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Relaxation of nonlinear systems driven by colored noise: An exact result

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An exact analytic method of calculating relaxation times of nonlinear systems driven by colored noise, applicable to a range of systems described by separable stochastic differential equations, is presented. Analog experiments and digital simulations performed on a quadratic model are shown to yield results in satisfactory agreement with the theoretical predictions.

I. INTRODUCTION

Jackson *et al.* have recently reported[1] a new, exact, analytic result for relaxation times in a class of stochastic systems driven by white Gaussian noise. In the present paper, we show how the calculations of Ref. 1 can readily be generalized to cover the case of other Gaussian noises, and we test the theoretical predictions against the results of analog experiments and digital simulations.

The calculations, which are applicable to any single-variable, separable, first-order stochastic differential equation, were originally motivated by a consideration of the effect of external noise on the simple quadratic model

$$\dot{x}=W(x-x^2/\Omega) \tag{1.1}$$

discussed by Leung.[2] The equation (Verhulst model) was used by Eigen and Schuster[3] to describe macromolecular self-replication under constraint, in which case x represents the number of molecules that duplicate themselves precisely, and Ω relates to the total size of the system. It is interesting to note in passing that the equation is also relevant to a wide range of other situations, including the spread of viral epidemics, the productivity of individual scientists, writers, and composers, and the freezing of supercooled liquids;[4] some examples are shown in Fig. 1.

In Ref. 1, the interest is centered on the consequence of parametric white noise introduced through the parameter W, such that

$$W=W_0+\xi(t)\ , \tag{1.2}$$

where W_0 is a constant, $\xi(t)$ is a Gaussian fluctuation with $\langle \xi(t)\rangle=0$, and

$$\langle \xi(t)\xi(t')\rangle=2D\delta(t-t')\ . \tag{1.3}$$

It was shown that, after normalizing (1.1) to the generic form with $W_0=\Omega=1$, the equation could be integrated to yield the trajectory

$$x(t)=\{1+B\exp[-t+\eta(t)]\}^{-1}\ , \tag{1.4}$$

where

$$B=[1-x(0)]/x(0) \tag{1.5}$$

is a constant that depends on the starting value $x(0)$, and $\eta(t)$ is a Wiener process. The expression (1.4) is a particular example of the more general class

$$x^{\alpha}(t)=f_{\alpha,t}(e^{\pm\eta(t)})\ , \tag{1.6}$$

which, provided that $f(y)$ is analytic at $y=0$, can be formally expanded, averaged, and Borel summed[5] to yield the moment of order α

$$\langle x^{\alpha}(t)\rangle=\pi^{-1/2}\int_{-\infty}^{\infty}d\phi\, f_{\alpha,t}[2\phi(Dt)^{1/2}]e^{-\phi^2}\ . \tag{1.7}$$

This expression can then be used to compute an appropriate relaxation time. In view of the nonexponential form of relaxation that is, in general, to be expected, we use the nonlinear relaxation time[6]

$$T=\frac{\int_0^{\infty}[\langle x^{\alpha}(t)\rangle-\langle x^{\alpha}(\infty)\rangle]dt}{\langle x^{\alpha}(0)\rangle-\langle x^{\alpha}(\infty)\rangle}\ , \tag{1.8}$$

where $x(\infty)$ is the final state of the system. Applying the result (1.7) to the trajectory (1.4) and evaluating the relaxation time (1.8), Jackson *et al.* showed[1] that, contrary to

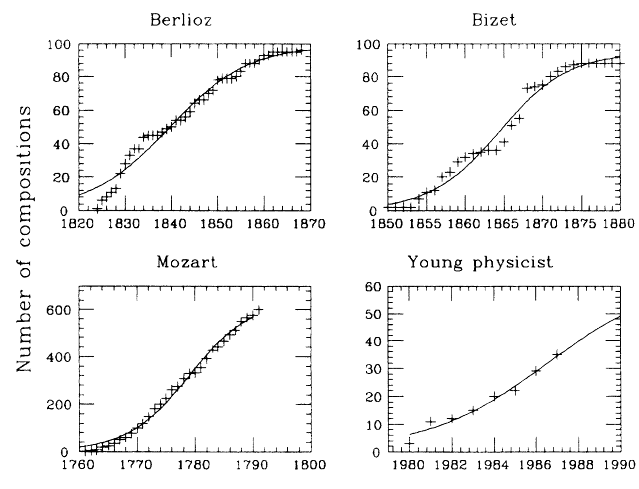

FIG. 1. Demonstration that Eq. (1.1) provides a reasonable description of the productivity of well-known composers and of a young physicist (real but anonymous). In each case, the number of compositions (published papers) is plotted as a function of time.

prior expectation,[2] T varies smoothly with D through the critical value $D_c=1$, where a noise-induced transition occurs in the stationary distribution.

