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Fluctuation phenomena in a multibranch potential

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Fluctuation phenomena, to be anticipated in all systems for which the internal fluctuating variable is a multivalued function of the observable quantity, are reported. Analog electronic experiments on a model optically bistable Fabry-Pérot cavity have shown that the observable may be described in terms of a Boltzmann distribution with a *multibranch* effective potential. The distribution was observed to possess up to four peaks (two twin peaks) within the range of bistability, and to become singular at outer boundaries corresponding to the branching points of the potential. The experimental results are compared with theory.

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The high level of scientific activity currently devoted to optical bistability (OB) is attributable to its prospective applications, and also to the richness and intrinsic interest of the underlying physics. In particular, OB systems provide opportunities for the investigation of a wide range of quite general fluctuational phenomena associated with coexisting stable states. Work of this kind has been carried out on lasers [1-4], on hybrid electro-optical systems [5] and, most recently, on a passive all-optical double-cavity membrane system (DCMS) [6,7]. The latter arrangement has turned out to be of considerable interest in that it exhibits zero-frequency spectral peaks in the intensity of the transmitted light together with the closely associated phenomenon of stochastic resonance [8].

An intriguing feature of attempts [6] to account for the effect of phase noise in the DCMS was the inference that the intensity distribution $P(I_T)$ of the transmitted light should in principle be describable in terms of a *multibranch* effective potential. Some remarkable consequences followed from this hypothesis. In particular, it was predicted that $P(I_T)$ should be confined within boundaries, that it should be singular at these boundaries and that it might possess up to four separate maxima. These predictions could not be tested in the DCMS itself because of various smearing effects, in particular those due to fluctuations in the input light intensity. Partly for this reason, and partly because the theory of the DCMS involves two periodic functions of the phase gain ϕ of light in the cavity, $M(\phi)$ and $N(\phi)$ [6], whose exact form is unknown, it was impossible to make a direct comparison of experiment and theory.

The main purpose of this Rapid Communication is to report the principal results of an analog electronic experiment that has enabled us to model an idealized cavity, in the absence of input intensity noise, and with $M(\phi)$ and $N(\phi)$ chosen in the (archetypal) forms corresponding to a Fabry-Pérot cavity. Thus it has been possible to investigate the interesting phenomena associated with a multibranch effective potential and to compare the results, both qualitatively and quantitatively, with theoretical predictions. As we demonstrate below, the experiment has confirmed the unusual singular and multipeaked character of $P(I_T)$ and, in addition, has revealed features of the inner workings of the cavity system, such as the phase distribution, that are inaccessible through conventional optical measurements. Because phase noise is inherent to all optical cavities, the results obtained are of wide relevance.

The model considered is a nonlinear Fabry-Pérot resonator displaying dispersive OB. In modeling the mechanism of OB we assume that the internal medium of the cavity has a refractive index which depends linearly on light intensity, and that relaxation of the intracavity phase gain of the radiation can be described by the Debye equation (see, e.g., [9] and references therein). (Note that this is actually a different OB mechanism to that of the air-spaced DCMS device [6].) Our system is then described by

$$\begin{aligned} &\dot{\phi}+(1/\tau)(\phi-\phi^{(0)})=IM(\phi)+I_m(t)\,,\\ &I_T=IN(\phi)\,,\\ &M(\phi)=A_M[1+\tfrac{1}{2}F(1-\cos\phi)]^{-1}\,,\\ &N(\phi)=A_N[1+\tfrac{1}{2}F(1-\cos\phi)]^{-1}\,. \end{aligned} \tag{1}$$

Here, I and I_T are the intensities of the incident and transmitted radiation, ϕ is the intracavity phase gain, which takes the value $\phi^{(0)}$ in the limit $I\rightarrow 0$, and F is the finesse of the cavity. The functions $M(\phi)$ and $N(\phi)$ relate the intensities of the transmitted light and of the intracavity field (which drives the optically nonlinear medium and causes the changes in ϕ) to that of the incident light; the expressions for $M(\phi)$ and $N(\phi)$ are standard [9] for a Fabry-Pérot cavity; and we have included the nonlinearity parameter of the medium within A_M. The function $I_m(t)$ describes the noise driving the intracavity phase gain ϕ of the radiation. It is assumed white and Gaussian.

$$\begin{aligned} &I_m(t)=\bar{I}_m+\delta I_m(t)\,,\\ &\langle\delta I_m(t)\rangle=0,\ \langle\delta I_m(t)\delta I_m(t')\rangle=2D\delta(t-t')\,. \end{aligned} \tag{2}$$

The noise (1) and (2) can originate in various ways, e.g., from thermal fluctuations in the nonlinear medium or through random vibrations of the mirrors resulting in variations of the intracavity optical length. The crucial point here is that the fluctuations of the transmitted light intensity I_T are only due to phase fluctuations, while the intensity of incident light I remains constant.

