langle x(t) \rangle$, we derive an exact result for the relaxation time. This predicts, in agreement with the simulation results, that there is no divergence in the relaxation time of the system described by Eq. (1) and provides insight into why an approximate calculation of T, based on a finite moment expansion, will erroneously predict such a divergence.

The analog electronic simulation is based on the design principles discussed in more detail elsewhere.[8] A suitably scaled circuit was built to model Eq. (1), and its transient response was measured under a range of different conditions with the aid of a Nicolet 1080 data processor. In practice, the circuit was set initially such that $x = 0.2\Omega$ and was then allowed to evolve towards its final steady state at $x = \Omega$. A typical trajectory measured in this way, is shown in Fig. 1(a); both here, and in what follows, all the results normalized to the generic form of (1) with $W_0 = \Omega = 1$ in order to provide more convenient comparisons between the different experimental and calculated forms of $x(t)$, $\langle x(t) \rangle$, and $T(D)$. A series of such $x(t)$ trajectories was ensemble averaged until the statistical quality of the result was considered adequate; the sensitivity of the circuit to drift, causing small changes in its parameters, meant that the number of blocks in the average was usually limited to 50. An example is shown in Fig. 1(b). The nonlinear relaxation time,[9] defined as

$$T = \frac{\int_0^\infty [\langle x(t) \rangle - \langle x(\infty) \rangle] dt}{x(0) - \langle x(\infty) \rangle} , \tag{4}$$

where $x(0)$ and $x(\infty)$ are the initial and final states, respectively, was then computed from the experimental

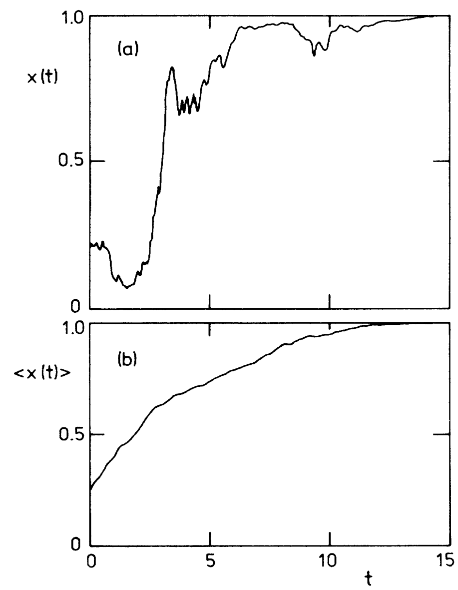

FIG. 1. (a) Typical trajectory of $x(t)$ for the system described by Eqs. (1)–(3) with constants $W_0=\Omega=1$ and the noise intensity $D=0.71$. (b) Ensemble average of 40 trajectories of the kind shown in (a). Note that the ordinate scale is slightly different.

$\langle x(t)\rangle$. (In practice, of course, the final state could not be at $t=\infty$, but was chosen to be such that x had become virtually independent of T.) The procedure was repeated for a range of values of D.

The experimental results are shown by the circled data of Fig. 2. For more convenient comparison with theory they have been normalized by dividing by the calculated (exact) deterministic ($D=0$) relaxation time T_0. The form of $T(D)$ predicted[3] by Leung is shown by the full curve, and can be seen to be in clear disagreement with the experimental data. In fact, we believe that Leung's equations lead to relaxation times that are smaller by about an order of magnitude that those he reports [for example, we find $T_0=-\ln x_0/(1-x_0)=2.0118$ for $x_0=0.2$]; these discrepancies in the absolute numerical values are of relatively minor importance, however, as compared to the relative *shapes* of Leung's result and of the experimental $T(D)$. The experiment suggests that T increases approximately linearly with D from $D=0$, whereas the theoretical prediction is a virtually constant T until very close to D_c, where $T(D)$ rises extremely rapidly and diverges as $D\rightarrow D_c$.