It is, of course, the case that by means of the transformation

$$y=\frac{1}{x}\ , \tag{1.9}$$

Eq. (1.1) can be reduced to the linear form

$$\dot{y}=(1-y)[W_0+\xi(t)]\ , \tag{1.10}$$

so that, in a sense, it is perhaps not particularly surprising *a priori* that Jackson *et al.* where able to obtain an exact expression for T in respect of this particular system.[7] However, though it is clearly straightforward to derive the moments of y, the latter are of no obvious help in obtaining the moments of x, which are the quantities that we seek. The point which should be emphasized is that, even where a stochastic differential equation is separable, and expressions like (1.4) can be found in terms of stochastic integrals for the quantities of interest, it still remains a nontrivial and very much more difficult task to obtain explicit time-dependent averages of these quantities.

II. RELAXATION WITH COLORED NOISE

In this section, we extend the analysis of Jackson *et al.* to more complex systems. We consider

$$\dot{x}=F(x,\xi,t)\ , \tag{2.1}$$

where ξ is now a stochastic variable of known statistical properties, but which is not necessarily Gaussian. We introduce an auxiliary variable y satisfying

$$\dot{y}=\xi\ , \tag{2.2}$$

and we suppose that the system described by (2.1) is separable, i.e., that the solution can be cast in the form

$$x(t)=G\left(\int\xi\,dt,t\right)\ . \tag{2.3}$$

If we are able to solve (2.2) for $P(y,t)$ the time-dependent probability distribution of y (see, for example, Ref. 8 for the solution in the presence of quadratic Ornstein-Uhlenbeck noise), we can, in principle, write

$$\langle x^{\alpha}(t)\rangle=\int P(y,t)G^{\alpha}(y,t)dy\ . \tag{2.4}$$

For general noise, the solution of (2.2) can be a formidable task. Nonetheless, if we are principally interested in the long-time dynamics (as in the present case, where we are studying a possible critical point), it can always be argued that the central limit theorem will allow us to write

$$P(y,t)\simeq N\exp\left[-\left[\frac{[y-\langle y(t)\rangle]^2}{2\langle y^2(t)\rangle}\right]\right] \tag{2.5}$$

for sufficiently large t. Equation (2.4) appears justified on intuitive grounds; as we demonstrate below, it can be proved formally for Gaussian noises of any correlation, provided that G is of the form

$$G=f\left(\exp\left(\pm\int\xi\,dt\right),t\right)\ . \tag{2.6}$$

Thus we treat systems for which

$$x^{\alpha}(t)=f_{\alpha,t}(e^{\pm\eta(t)})\ , \tag{2.7}$$

where $\eta(t)$ is now a general Gaussian stochastic process of zero mean; for the particular case of white noise, it will be a Wiener process of standard deviation $2Dt$, as defined by (1.3). We again require that $f(y)$ by analytic at $y=0$. Equation (2.1) can be *formally* expanded as

$$\langle x^{\alpha}(t)\rangle=\sum_{n=0}^{\infty}\left[\frac{f^{(n)}(0)}{n!}\right]\langle\exp[\pm n\eta(t)]\rangle\ , \tag{2.8}$$

where $f^{(n)}(0)$ denotes the nth derivative of $f(y)$ evaluated at $y=0$. The average of the exponential is defined via its Taylor expansion. We emphasize that (2.8) represents a formal expansion; properly, we should write