The statistical distribution of the phase $p(\phi)$ has the form

$$p(\phi)=Z^{-1}\exp[-U(\phi)/D]\,,$$
$$Z=\int d\phi \exp[-U(\phi)/D]\,, \tag{3}$$

where $U(\phi)$ is the effective potential for the dynamics of the phase:

$$U(\phi)=-I\int_0^{\phi} d\phi' M(\phi')+\frac{1}{2\tau}\phi^2-\frac{1}{\tau}(\phi_0+\bar{I}_m\tau)\phi\,. \tag{4}$$

In the range of optical bistability the potential $U(\phi)$ has two minima lying at $\phi=\phi_{1,2}$ where $\phi_{1,2}$ are the stable solutions of (1) with $I_m(t)=\bar{I}_m$.

The quantity of greatest physical interest is the probability distribution $P(I_T)$ of the transmitted light intensity, rather than $p(\phi)$, since it is $P(I_T)$ that can be measured in optical experiments. This quantity can be calculated immediately from (1), (3), and (4) and has the form

$$P(I_T)=K(I_T)\exp[-V(I_T)/D]\,,$$
$$V(I_T)=U[\phi(I_T)]\,, \tag{5}$$
$$K(I_T)=I_T^{-2}Z^{-1}\frac{2IA_N}{F|\sin\phi|}\,.$$

It is evident from Eqs. (1)–(5) that there are dramatic differences between the phase distribution and that of the transmitted light intensity. This is due to the periodicity of the transmission coefficient $N(\phi)$ in (1), $N(\phi+2\pi)=N(\phi)$. As a consequence, the whole ϕ axis is mapped by the relation $I_T=IN(\phi)$ on the interval $(I_{T\,\min},I_{T\,\max})$ of I_T

$$I_{T\,\min}=IA_N/(1+F)\,,\quad I_{T\,\max}=IA_N\,, \tag{6}$$

and the distribution $P(I_T)=0$ for all I_T lying outside this range. At the same time, the effective potential for the transmitted light $V(I_T)$ is a multibranch function, and the values of $I_{T\,\min}$ and $I_{T\,\max}$ (i.e., those for $\phi=n\pi$) correspond to its branching points. A comparison of $U(\phi)$ and $V(I_T)$ is shown in Figs. 1(a) and 1(b).

It is the multibranch character of the effective potential $V(I_T)$ that accounts for some very unusual features seen in the distribution of the transmitted light. The distribution still has two maxima at $I_T=I_{T1,2}\equiv IN(\phi_{1,2})$ corresponding [10] to the minima of $V(I_T)$, but they can now lie on different branches of $V(I_T)$ [see Fig. 1(b)]. The branching points give rise to the singularities of the distribution, because they correspond to an infinite "density of states" for the transmitted light intensity near boundaries [see the prefactor in (5)]. Correspondingly, $P(I_T)$ diverges for $I_T\rightarrow I_{T\,\min},I_{T\,\max}$. It should thus be possible to obtain a *four-peaked* distribution for a double-well potential under OB conditions. While investigating this interesting theoretical prediction it is very important to bear in mind that the distribution as determined from experiment $P_{\rm expt.}(I_T)$ is the coarse-grained "bare" distribution $P(I_T)$:

$$P_{\rm expt}(I_t)=\frac{1}{2\Delta I_T}\int_{I_T-\Delta I_T}^{I_T+\Delta I_T} P(I_T')dI_T'\,. \tag{7}$$

Thus in experiments, rather than singularities, we can expect to see very strong additional peaks near boundaries of the light distribution. We note that the onset of the four-peaked distribution depends on the system becoming optically bistable (for optically monostable systems, the distribution would be expected to have three peaks). Thus, in the model, (1) and (2), there is no special requirement on the finesse for the four-peaked distribution to arise; in real systems, however, which have losses, a minimum value of the finesse would need to be exceeded.

