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Zero-Dispersion Nonlinear Resonance in Dissipative Systems

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It is shown theoretically and by analog electronic experiment that, in dissipative oscillatory systems for which the frequency of eigenoscillation displays an extremum as a function of energy, the dynamics of nonlinear resonance can differ markedly from the conventional case. Transitions between the conventional and novel types of nonlinear resonance, as parameters vary, correspond to changes in the topology of basins of attraction. With added noise, they can result in drastic changes in fluctuational transition rates between small- and large-amplitude regimes. [S0031-9007(96)00357-2]

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Nonlinear oscillators are well known to exhibit a remarkably rich variety of interesting phenomena (see, e.g., [1–3]). A distinct class of oscillatory systems manifesting unusual phenomena both in their deterministic dynamics and in the presence of noise was identified and studied recently [4–9]. Their common feature is an extremum in the dependence of a frequency of eigenoscillation on its energy. Many real physical systems such as SQUIDs [10], relativistic oscillators [11], electrical circuits [1], polymeric molecules, and others (see the discussion in [4–8]) can be described by models of this type. If a periodic force, or noise and associated dissipation, or their combination are added, interesting phenomena can arise. It has been predicted [7] that, if a weak periodic force of frequency close to the extremal eigenfrequency is applied in the absence of dissipation, a novel type of nonlinear resonance, *zero-dispersion nonlinear resonance* (ZDNR), can occur.

Real systems are usually subject to dissipation. Provided that this is weak, however, there can still be a large-amplitude response corresponding approximately to the resonant eigenoscillation of the dissipationless system, and thus the concept of nonlinear resonance [3,12] is still valid. Often, one is interested only in stable regimes which can be characterized, e.g., by a multivalued frequency-response curve [1], plotting the response amplitude against the driving frequency. Sometimes, however, the dynamics of the system is no less important, and this is especially true in those cases where external noise is also present. One of the most important ways of characterizing the dynamics is in terms of *basins of attraction* (BAs) of the stable states (attractors) in the phase space [2]. We show in this Letter that, for the class of systems considered, the BAs undergo qualitative (topological) changes as parameters vary which, in the presence of noise, may be expected to result in drastic changes in interattractor fluctuational transition rates.

First, however, we discuss the problem in the absence of noise, taking as an example a one-dimensional potential system subject to a weak linear friction and a weak periodic force, such that

$$\dot{q} = p, \qquad \dot{p} = -\frac{dU(q)}{dq} - \Gamma p + h\cos(\omega_f t). \quad (1)$$

Our goals will be to find period-1 orbits [13] and to describe the transition regimes. With these aims, we transform to the slow variables action I and phase difference $\tilde{\psi} = \psi - \omega_f t$ between the force and the response; we neglect high-frequency oscillatory terms. Then (1) can be reduced to

$$\dot{I} = -hq_1 \sin(\tilde{\psi}) - \Gamma I,$$

$$\dot{\tilde{\psi}} = \omega - \omega_f - \frac{dq_1}{dI} h \cos(\tilde{\psi}), \qquad (2)$$

where $\omega \equiv \omega(I)$ is the eigenfrequency corresponding to I, and q_1 is the first harmonic in the Fourier expansion of q:

$$q \equiv q(I, \psi) = 2 \sum_{n=0}^{\infty} q_n(I) \cos(n\psi). \qquad (3)$$

The period-1 orbits are located by finding the stationary solutions of (2) with nonzero action. These can be of two types,

$$\tilde{\psi}_{\rm st} = -\arcsin\{\Gamma I_{\rm st}/hq_1(I_{\rm st})\},$$
$$\tilde{\psi}_{\rm st} = \arcsin\{\Gamma I_{\rm st}/hq_1(I_{\rm st})\} - \pi, \qquad (4)$$

where $I_{\rm st}$ satisfies the equation

$$\omega(I_{\rm st}) - \omega_f = \pm h \frac{dq_1(I_{\rm st})}{dI_{\rm st}} \left[1 - \left(\frac{\Gamma I_{\rm st}}{hq_1(I_{\rm st})}\right)^2\right]^{1/2} \qquad (5)$$

in which the plus and minus refer to the upper and lower equations of (4), respectively. Equation (5) can be solved explicitly for very small h, or numerically for larger h (Fig. 1).

Looking at Fig. 1 one can easily understand that, just as in the dissipationless case [7], there will be a range of ω_f for which the extremum of the full line comes above the abscissa ($\omega = \omega_f$) line but still intersects the upper dotted line near I_m; i.e., a large amplitude response can exist even if ω_f is beyond the spectrum of eigenfrequencies.

