Fluctuation-Induced Transitions between Periodic Attractors: Observation of Supernarrow Spectral Peaks near a Kinetic Phase Transition

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(Received 19 December 1989)

Fluctuation-induced transitions between coexisting periodic attractors have been investigated with an analog electronic circuit model. Calculations and measurements of the spectral densities of fluctuations have revealed superimposed twin-peaked partial spectra and a supernarrow spectral peak whose intensity depends critically on the distance from the phase transitions where the populations of the attractors equalize.

PACS numbers: 64.60.−i, 05.40.+j, 05.70.Ce, 05.70.Jk

Nonlinear systems that possess two or more stable dynamical states (periodic attractors) in an external periodic force have been the subject of intensive investigation during the last decade. Such studies have included, for example, work on a variety of bistable and multistable optical systems[1] and, of particular interest, a relativistic electron confined to a cavity in a magnetic field and excited by cyclotron resonant radiation.[2] The fluctuations (noise) that are unbiquitous to real physical systems give rise to transitions between coexisting attractors. In the case of weak noise, the resulting stationary populations of the stable states usually differ drastically and only for certain values of the parameters of a bistable system are they of the same order of magnitude. Such behavior is strikingly similar to that of thermal equilibrium systems with coexisting phases (e.g., liquid and vapor) which exist with an overwhelming probability in either one or the other of two phases; only very close to the phase transition itself do the populations of the phases (e.g., molar volumes) become of the same order of magnitude. By analogy, therefore, the parameter range where the attractors of the bistable system are approximately equally populated may be referred to as the range of a kinetic phase transition, and one may expect some specific fluctuation phenomena to arise there.

In this Letter we report the principal results of an investigation of fluctuational transitions and of a kinetic phase transition in a bistable periodically driven system for which the attractors are stable periodic states of forced vibration, with differing amplitudes and phases. We have studied escape rates, and, to reveal features of fluctuations in the transition range, we have investigated their spectral densities. The onset of a multipeak structure, including a spectral peak very much narrower than the reciprocal relaxation time of the system, has been observed. The specific model studied, a nearly resonantly driven nonlinear oscillator (single-well Duffing oscillator), is an archetypal example of systems that display bistability in a periodic field. This model describes, in particular, the relativistic electron[2] referred to above; it is also widely used for the analysis of nonlinear optical systems.[1,3] For stronger periodic driving amplitudes than those considered here, it displays[4] dynamical chaos.[5]

For the widely applicable case where the noise acting on a system is Gaussian and weak, with dimensionless intensity $\alpha \ll 1$, the raito of the populations w_1 and w_2 of the attractors 1 and 2 is exponential:[6]

$$w_1/w_2 = \text{const} \times \exp[(R_1 - R_2)/\alpha], \quad w_1 + w_2 = 1\,. \quad (1)$$

Here, R_1 and R_2 are the characteristic "activation energies" for the probabilities W_{12} and W_{21} of the transitions $1 \rightarrow 2$ and $2 \rightarrow 1$; they depend on parameters of the system and on the shape of the power spectrum of the noise, in the general case of colored noise,[7] but are independent of α. In practice, at small α, for almost all values of parameters of the system, $|R_1 - R_2| \gg \alpha$: thus, one of the attractors is occupied and the other is empty.

However, there will be a relation between the parameters such that $|R_1 - R_2| \leq \alpha$, in which case the populations of the attractors will become comparable. The condition $R_1 = R_2$ defines the point at which a kinetic phase transition occurs in a periodically driven dynamical system subject to weak noise: On opposite sides of this point the system, with a probability $\simeq 1$, occupies different states. The width of the phase transition range is $\sim\alpha$. Within this range one expects not only that small fluctuations will occur about the stable states, but also that comparatively large fluctuations related to transitions between the stable states will become prominent. The characteristic time scale for these fluctuations and for the relaxation of the population difference $w_1 - w_2$ is given by the reciprocal transition probabilities $W_{12}^{-1} \sim W_{21}^{-1}$. A direct and convenient way of investigating the fluctuations is through a determination of their spectral density. It is to be expected that the slow $[W_{ij} \propto \exp(-R_i/\alpha)$ being exponentially small compared to all other relaxation rates] relaxation of w_1, w_2 should give rise to peaks in the spectral density of fluctuations with a characteristic width $\sim W_{ij}$, i.e., to exponentially narrow spectral peaks. It is a characteristic feature of systems with coexisting periodic attractors that such "supernarrow" peaks will arise[6,8,9] at the frequency of the driving force and its overtones, and at zero frequency; we may note their close correspondence to the zero-frequency spectral peaks for Brownian particles fluctuating in static double-well potentials[6,10,11] and for generalized multistability[12] in periodically driven systems.