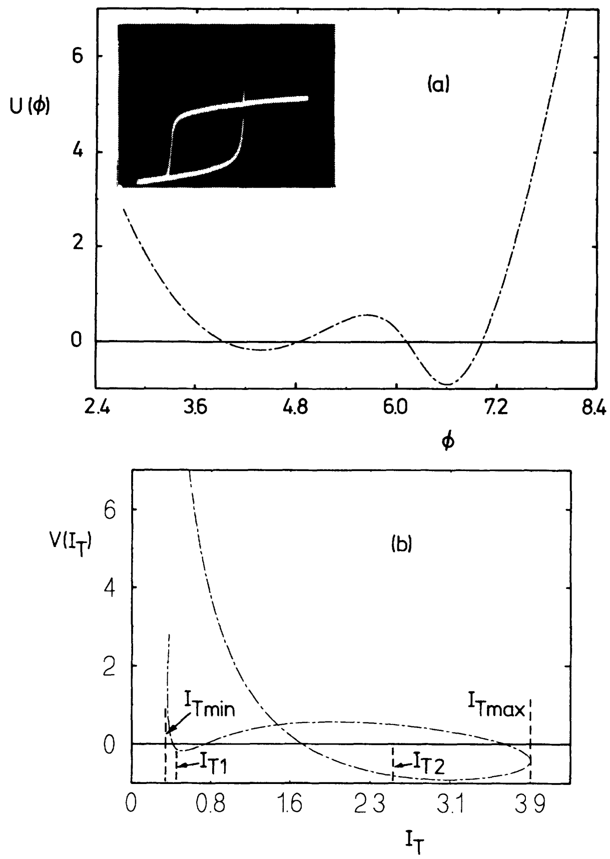

FIG. 1. (a) The potential $U(\phi)$ for the distribution over the phase in the nonlinear Fabry-Pérot cavity described by Eq. (1) ($\tau IA_M=3.925$, $\phi^{(0)}=3.4$ rad, $F=10.1$); (b) the corresponding effective multibranch potential $V(I_T)$ for the probability distribution of transmitted light intensity, which bounded at $I_{T\,\min},I_{T\,\max}$. The positions $I_{T1,2}$ of the inner maxima in $P(I_T)$ are close to but do not coincide exactly [10] with the corresponding minima in $V(I_T)$ shown in (b). The inset in (a) shows the experimental hysteresis loop measured for the electronic model using the same parameters.

The theoretical predictions, including the appearance of the four-peaked distribution, have been tested with the aid of an electronic analog model of (1) and (2). The circuit, constructed with the conventional design [11] and accuracy, was driven by a pseudo-white-noise generator [12]. The fluctuating voltages in the circuit representing $I_T(t)$ and $\phi(t)$ in (1) were digitized and the corresponding distributions $P(I_T)$ and $P(\phi)$ were computed by means of a Nicolet LAB80 data processor. More complete details of the circuit model will be given elsewhere.

The experimental hysteresis loop in the absence of noise is shown in the inset in Fig. 1. It contains a characteristic [9] spike at the bifurcation value of the incident radiation intensity corresponding to the disappearance of the lower-transmission branch: the transient transmission exceeds that on the upper branch because, on its way from the lower-I_T stable state [$I_T = I_{T1}$ in Fig. 1(b)] to the higher-I_T one [$I_T = I_{T2}$ in Fig. 1(b)], along the input-output characteristic, it is obvious from Fig. 1(b) that the system must pass the value $I_{T\max} > I_{T2}$. When noise was added to the system, transitions occurred between the stable states. Their reciprocal average lifetimes were measured and, like those from the earlier experiments on the DCMS [6], were found to be of the activation type; a Lorentzian-shaped zero-frequency peak was observed in the measured spectral density. To this extent, the results obtained were in qualitative agreement with those from the DCMS. In the latter work, however, it was not possible to compare the shape of the distribution of the transmitted light intensity quantitatively with theory because of the input intensity fluctuations and because, as mentioned above, the forms of $M(\phi)$ and $N(\phi)$ for the DCMS are unknown and presumably do not correspond to those assumed in the simple model (1). Neither was it possible to observe the predicted four-peaked distribution $P(I_T)$ for the DCMS.

The distribution $P(I_T)$ measured for the electronic model is shown by the jagged curve of Fig. 2(a). Unlike the (smeared) results obtained in [6], the multipeaked structure of $P(I_T)$, expected to result from the branching of the effective potential $V(I_T)$ given by (5), is clearly resolved. Three maxima are immediately apparent; closer inspection reveals that the left-hand one is actually a double peak, i.e., there are four maxima in total. It is also evident that the outer peaks are highly asymmetric: At their outer boundaries they are, in fact, singular within the resolution of the experiment, just as predicted. The smooth curve, representing the theory, is in good quantitative agreement with the measurements. The distribution of phase $p(\phi)$ has also been measured for the electronic model and is shown by the jagged curve in Fig. 2(b). In striking contrast to $P(I_T)$, it exhibits a conventional double-peaked structure, consistent with the ordinary single-valued double-well effective potential (4) from which it is derived. Again, the measurements are in excellent quantitative agreement with the theoretical prediction (smooth curve).