For further analysis, we turn to an archetypal example [5] of a potential whose dependence of eigenfrequency on energy possesses an extremum [14],

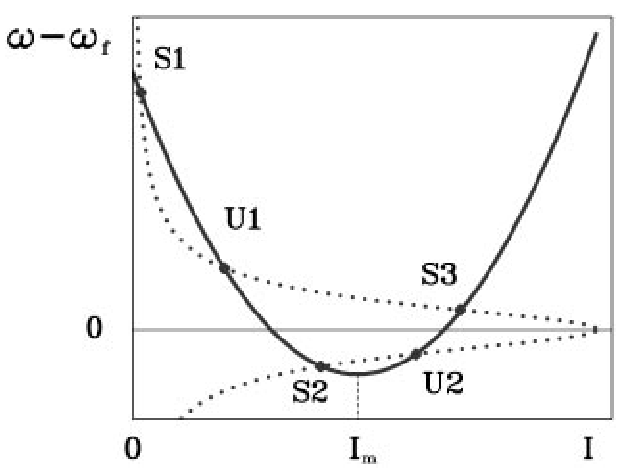

FIG. 1. Typical dependence of $\omega - \omega_f$ on I for a system with a minimum in $\omega(E)$ (full line), and solutions of (5) for stationary actions. The dotted line represents $\pm \hbar q_1'[1 - (\Gamma I/hq_1)^2]^{1/2}$. The intersections corresponding to stable points (attractors) and unstable ones (saddles) are labeled S1–S3 and U1 and U2, respectively.

$$U(q) = \frac{1}{2}\omega_0^2 q^2 + \frac{1}{3}\beta q^3 + \frac{1}{4}\gamma q^4,$$
$$\frac{9}{10} < \beta^2/\gamma\omega_0^2 < 4, \qquad (6)$$

with $\omega_0 = 1, \beta = 5/3, \gamma = 1$. With these coefficients in (6), there is a minimum in $\omega(I)$: $\omega_m = 0.805, I_m = 0.187$, $\omega_m'' \equiv d^2\omega(I_m)/dI_m^2 = 10.5$, $q_{1m} \equiv q_1(I_m) = 0.325$. Figure 2 shows the bifurcation diagram in the plane of the driving force parameters for $\Gamma = 0.011$. Its structure is typical of that expected for any system with an extremum in eigenfrequency as a function of energy (or action). The theoretical lines were obtained from the condition that curves corresponding to the left- and right-hand sides of (5) touch rather than cross each other (cf. Fig. 1); they are in good agreement with the analog electronic measurements (data points) based on the use of a standard technique [15] to model (1) and (6) which represent the first experimental observations of ZDNR. Within the region bounded by the full lines (except *very* close to cusps of full lines), point S1 in Fig. 1 is well separated in action from S2/S3. The response corresponding to S2/S3 is always strongly nonlinear. The response corresponding to S1 is linear in the region far below the upper full line. It starts to be nonlinear when closer to the line, but even then it is still significantly

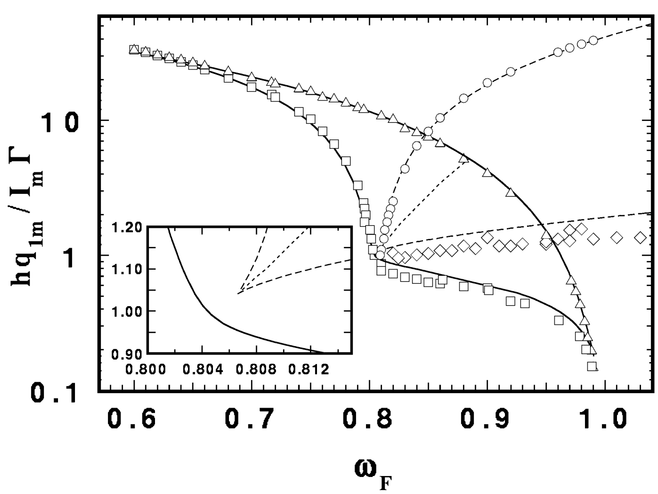

FIG. 2. The bifurcation diagram in the plane of the driving force parameters for (1) and (6) with $\omega_0 = \gamma = 1.0, \beta = 5/3, \Gamma = 0.011$ (the normalization of the vertical axis is $q_{1m}/\Gamma I_m = 159.0$). Full lines bound the region within which both the linear (S1) and one or both of nonlinear (S2, S3) responses can exist: The upper line (theory) and triangles (analog electronic simulation) mark the boundary of linear response, and the lower line and squares mark that for the nonlinear responses. Dashed lines bound the region where *both* nonlinear responses (S2, S3) coexist: The upper dashed line (theory) and circles (simulation) mark the boundary for the lower action attractor (S2), and the lower dashed line and diamonds mark that for the larger action attractor (S3). The calculated ZDNR-NR transition is shown by the dotted line. The inset provides an enlarged plot of the region near the cusp.

smaller than S2/S3; in order to distinguish S1 from S2/S3, we shall refer to it as "linear" within the whole region bounded by the full lines.