The equation of motion of the nonlinear oscillator investigated in the present paper is

$$\ddot{q}+2\Gamma\dot{q}+\omega_0^2 q+\gamma q^3=h\cos(\omega_h t)+f(t)\,. \qquad (2)$$

The force $f(t)$ is assumed to be a white Gaussian noise with correlator

$$\langle f(t)f(t')\rangle=2\Gamma B\delta(t-t')\,. \qquad (3)$$

In the underdamped case, $\Gamma \ll \omega_0$, and for nearly resonant (but not too strong) periodic forcing, $|\omega_h-\omega_0| \ll \omega_h$, it is convenient in analyzing the motion to transform to the rotating frame. It can then be shown[6,13] that the dynamics of the system is determined by two dimensionless parameters, Ω and β, which characterize, respectively, the frequency detuning and the strength of the periodic field,

$$\Omega=\frac{\omega_h-\omega_0}{\Gamma},\quad \beta=\frac{3|\gamma|h^2}{32\omega_h^3|\omega_h-\omega_0|^3}\,. \qquad (4)$$

The dimensionless noise intensity α is given by

$$\alpha=3|\gamma|B/16\omega_h^3\Gamma\,. \qquad (5)$$

The range of β and Ω within which the oscillator is bistable is enclosed by solid lines[6] of Fig. 1. The transition probabilities between the attractors were considered in Ref. 6; explicit expressions were obtained for the "activation energies" $R_{1,2}(\beta,\Omega)$ for both $\Omega^{-1}\to 0$ and near the spinode point K ($\beta=\frac{8}{27}$, $\Omega=\sqrt{3}$). These results may be used to interpolate the phase-transition line $\beta_0(\Omega)$, defined by $R_1(\beta,\Omega)=R_2(\beta,\Omega)$, between the arrow and dashed curve in Fig. 1.

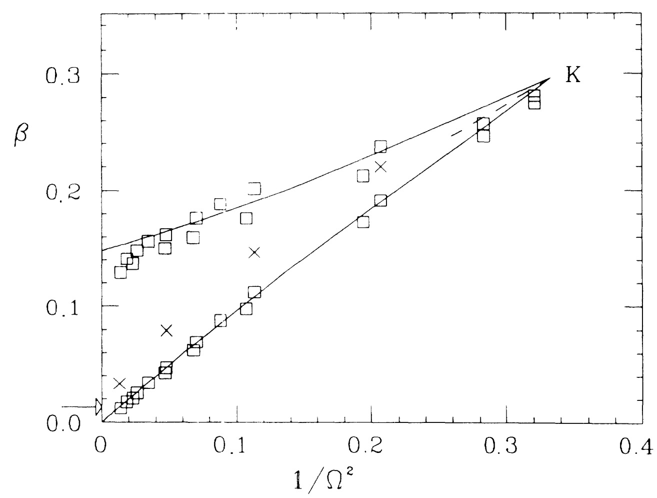

FIG. 1. Phase diagram showing the region of bistability of the system (2) in terms of the reduced parameters (4); the experimental data (squares) are compared with theory (solid curves). The theoretical phase transition interpolates from the dashed curve to the arrowed point; the measured points at which the transition rates between attractors are equal are shown by crosses.

The spectral density fluctuations of the coordinate of the oscillator is determined by

$$Q(\omega)=\frac{1}{\pi}\mathrm{Re}\int_0^\infty dt\exp(i\omega t)\bar{Q}(t)\,,$$
$$\bar{Q}(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\tau[q(t+\tau)-\langle q(t+\tau)\rangle][q(\tau)-\langle q(\tau)\rangle] \qquad (6)$$

[note that $\langle q(t)\rangle$ oscillates periodically in time]. In the case of weak noise intensity, the function $Q(\omega)$ can be broken[6,9] into two "partial" contributions $Q_i(\omega)$ formed by small fluctuations about the attractors $i=1,2$ (cf. Ref. 14) and, in addition, the term $Q_{tr}(\omega)$ due to fluctuational transitions between the attractors, so that

$$Q(\omega)=\sum_i w_i Q_i(\omega)+Q_{tr}(\omega)\,. \qquad (7)$$

The expression for $Q_i(\omega)$ may be obtained easily by linearizing the oscillator's equation of motion near the stable state i,

$$Q_i=\frac{\Gamma B}{4\pi\omega_n^2}\frac{(\omega-\omega_h)^2+2(\omega-\omega_h)\Gamma\Omega(2|u_i|^2-1)+\Gamma^2(v_i^2+2\Omega^2|u_i|^4)}{[(\omega-\omega_h)^2-\Gamma^2 v_i^2]^2+4\Gamma^2(\omega-\omega_h)^2}\,, \qquad (8)$$