$$\langle x^{\alpha}(t)\rangle=\sum_{n=0}^{\infty}\frac{d^nf(\beta e^{\pm\eta(t)})}{d(\beta e^{\pm\eta(t)})^n}\bigg|_{(\beta e^{\pm\eta(t)})=0}\frac{\beta^n}{n!}\times\langle\exp[\pm n\eta(t)]\rangle\ , \tag{2.9}$$

and then set $\beta=1$. Let us now make a mild assumption about the stochastic process appearing in (2.8), that

$$\langle[\eta(t)]^2\rangle=2Dt^*\ , \tag{2.10}$$

where t^* is some function of time. Using the fact that $\eta(t)$ is a Gaussian process, we can readily show that

$$\langle[\eta(t)]^m\rangle=\begin{cases}0, & \text{for } m \text{ odd}\\ \dfrac{m!(Dt^*)^{m/2}}{(m/2)!}, & \text{for } m \text{ even}\ .\end{cases} \tag{2.11}$$

The exponential term in (2.8) therefore becomes

$$\begin{aligned}\langle\exp[\pm n\eta(t)]\rangle&=\sum_{m=0}^{\infty}\frac{n^m}{m!}\langle[\eta(t)]^m\rangle\\&=\sum_{m=0}^{\infty}{}'\frac{n^m}{(m/2)!}(Dt^*)^{m/2}\\&=\sum_{m=0}^{\infty}{}'\frac{1}{(m/2)!}(n^2Dt^*)^{m/2}\\&=\exp(n^2Dt^*)\ ,\end{aligned} \tag{2.12}$$

where the primes indicate sums over even m only.

Using the identity

$$\exp(n^2Dt^*)=\pi^{-1/2}\int_{-\infty}^{\infty}d\phi\exp[-\phi^2+2\phi(Dt^*)^{1/2}n]\ , \tag{2.13}$$

substituting (2.8), and summing the series, we obtain

$$\langle x^{\alpha}(t)\rangle=\pi^{-1/2}\int_{-\infty}^{\infty}d\phi\,f[2\phi(Dt^*)^{1/2}]e^{-\phi^2}\ . \tag{2.14}$$

This exact expression for the time dependence of the moment of power α is of the same form as (1.7) for white noise, except that t^*, an arbitrary function of time, has replaced t. For the particular case where the external noise $\xi(t)$ in (1.2) is exponentially correlated, such that

$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau} e^{(-|t-t'|)/\tau} , \qquad (2.15)$$

we have

$$t^* = t + \tau(e^{-t/\tau} - 1) \qquad (2.16)$$

(where, as $\tau \to 0$, we recover the white noise 1.7). We thus obtain, finally,

$$\langle x^{\alpha}(t)\rangle = \pi^{-1/2} \int_{-\infty}^{\infty} d\phi\, e^{-\phi^2} \times [1 + B \exp(-t - 2\phi\sqrt{Dt^*})]^{-\alpha} , \qquad (2.17)$$

where the constant B is defined, as before, by (1.5). The nonlinear relaxation time T can then be found from (1.8) for any given starting position $x(0)$, noise intensity D, and noise correlation time τ.

In the case of harmonic noise[9]

$$\ddot{y} = -\gamma \dot{y} - \omega_0^2 y + \omega_0^2 \sqrt{2D} f(t) , \qquad (2.18)$$

with $f(t)$ white and Gaussian with zero average and standard deviation 1, (2.16) would need to be replaced by

$$t^* = \frac{\omega_0^2}{\lambda_2 - \lambda_1} \left[t \frac{(\lambda_2^2 + \lambda_1^2)}{\lambda_1 \lambda_2} + \frac{\lambda_2^3 (e^{\lambda_1 t} - 1) + \lambda_1^3 (e^{\lambda_2 t} - 1)}{(\lambda_1 \lambda_2)^2} \right] , \qquad (2.19)$$

where

$$\lambda_{1,2} = \frac{-\gamma \pm (\gamma^2 - 4\omega_0^2)^{1/2}}{2} . \qquad (2.20)$$

In a similar way, corresponding relations could readily be written down for any other correlation for which it was necessary to calculate moments from Eq. (2.17).