The evolution of the phase space with increasing ω_f, calculated for fixed h, is shown in Fig. 3. One can see a distinct difference in the structure of the BAs of the nonlinear responses at different ω_f. At smaller ω_f [Figs. 3(a) and 3(b)], the phase difference between attractor and saddle [16] for each BA is negligible, whereas for larger ω_f [Figs. 3(c)–3(e)] it is of the same order as the characteristic width of the BA. This holds true throughout the whole region enclosed by the full lines in Fig. 2 (except *very* close to bifurcation lines), the two types of behavior being separated by the dotted line. In analogy with the dissipationless case [7], the parameter ranges to the left and right of the dotted line can be defined as the zero-dispersion (ZDNR) and conventional (NR) stages of nonlinear response [and the definition can be formulated in a similar way for the original system (1) in terms of a stroboscopic Poincaré section].

In the dissipationless case, the transition between the ZDNR and NR stages as parameters change occurs [8] through *separatrix reconnection* [17], resulting in a different topology of separatrices between regions of trapped and untrapped motion: The separatrices are homoclinic or heteroclinic for ZDNR or NR, respectively [7,8]. In the presence of dissipation, the transition occurs typically via a *saddle connection* [2] as can be seen from Figs. 3(b) and 3(c) [18]. It also results in a change of topology of the BAs of the nonlinear responses. Just before the bifurcation, at the ZDNR stage, the basin corresponding to the larger action attractor (S3) encompasses the other one (S2), whereas the opposite applies for the NR stage just after the bifurcation, see Figs. 3(b) and 3(c). Beyond the close vicinity of the cusp, the frequency $\omega_f^{(tr)}$ of the ZDNR-NR transition can be shown to satisfy an asymptotic ($h \to 0$) formula which is valid for the general case rather than for (6) only:

$$\omega_f^{(tr)} = \omega_m + \mathrm{sgn}(\omega_m'')\,(|\omega_m''|/2)^{1/3} \times \left(\tfrac{3}{2}\, h q_{1m}\{(1-\eta^2)^{1/2} - \eta[\pi/2 - \arcsin(\eta)]\}\right), 2/3 \qquad (7)$$

$$\eta \equiv \frac{\Gamma I_m}{h q_{1m}} < 1\,,$$

$$\left[\frac{\Gamma}{|\omega_m''| I_m^2}\right]^{2/3} \ll \left[\frac{\omega_f^{(tr)}\omega_m}{\omega_m'' I_m^2}\right]^{1/2} \ll 1\,.$$

It can be found more exactly by numerical solution of (2) (Fig. 2).

It is interesting to note one more nontrivial bifurcation that has no analog in the dissipationless case: the alternation between the BA of one attractor, either encom-

