Symmetry Breaking of Fluctuation Dynamics by Noise Color

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Activated escape is investigated for systems that are driven by noise whose power spectrum peaks at a finite frequency. Analytic theory and analog and digital experiments show that the system dynamics during escape exhibit a symmetry-breaking transition as the width of the fluctuational spectral peak is varied. For double-well potentials, even a small asymmetry may result in a parametrically large difference of the *activation energies* for escape from different wells.

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The problem of diffusion and activated escape in systems away from thermal equilibrium is attracting increasing attention in diverse contexts, from crystal growth [1] to switching of current in microstructures [2], modes in lasers [3], and biological systems [4]. In many cases the system of interest is nonequilibrium because it is driven by nonthermal noise. Such noise can give rise to new and interesting phenomena, e.g., the onset of directed diffusion in a spatially periodic potential (ratchet) [5], in which case the diffusion rate and direction depend on the *shape* of the noise power spectrum $\Phi(\omega)$ [6,7].

One would expect the escape probabilities to be determined by $\Phi(\omega)$ for $\omega \lesssim t_r^{-1}, \omega_v$ where t_r and ω_v are the characteristic relaxation time and vibrational frequency of the system. However, the situation is more complicated in the important case of quasimonochromatic noise (QMN) where $\Phi(\omega)$ has a peak at a comparatively high frequency $\omega_0 \gg t_r^{-1}, \omega_v$ [8–10]. Here, the optimal fluctuation for escape may correspond to "fluctuational preparation of the barrier," analogous to phonon-induced barrier preparation in solids [11]. The system is forced by the noise to fluctuate at frequency $\approx\omega_0$. As the amplitude of such random vibrations increases, the shape of the effective potential for their center of motion q_c alters, as illustrated for the case of a bistable potential in Fig. 1. For a large enough amplitude, the initially occupied metastable potential well for q_c may disappear, so that the system then escapes from it. Such an adiabatic escape scenario, during which the system is most likely to move along an oscillating path, was predicted [8] and observed [9] for the case where the half-width of the peak in the noise spectrum $\Gamma \ll t_r^{-1}$. Recent numerical analysis indicates [10] that, as Γ increases to $\sim t_r^{-1}$, the most probable escape path (MPEP) becomes smooth.

In this Letter we show that the transition with changing noise spectrum between different types of MPEPs, and thus between escape scenarios, is a symmetry-breaking transition. The symmetry is broken in *time*: it corresponds to the onset of fast oscillations of the MPEP, with an arbitrary initial phase, to leading order. We find the location of this transition and show that, at the transition, the activation energy of escape displays a nonanalytic dependence on the noise color parameter.

The second effect that we discuss occurs when there is an asymmetry in the potential. Even a small asymmetry may lead to a parametrically large difference in the activation energies of interwell transitions in opposite directions, giving rise to a dc current in a periodic potential or to strong effective localization in one of the wells of a double-well potential.

We will consider such effects for an overdamped system fluctuating in a double-well potential $U(q)$ (cf. Fig. 1), with equation of motion

$$\dot{q} = -U'(q) + f(t). \tag{1}$$

Here, $f(t)$ is a quasimonochromatic noise. An example of QMN is provided by Brownian motion of an underdamped harmonic oscillator, which is described by the equation

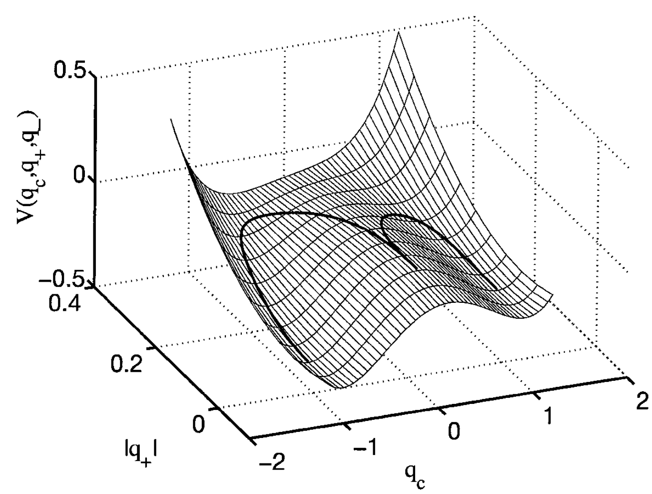

FIG. 1. The average potential V (7) for motion of the vibration center q_c as a function of the vibration amplitude $|q_-| = |q_+|$. The initial double-well potential $U(q)$ is given by $V(q, 0, 0)$. For an asymmetric $U(q)$, only one of the wells disappears with increasing $|q_\pm|$. The plot refers to a double-well potential $U(q) \equiv V(q, q_\pm = 0) = -q^2/2 + q^4/4 - 0.15q$. The bold curves show most probable escape paths for $\Gamma = 0.3$.