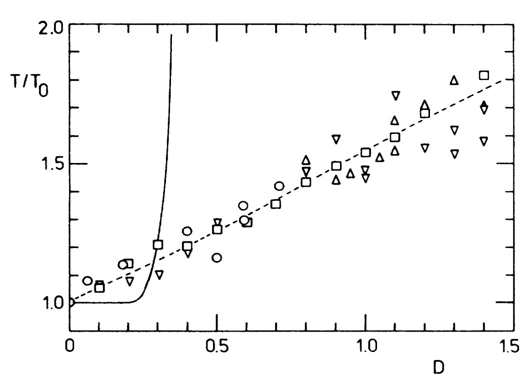

FIG. 2. The normalized nonlinear relaxation time T/T_0 of the system described by Eqs. (1)–(3) with $W_0=\Omega=1$, plotted as a function of the noise intensity D, after being determined by analog experiment (circles); digital simulation, using two variants of the algorithm (point-up and point-down triangles, see text); and stochastic calculation based on Eq. (6) (squares). The solid curve represents the theoretical prediction by Leung (Ref. 3) and the dashed curve is derived from Eq. (10).

To try to resolve the disagreement, we have also carried out digital simulations of (1), using the algorithm described in Ref. 10. One of the resultant stochastic trajectories and the corresponding moment average are shown in Fig. 3(a). In practice, if h is the integration time step, some of the integrations were done with the algorithm[10] at order h ($\triangle$ in Fig. 2), others with the algorithm[10] at order h^2 (∇ in Fig. 2). This is because, if increasing the order makes the algorithm more "precise," it also decreases the

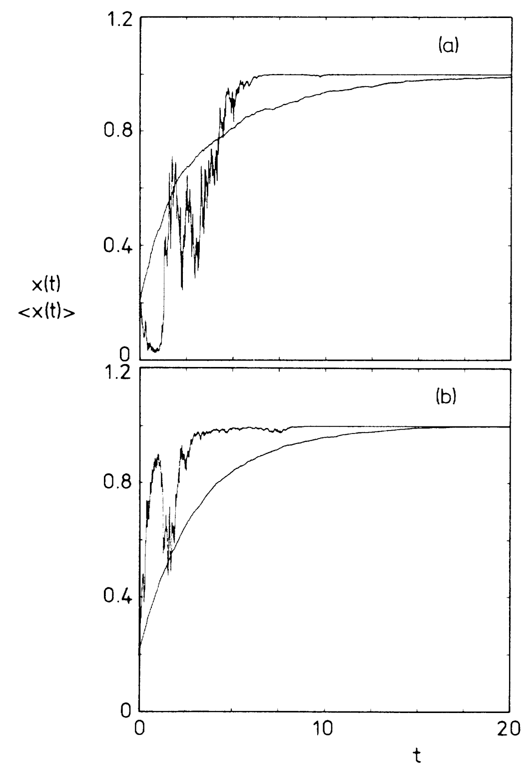

FIG. 3. Computations of the time evolution of the system described by Eqs. (1)–(3) with $W_0=\Omega=1$: (a) by digital simulation, with $D=1.5$, and (b) by stochastic calculation based on Eq. (6) with $D=1.0$. In each case, the jagged curve represents a typical individual $x(t)$ trajectory and the smoother curve is an $\langle x(t)\rangle$ ensemble average over many such trajectories: 400 in (a) and 1000 in (b). The computations actually extended out to $t=40$, and the full range $0<t<40$ was used for the calculation of relaxation times, but the (uninteresting) $20<t<40$ range has been omitted from the figure in order to exhibit to better effect the region where $x(t)$ and $\langle x(t)\rangle$ are changing rapidly.

stability. In particular we wanted to check that our results were accurate for both small and large D. Figure 2 shows that, within statistical uncertainty, this is indeed the case. For all simulations, the time step was 10^{-3}, the trajectories were followed for 40000 integration time steps, and the final result was obtained in each case by averaging 400 trajectories. The results obtained are consistent with those from the analog experiment.