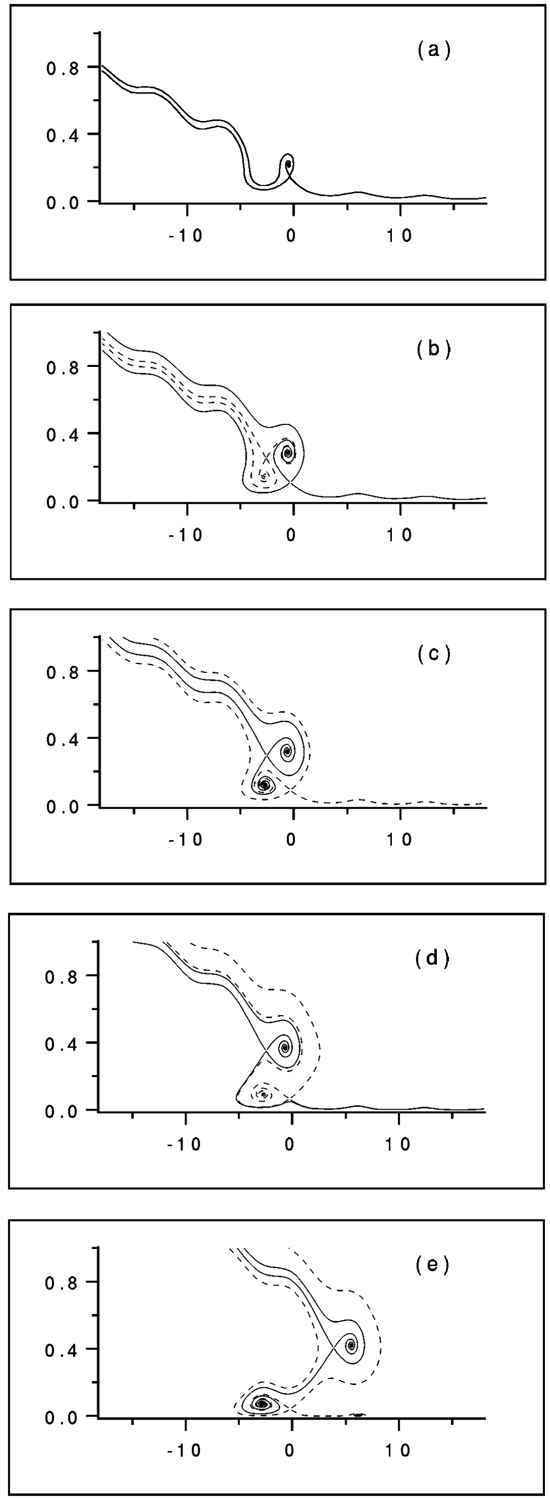

FIG. 3. Evolution with driving frequency ω_f of the basins of attraction (BAs) of nonlinear responses in a 2π band of the phase space of the slow variables (I ordinate, $\tilde{\psi}$ abscissa) for the same system as in Fig. 2 but averaged over the high-frequency oscillations [Eqs. (2) and (3)] for $h = 0.0143$ and (a) $\omega_f = 0.8$, (b) 0.83, (c) 0.85, (d) 0.88, and (e) 0.92. The boundaries of the BAs of S2 (S3) and trajectories emerging from the saddles are drawn by full (dashed) lines. One can obtain the complete phase space by repeating the above picture with a period 2π in $\tilde{\psi}$.

passing the other one or simply moving around it [Figs. 3(c)–3(e)], as the frequency is changed (for smaller Γ, a similar alternation takes place at the ZDNR stage also). These bifurcations are also of the saddle connection type, and, together with the ZDNR-NR transition, they are characteristic of any oscillatory system whose variation of eigenfrequency with energy possesses an extremum. The ZDNR-NR and encompassing or moving around bifurcations are of particular interest because they may be expected to give rise to unusual fluctuational phenomena (see below). Yet other global bifurcations are also possible; they will be considered in detail elsewhere.

In the presence of noise, previously stable states become metastable. Escape from an attractor takes place with overwhelming probability via one of the saddle points of its BA (see, e.g., [19,20]). Thus the transition of a saddle point from the BA of one nonlinear response to the other at the ZDNR-NR bifurcation would be expected to result in a jump-wise change in the probabilities of fluctuational transitions between the nonlinear and linear responses [in Figs. 3(b) and 3(c), the trajectory outgoing to the right from the saddle point with a lower action goes to the attractor corresponding to linear response]: For ZDNR, just before the bifurcation, there are no direct transitions between attractors S1 and S2, whereas for NR, just after the bifurcation, there are no direct transitions between S1 and S3. A similar effect should occur at global bifurcations of the "encompassing or moving around" type. Note that these various changes cannot be described in terms of frequency-response curves [1] because nothing happens to the attractors themselves at a global bifurcation.

Thus the sequence of global bifurcations undergone by a periodically driven oscillatory system whose dependence of eigenfrequency on energy possesses an extremum should manifest itself in some very unusual dependences of the fluctuational interattractor transition probabilities on parameters of the driving force. They, too, will be studied in detail elsewhere; here, we consider briefly the stationary fluctuations in the system. It can be shown that, for small enough h, there is a repopulation of the attractors associated with the ZDNR-NR transition: During the ZDNR stage, the population of attractor S3 is bigger than that of S2, whereas it is the other way round for NR. As shown in [9], this repopulation produces corresponding changes in the fluctuation spectra which are at their most pronounced for intermediate noise intensities and amplitudes of the driving force. The spectral density of fluctuations of the coordinate consists typically of a narrow transition peak at the frequency of the driving force which is due to fluctuational transitions between the metastable states (cf. [21]), Stokes and anti-Stokes bands which are due to small fluctuations around S2, S3, and a comparatively broad peak at ω_0 caused by small fluctuations near S1. The repopulation of S2, S3 reverses the relative amplitudes of the Stokes and anti-Stokes bands and can also result in a rather unusual dependence of the intensity of the transition peak on the frequency of the driving force: Quite unlike the case of the Duffing oscillator [21], it can have a two-humped structure.

Zero-dispersion nonlinear resonance and related phenomena offer a rich manifold of interesting problems, only a few of which have been identified or discussed above. Others that would be likely to repay early investigation are associated with the dynamical chaos that is to be anticipated for larger amplitudes of the driving force, and its interaction with external noise.

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