$$v_i^2=1+\Omega^2(3|u_i|^4-4|u_i|^2+1)\,.$$

Here, $|u_i|^2$ is the square dimensionless amplitude of forced oscillations in attractor i: For the states with lower ($i=1$) and higher ($i=2$) amplitudes, it is given, respectively, by the smallest and largest roots of the equation

$$\phi(|u_i|^2)=0,\quad \phi(x)=x(x-1)^2+\Omega^{-2}x-\beta \qquad (9)$$

[where (8) and (9) are written for the case $\gamma>0$, $\Omega>0$; bistability arises only for $\gamma\Omega>0$]. The expression for $Q_{tr}(\omega)$

may be shown to be of the form

$$Q_{tr}(\omega)=\frac{2\omega_h(\omega_h-\omega_0)}{3\pi\gamma}|u_1-u_2|^2 w_1 w_2\frac{W_{12}+W_{21}}{(W_{12}+W_{21})^2+(\omega-\omega_h)^2}, \tag{10}$$

where

$$u_j=\sqrt{\beta}(|u_j|^2-1+i\Omega^{-1})^{-1}.$$

It is evident from (10) that $Q_{tr}(\omega)$ has a Lorentzian peak centered at the frequency of the external field ω_h. The width of the peak is exponentially small. The intensity of the peak is an extremely rapid function of the distance from the phase-transition point. On a scale coarse grained over the range $\sim\alpha\ll 1$, the coefficient w_1w_2 in (10) is exponentially sharp,

$$\ln(w_1w_2)\simeq-|R_1'-R_2'|\,|\beta-\beta_0(\Omega)|/\alpha, \tag{11}$$

where $R_{1,2}'\equiv(\partial R_{1,2}/\partial\beta)_{\beta=\beta_0(\Omega)}$. Thus, the intensity depends on the distance $\beta-\beta_0(\Omega)$ to the phase-transition point nonanalytically; its first derivative has a cusp, which is a characteristic feature of first-order phase transitions.

The theoretical predictions, including the appearance of the supernarrow peak (10), have been tested with the aid of an electronic analog model of (2). The circuit, of conventional[15] design and accuracy, was driven by a sinusoidal periodic force from an HP3325B frequency synthesizer and optionally, in addition, pseudowhite Gaussian noise from a feedback shift-register noise generator.[16] The fluctuating voltage in the circuit representing $q(t)$ in (2) was digitized, and the power spectral density $Q(\omega)$ of the fluctuations $q(t)-\langle q(t)\rangle$ was computed and averaged by means of a Nicolet 1080 data processor. Fuller details of the circuit model and data anlaysis will be given elsewhere.

The bistability of the oscillator in the absence of noise was observed. Its measured boundaries (square data points in Fig. 1) were found to be well described (solid curves) by the theory of Ref. 6. When weak noise was added, transitions occurred between the attractors. The dependence of the reciprocal average lifetimes $\langle T_i\rangle^{-1}=W_{ij}$ of the attractors on noise intensity was found to be of the activation type, and the dependence of the exponents on parameters was investigated. The values of β and Ω for which $\langle T_1\rangle=\langle T_2\rangle$ are shown by the crosses in Fig. 1.

Some typical spectra showing the measured evolution of $Q(\omega)$ with decreasing β are plotted (histograms) in Fig. 2 and compared with the theoretical predictions (solid curves). The agreement between experiment and theory is excellent; there are no adjustable parameters.

The most striking feature of the measured spectra in Fig. 2 is the supernarrow peak that rises in the range where $\beta\simeq\beta_0(\Omega)$. Its width (unresolvable by the data-analysis system) is very much smaller than either the widths of the other peaks, or the experimentally determined damping constant Γ, or the frequency detuning $\omega_h-\omega_0$. The dependence of the intensity of this peak on the distance from the phase-transition line is found to be exponential, as shown in Fig. 3; note the logarithmic ordinate scale. This cusplike dependence is well described by the analytic function (11) [the value of the coefficient