III. ANALOG EXPERIMENT

The theoretical results of the preceding section have been tested in part by means of an analog experiment on an electronic circuit model of (1.1). The circuit was based on the principles discussed in more detail elsewhere.[10] It employed two analog multipliers[11] and two standard operational amplifiers, arranged according to the (slightly simplified) block diagram shown in Fig. 2; the trimming and trajectory initialization circuitry was of the usual type and has been omitted in the interest of clarity. The output from the (differential input) multipliers was internally scaled by a factor of 0.1. This was compensated for in one case by the prior multiplication of x by 10; in the other case, the factor was allowed for by scaling the integrator time constant accordingly. To optimize its performance, the circuit was scaled[9] in both x and t, but the results to be presented below have all been normalized so as to be consistent with Eqs. (1.1), (1.2), and (2.9), with $W = \Omega = 1$. The external noise, obtained from a homemade noise generator,[12] was passed through an active single-pole filter to provide a Gaussian exponentially correlated output of correlation time τ in accordance with (2.15).

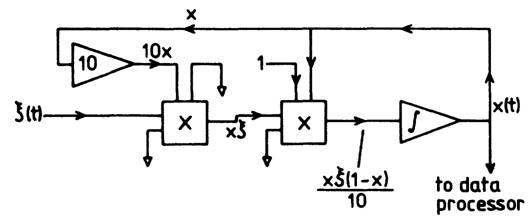

FIG. 2. Block diagram of the analog electronic circuit model of Eq. (1.1).

In practice, the circuit was set initially to a nonequilibrium value of x, and then released. Simultaneously, a Nicolet 1080 data processor was triggered to digitize the trajectory of $x(t)$. Many such trajectories were ensemble-averaged to obtain the first moment $\langle x(t)\rangle$, which was then used to compute the experimental value of T from (1.8). The procedure was repeated for different values of D and τ, all for an initial value of $x = 0.2$. A typical trajectory and its corresponding ensemble average are shown in Fig. 3.

Some typical experimental data obtained by this method are shown by the points in Fig. 4. The measured values of T have been scaled by division by the deterministic relaxation time $T_0 = -\ln x_0/(1-x_0) = 2.0118$ for $x_0 = 0.2$, and plotted as a function of τ for the fixed noise intensity $D = 0.5$. The full curve represents the theoretical prediction derived from (2.11). Although the data are scattered, their random error being about $\pm 5\%$ on top of a systematic error of about $\pm 4\%$, it is evident that they are in satisfactory agreement with the theoretical prediction.

In fact, the system (1.1) is a relatively difficult one to investigate by analog experiment. This is partly because of its particular sensitivity to internal noise in the active components, which tends to kick $x(t)$ away from its asymptotic value at large τ, and partly because of its sen-

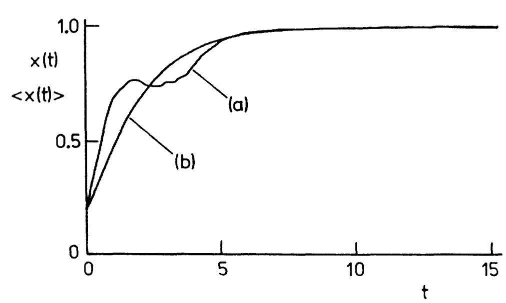

FIG. 3. Experimental results obtained from the analog electronic circuit of Fig. 2: (a) a typical trajectory, obtained for $D = 0.5$, $\tau = 1.0$; (b) ensemble average of 100 trajectories digitized under the same conditions.

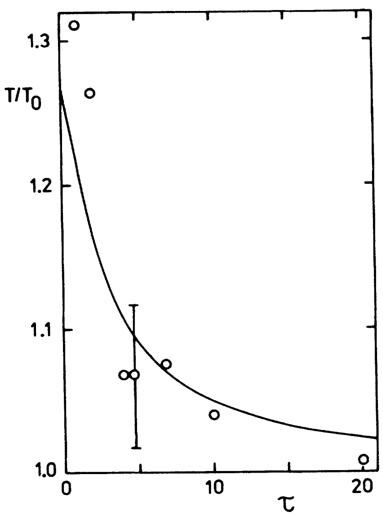

FIG. 4. Nonlinear relaxation time T of the electronic circuit model of Eq. (1.1), measured as a function of noise color τ for a noise intensity $D=0.5$ (points) compared with the theoretical prediction (full curve) derived from Eq. (2.17). The results have all been normalized by division by the deterministic relaxation time T_0. The experimental data are subject to a random error of $\sim\pm5\%$, as indicated by the single bar.

sitivity to small drifts in the trimming circuitry. For the latter reason, the number of trajectories included in each of the ensemble averages was kept relatively small. Despite these shortcomings, we can certainly conclude from the results of Fig. 4 that a real physical system does indeed behave very much in the manner predicted by the theory presented in the previous section.

