FLUCTUATIONAL TRANSITIONS AND CRITICAL PHENOMENA IN A PERIODICALLY DRIVEN NONLINEAR OSCILLATOR SUBJECT TO WEAK NOISE

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ABSTRACT

Fluctuation-induced transitions between coexisting periodic attractors in a periodically driven nonlinear oscillator have been investigated theoretically and by analogue electronic experiment. Calculations and measurements of the corresponding activation energies are in good agreement, and have enabled the position of the kinetic phase transition (KPT) line to be established over its full range.

The kinetics of an oscillator is a long-standing and important problem of classical and quantum statistical physics, for two reasons. On the one hand, many real physical systems can be well modelled by such oscillators and, on the other, comparatively simple solutions can be obtained for models of this kind, in particular for an underdamped nonlinear oscillator (see [1]). Interesting new phenomena arise if an underdamped nonlinear oscillator is driven by a nearly resonant force. Since the frequency of the eigenvibrations of the oscillator depends on their amplitude, there is a certain range of the forcing amplitude in which there are co-existing stable states of vibration with comparatively small and large amplitudes respectively. The corresponding eigenfrequencies are self-consistently in comparatively bad or good resonance with the field frequency ω_F. The bistability of a periodically driven oscillator has been observed for cyclotron motion of an electron in a quadrupole trap [2]. It has been discussed also in the context of optical bistability [3], and in acoustics and engineering.

An oscillator bistable in a periodic field provides an example of a bistable system far from thermal equilibrium, for which the co-existing attractors are limit cycles. The quantities of particular interest and importance for nonequilibrium bistable systems are the probabilities W_{nm} of fluctuational transitions between the stable states ($n, m = 1, 2$). For weak intensity of the fluctuations (induced by external noise, or resulting from coupling to a thermal bath) these probabilities are very much smaller than the characteristic reciprocal relaxation time(s) of the system t_r^{-1}. In the general case, the probabilities W_{12} and W_{21} of the transitions $1 \to 2$ and $2 \to 1$ are strongly different (exponentially different, for Gaussian

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noise), as are the stationary populations $w_{1,2}$ of the stable states ,

$$w_1 = W_{21}/W, \quad w_2 = W_{12}/W, \quad W = W_{12} + W_{21} \tag{1}$$

Only within a narrow range of the parameters of the system are w_1 and w_2 of the same order of magnitude. In this range a kinetic phase transition occurs and although the fluctuations about each stable state are small, large fluctuations then come into play, related to transitions between the states. A feature of these fluctuations is that they are infrequent, the characteristic time scale being $\sim W^{-1}$, and they would therefore be expected to give rise to high supernarrow peaks (of width $\sim W \ll t_r^{-1}$)in the spectral density of the fluctuations (SDF) and in the susceptibility at the frequency of the co-existing limit cycles and its overtones (we note that a nonequilibrium system can amplify a weak signal, i.e., the imaginary part of the susceptibility can be negative) [1]. The intensity of the peaks should (and for the SDF does [4]) display an extremely sharp dependence on the distance to the transition point. Since the supernarrow peaks in the susceptibility are due to fluctuations (caused by the noise driving the oscillator), then, in a certain range of the noise intensity B, the amplitude of a signal due to an additional weak force should *increase* with increasing B. This is a prerequisite for stochastic resonance (SR) - an interesting phenomenon that has attracted much attention recently in view of various applications [5]. In contrast to the "conventional" SR observed at low frequencies, in the case of a periodically driven oscillator there arises [6] a high-frequency form of stochastic resonance (HFSR).

We present below some results on fluctuations in a nonlinear oscillator obtained recently by means of analog simulation and theoretically. The model investigated is described by the equation

$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = F\cos\omega_F t + f(t) \tag{2}$$

$$\Gamma, |\delta\omega| \ll \omega_F, \quad \delta\omega = \omega_F - \omega_0, \quad \langle f(t)f(t')\rangle = 2\Gamma B\delta(t-t')$$

Here, $f(t)$ is Gaussian white noise. For small friction coefficient Γ and small frequency detuning of the driving force with respect to the eigenfrequency of the oscillator ω_0, and also for a comparatively small field amplitude F, the motion of the oscillator is primarily vibrations at a frequency ω_F with slowly varying amplitude and phase. The dynamics of the latter depends on the values of two dimensionless parameters, β and η, and also on the dimensionless noise intensity α,

$$\beta = 3|\gamma|F^2/32\omega_F^3|\delta\omega|^3, \quad \eta = \Gamma/|\delta\omega|, \quad \alpha = 3|\gamma|B/16\omega_F^3\Gamma \tag{3}$$