In conclusion, we point out that, although the present investigations have related specifically to the model (1) of optical bistability in a Fabry-Pérot cavity, very similar phenomena to those discussed above—notably, the appearance of multipeaked distributions confined between singular outer boundaries—may also be expected to arise in other types of passive optical resonator and also in lasers, as well as in nonoptical systems, whenever the fluctuating quantity (corresponding to the phase gain in the above analysis) is a multibranch function of the observed quantity (the transmitted light intensity).

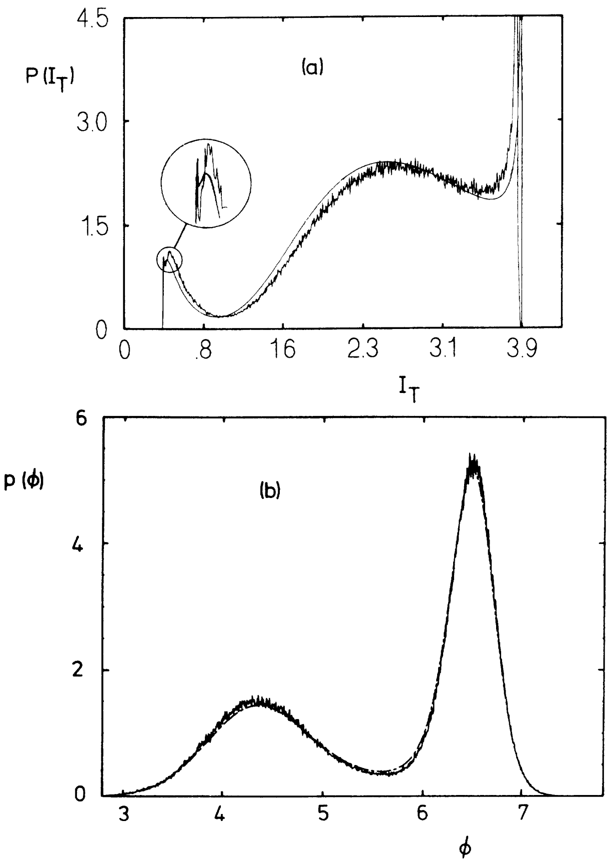

FIG. 2. Experimental probability distributions (jagged curves), measured for the electronic model of (1) with $\tau IA_M = 3.4$, compared with the corresponding theoretical predictions (smooth curves): (a) the distribution of transmitted light intensity $P(I_T)$ displays four separate maxima confined between singular outer boundaries (note that the left-hand peak is actually a double one, in both experiment and theory, as shown in the expanded region); (b) the variation of the phase takes the form of a standard double-peaked distribution $p(\phi)$.

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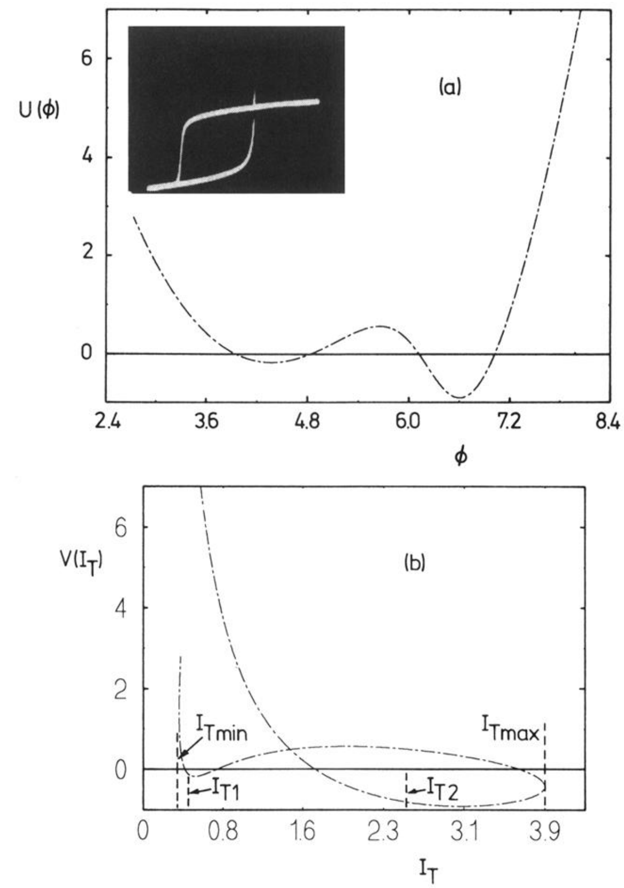

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