IV. DIGITAL SIMULATION

In order to test the theoretical results of Sec. II more precisely and over a wide range of D and τ then was possible in the analog experiments, a digital simulation of (1.1) was also carried out. The technique, which has already been described in detail elsewhere,[13] was used to compute a sequence of $x(t)$ trajectories. These were then ensemble-averaged and used to obtain T for the first moment from Eq. (1.8), in much the same way as in the analog experiment of the preceding section; the trajectories and corresponding moment averages looked very similar to those for the electronic circuit, shown in Fig. 3. An important difference, however, was that the number of trajectories (400) in each ensemble average could be made much larger because there were no problems due to parameter drift, so that the statistical quality of the final data was correspondingly better.

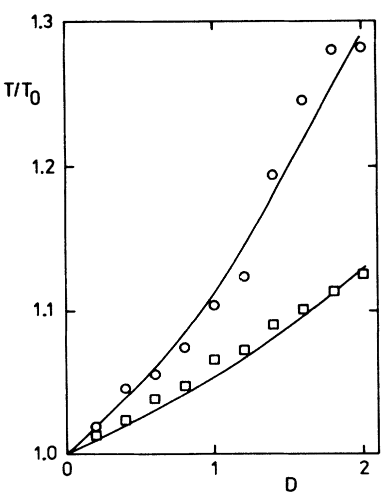

FIG. 5. Normalized nonlinear realxation time T/T_0 obtained by digital simulation for noise correlation times $\tau=10$ (circles) and $\tau=20$ (squares), plotted as a function of noise intensity D. The full curves, representing the theoretical prediction in each case, are derived from Eq. (2.17).

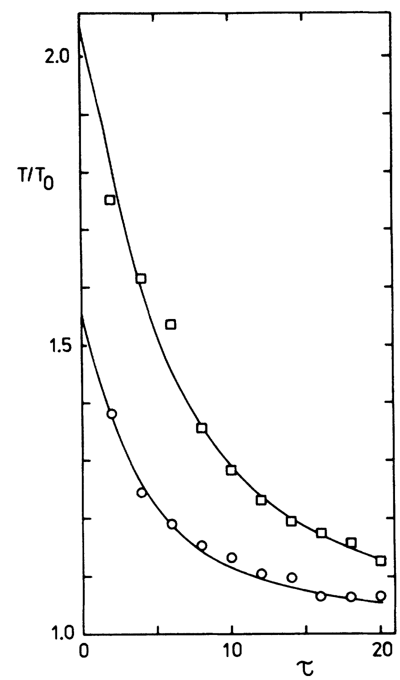

FIG. 6. Normalized nonlinear relaxation time T/T_0 obtained by digital simulation for noise intensities $D=1$ (circles) and $D=2$ (squares), plotted as a function of noise correlation time τ. The solid curves, representing the theoretical prediction in each case, are derived from Eq. (2.10).

Some typical digital simulation results are shown in Figs. 5 and 6. Figure 5 plots the relaxation time T against the noise intensity for two values of the noise correlation time τ; Fig. 6 plots T against τ for two values of D. In each case, the results have been scaled, as before, by division by the deterministic relaxation time T_0, and the full curves represent theoretical predictions derived from Eq. (2.17). The agreement between digital simulation and theory is clearly excellent.

V. CONCLUSION

The theory presented in Sec. II is strongly supported by the results of the analog experiment and digital simulation for the particular case of the quadratic model (1.1).

We conclude that Eq. (2.14) enables the time dependence of the moments to be calculated exactly for any single-variable, separable, stochastic differential equation, *for arbitrary intensity and correlation* of the Gaussian external noise. This result is of interest, not only for its own sake, but also because it is rare indeed in the realms of colored noise theory to obtain results that are exact.

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