As a check on the validity of these results, we have also calculated T by a third, quite independent, method. It can be seen immediately by inspection that a special simplifying feature of (1) is that the variables separate, so that it is straightforward to demonstrate that for $\Omega=1$,

$$\frac{x(t)}{1-x(t)}\frac{1-x(0)}{x(0)}=\exp\left[t-t_0+\int_{t_0}^{t}\xi(t)dt\right], \quad (5)$$

whence, taking the initial time $t_0=0$, we obtain

$$x(t)=\{1+B\exp[-t+\eta(t)]\}^{-1}, \quad (6)$$

where $\eta(t)=\int_0^t dt'\xi(t')$ and $B=[1-x(0)]/x(0)$. Here, the final term in the exponent, $\eta(t)$, is of course a stochastic variable (a Wiener process).[11] The right-hand side of this equation can thus be numerically integrated forward in time in a perfectly straightforward manner, thereby yielding the trajectory $x(t)$; ensemble averaging a series of such trajectories [Fig. 3(b)], as before, we can quickly find values of T from (4). The results of this procedure are shown by the square data points of Fig. 2 and are clearly in excellent agreement with results obtained by both of the other two methods. It would appear, therefore, that T does not diverge as predicted.[3]

To understand this discrepancy, we now derive an exact result for T, applicable to any nonlinear separable equation which, like (1), can be integrated to yield, as in Eq. (6),

$$x(t)=f(e^{\pm\eta(t)}), \quad (7)$$

where $f(y)$ is analytic at $y=0$. Equation (7) can formally be expanded in the form

$$\langle x(t)\rangle=\sum_{n=0}^{\infty}[f^{(n)}(0)/n!]\langle\exp[\pm n\eta(t)]\rangle, \quad (8)$$

and the average of the exponential evaluated by averaging its Taylor expansion term by term. Noting that $\langle[\eta(t)]^m\rangle$ vanishes for m odd and for m even is equal to $m!(Dt)^{m/2}/(m/2)!$, yields

$$\langle\exp[\pm n\eta(t)]\rangle=\exp(Dtn^2).$$

When substituted into Eq. (8), this yields an asymptotic expansion for $\langle x(t)\rangle$ which, using the identity

$$\exp(Dtn^2)=\pi^{-1/2}\int_{-\infty}^{\infty}d\phi\exp\{-[\phi^2+2\phi(Dt)^{1/2}n]\},$$

can be summed to yield

$$\langle x(t)\rangle=\pi^{-1/2}\int_{-\infty}^{\infty}d\phi f[2\phi(Dt)^{1/2}]\exp(-\phi^2). \quad (9)$$

When substituted into Eq. (4), this yields an exact theoretical result for the relaxation time T.

For the model of Eq. (1), the above analysis yields

$$\langle x(t)\rangle=\pi^{-1/2}\int_{-\infty}^{\infty}d\phi\{1+B\exp(-[t+2\phi(Dt)^{1/2}])\}^{-1}\times\exp(-\phi^2). \quad (10)$$

This result, if it were to be plotted in Fig. 3, would be indistinguishable from the digital results (apart from their small statistical fluctuations) for $\langle x(t)\rangle$. The exact result for T, obtained by substituting Eq. (10) in Eq. (4), is shown by the dashed line in Fig. 2. The remarkable agreement between this and the simulation results provides conclusive evidence that the relaxation time does not diverge in this system, despite the fact that the probability density for x switches from a monomodal to a bimodal distribution at $D=1.0$. The origin of Leung's prediction lies in the divergence of the asymptotic expansion leading to Eq. (9). For the system of Eq. (1), keeping only the first N terms in the expansion leads to a divergence in T at $D_c=1/N$, whereas the exact sum contains no such divergence. This feature expresses itself in Eq. (10), where it is evident that the limit $\tau\to\infty$ and the integration *do not commute*.

The results of the paper show that, for the model of Eq. (1), the noise induced transition from a monomodal to a bimodal distribution at $D=1$ is not reflected in the relaxation time T. In deriving this result, we have obtained an exact analytic formula which can be applied to a wide range of separable stochastic differential equations. The analysis may readily be extended[12] to cover the case of nonwhite Gaussian noise of arbitrary correlation.

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