(the kinetics of slow variables is basically the same whether the model (2) is used or fluctuations are assumed to be due to the coupling to a thermal bath; the main point is that the power spectrum of the noise $f(t)$ should be flat in a range $\gtrsim |\delta\omega|$ about ω_F). The range of β, η where the two attractors coexist as calculated and measured in the experiment is enclosed by the approximately triangular region shown in Fig.1. On the upper branch of $\beta(\eta^2)$ the small-amplitude limit cycle (the stable state 1) merges with an unstable one and disappears, whereas on the lower branch this occurs to the large-amplitude limit cycle (the stable state 2). A fluctuating periodically driven nonlinear oscillator was one of the first physical systems without detailed balance for which the probabilities of transitions between coexisting stable states were analyzed [7]. It follows from the theory [7] that, to logarithmic accuracy, the dependence of W_{nm} on the noise intensity is of activation type,

$$W_{nm} = \text{const} \times \exp(-R_n/\alpha) \quad (4)$$

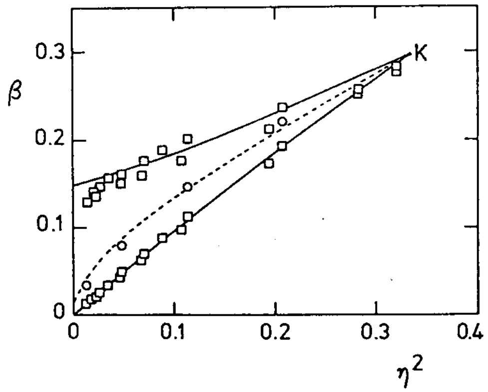

Fig.1 The phase diagram as calculated (full lines) and measured (squares [4]). The calculated KPT line (dashed) is compared with experimental measurements (circles).

The values of the activation energies $R_{1,2}$ for the transitions from the states 1,2 depend on the parameters β, η and are given by the solution of a variational problem. Such an activation dependence on the noise intensity for a system without detailed balance has indeed been observed in our experiments. The experimental mean first passage times (MFPT) and activation energies $R_{1,2}$ are shown in Figs.2 and 3. As expected, the activation energy for escape from the small-amplitude limit cycle decreases with increasing β (i.e., with increasing field amplitude) until R_1 becomes equal to 0 at the bifurcation point where the stable state disappears. Conversely, R_2 increases with increasing β. The data are in reasonably good agreement with the results of the numerical solution of the variational problem for $R_{1,2}$: there are no adjustable parameters in either the theory or the experiment.

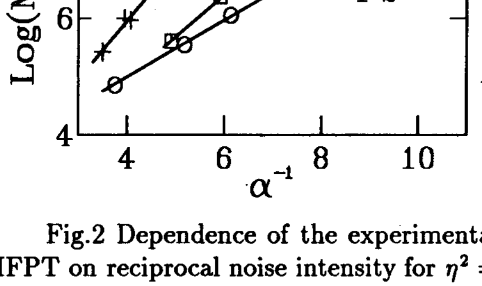

Fig.2 Dependence of the experimental MFPT on reciprocal noise intensity for $\eta^2 =$ 0.033.

It follows from the expressions (1),(4) that for the most of the values of β, η the ratio of the stationary populations of the states $w_1/w_2 \propto \exp(-(R_2 - R_1)/\alpha)$ is either exponentially large or small, and only for $R_1 \approx R_2$ are the populations of the same order of magnitude.

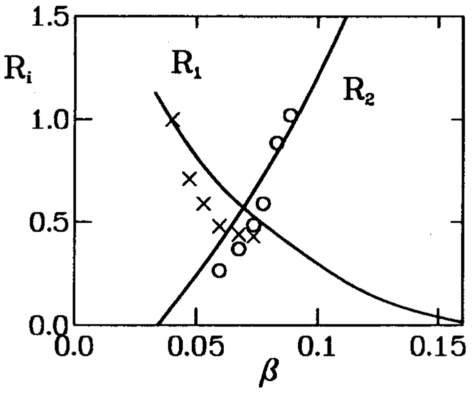

Fig.3 The experimental and theoretical activation energies for $\eta^2 = 0.033$.

The KPT phase-transition line $\beta_c(\eta^2)$ as given by the condition $R_1 = R_2$ is shown dashed in Fig.1. It was found analytically for small values of η^2 (we note that β_c is a nonanalytic function of η^2 for small η^2, $\beta_c(\eta^2) - \beta_c(0) \propto \eta$) and in the vicinity of the spinode point K, and it was evaluated numerically in between using the calculated values of $R_{1,2}$. The experimental points (circles), which were obtained from the condition $W_{12} = W_{21}$, are seen to be in good agreement with the theory. The supernarrow peaks in the SDF, corresponding to those anticipated [7] in the susceptibility of the system, were observed previously [4].

In conclusion, we note that the calculated and measured activation energies are in satisfactory agreement and that they have enabled us to establish the position of the KPT line (Fig.1) over the full range of η^2 from zero up to the spinode point K.

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