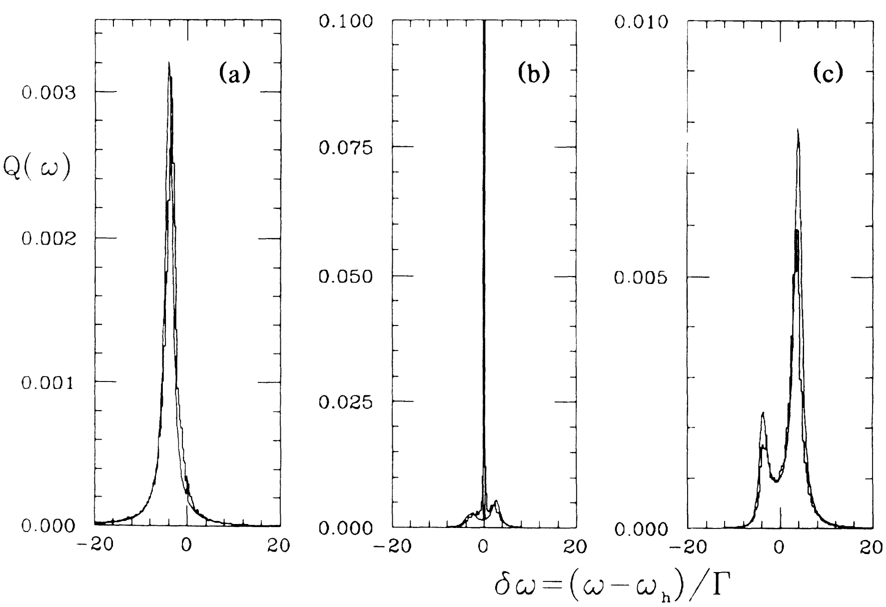

FIG. 2. Spectral densities of fluctuations measured (histograms) for the electronic circuit model of (2) with $\Omega=4.574$, $\alpha=8.69\times10^{-2}$ for (a) $\beta=0.048$, (b) $\beta=0.078$, and (c) $\beta=0.150$. The solid lines represent theoretical predictions.

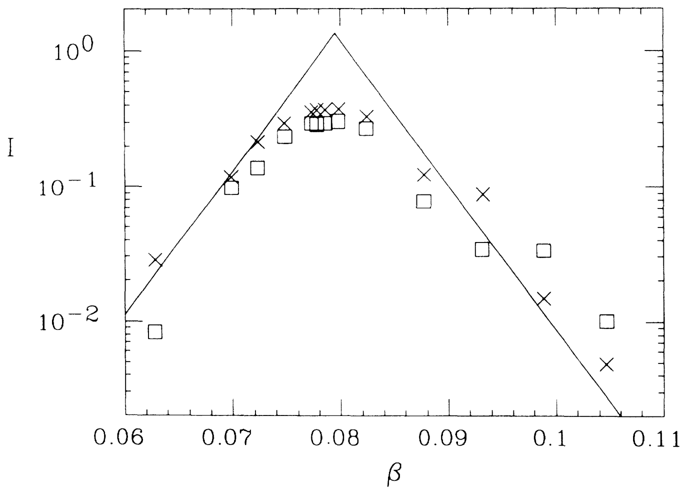

FIG. 3. Variation of the intensity I of the supernarrow peak with distance from the phase-transition line for $\Omega=4.574$, $\alpha=8.69\times10^{-2}$. The squares are direct measurements; the crosses are derived from (10), based on measured transition rates; the solid line also represents (10), but for $\ln(w_1\omega_2)$ given by (11) with measured $R_{1,2}$.

$|R_1' - R_2'|_{\beta=\beta_0(\Omega)}$ having been calculated from the experimental data for $R_{1,2}(\beta,\Omega)$].

The rest of the spectrum $Q(\omega)$ in Fig. 2 is also of considerable interest. Depending on the values of β and Ω^{-1}, it was observed to contain between one and four peaks. This structure arises because, in the region of the kinetic phase transition, the two partial spectra are superimposed and each of them can be, in general, twin peaked. The latter structure can be viewed as the result of the modulation of forced vibrations at frequency ω_h in a given state by the relatively slow (with characteristic frequency $\sim\omega_h-\omega_0$) fluctuational vibrations about this state.

We note, in conclusion, that the supernarrow peak in the spectral density of fluctuations of periodically driven systems found in the present work, and the corresponding peak in the susceptibility predicted in Ref. 6, may be used not only for determination of the phase-transition point and for revealing features of the transition (as here), but also for tunable and extremely narrow-band filtering and detection of optical signals. In common with the zero-frequency peak studied previously,[11] the supernarrow peak observed here provides, in principle, a basis for signal-to-noise enhancement of weak periodic signals. Such stochastic resonance[17] would occur,[18] not only at low frequencies, but close to the tunable frequency of the driving field.

One of us (M.I.D.) acknowledges the hospitality of Lancaster University during his memorable visit. The work was supported by the Science and Engineering Research Council (United Kingdom) and by the U.S.S.R. Academy of Sciences (U.S.S.R.).

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