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$$\hat{M}f(t) = \xi(t), \qquad \hat{M} = \frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2, \quad (2)$$

where ω_0 and Γ are the oscillator frequency and damping, respectively, $\xi(t)$ is Gaussian white noise, and $\langle\xi(t)\xi(t')\rangle = 2\tilde{D}\delta(t-t')$ (for thermal noise, the characteristic intensity $\tilde{D} = 2\Gamma k_B T$). For small Γ/ω_0, the power spectrum $\Phi(\omega)$ of the noise $f(t)$ has a narrow Lorentzian peak at ω_0, of half-width Γ. The probability density of realizations of $f(t)$ is $\propto \exp(-\mathcal{R}_0[f]/\tilde{D})$ [12], with

$$\mathcal{R}_0[f] = \frac{1}{4}\int_{-\infty}^{\infty} dt\,[\hat{M}f(t)]^2. \quad (3)$$

The probabilities W_{nm} of fluctuational transitions between the potential wells $n = 1, 2$ of $U(q)$ can be obtained [8,13] (see also [14]) using Feynman's idea [12] that the trajectory of the system $q(t)$ is uniquely determined by the force $f(t)$. Different realizations of $f(t)$ can lead to a transition, but for small noise intensity $\tilde{D}$, their probability densities are exponentially different. To logarithmic accuracy, W_{nm} is determined by the probability density of the most probable realization, i.e., the one that minimizes $\mathcal{R}_0[f]$ subject to the constraint (1). In such an optimal fluctuation, the system is driven along an instantonlike optimal path $q(t)$ from the bottom q_n of the nth well, which is occupied for $t\to-\infty$, to the top of the potential barrier q_b between the wells for $t\to\infty$, and the driving optimal force $f(t)\to 0$ for $t\to\pm\infty$ [8]. The optimal trajectories $q(t), f(t)$ provide a minimum to the functional

$$\mathcal{R}[f,q] = \mathcal{R}_0[f] + \int_{-\infty}^{\infty} dt\,\lambda(t)[\dot{q} + U' - f(t)], \quad (4)$$

where $\lambda(t)$ is a Lagrange multiplier. The escape rate

$$W \propto \exp(-R/\tilde{D}), \qquad R = \min\mathcal{R}[f,q]. \quad (5)$$

We will investigate QMN-activated escape in the most interesting case where Γt_r can be arbitrary, but the characteristic noise frequency $\omega_0 \gg \Gamma, 1/t_r$ [$t_r = \max_n[1/U''(q_n)]$ is the relaxation time of the system]. In this case the Euler equations for the functional (4) can be analyzed [8] using a standard averaging method. The motion of the system is a superposition of fast vibrations at frequency ω_0 and slow motion of the vibration center q_c,

$$q(t) = q_c(t) + \sum_{\alpha=\pm} q_\alpha \exp(i\alpha\omega_0 t). \quad (6)$$

The two motions are mixed by the nonlinearity of the potential $U(q)$. The time evolution of q_c and of $q_+ = q_-^*$ occurs on the scale $1/\Gamma, t_r$ and is determined by the time-averaged potential $V(q_c, q_+, q_-)$,

$$V = \frac{1}{2\pi}\int_0^{2\pi} d\phi\,U(q_c + q_+e^{i\phi} + q_-e^{-i\phi}). \quad (7)$$

To lowest order in $\Gamma/\omega_0, 1/\omega_0 t_r$, the variational problem (4) can be reduced [15] to a much simpler one, $R = \min\bar{\mathcal{R}}[q_c, q_\pm]$, with

$$\bar{\mathcal{R}}[q_c, q_\pm] = \frac{1}{2}\omega_0^4\int dt\,L(\dot{q}_c, q_c; \dot{q}_\pm, q_\pm),$$
$$L = \frac{1}{2}(\dot{q}_c + V_c')^2 + 4(\dot{q}_+\dot{q}_- + \Gamma^2 q_+q_-), \quad (8)$$

where $V_c' = \partial V/\partial q_c$. The functional $\bar{\mathcal{R}}$ takes the form of a classical action for a particle with coordinates $q_c, q_\pm$ and a Lagrangian L. Escape is determined by the extreme trajectory which starts from $q_c = q_n, q_\pm = 0$ for $t\to-\infty$ and approaches $q_c = q_b, q_\pm = 0$ as $t\to\infty$.

From (8), the equation for $q_\pm$ is of the form

$$-\ddot{q}_\alpha + \frac{1}{4}(\dot{q}_c + V_c')V''_{c,-\alpha} + \Gamma^2 q_\alpha = 0, \qquad \alpha = \pm, \quad (9)$$

where $V''_{c,\alpha} = \partial^2 V/\partial q_c\partial q_\alpha$. The potential V (7) depends on q_+, q_- only in terms of the product q_+q_-. Therefore the phase $\arg q_+ = -\arg q_-$ is a cyclic variable. From (8), $\bar{\mathcal{R}}$ is minimal for $(d/dt)\arg q_+ = 0$, and so by choosing the time origin in (6) one can make $q_+(t) = q_-(t)$. Another important consequence is that Eq. (9) has a trivial solution $q_\pm(t) = 0$. It corresponds to a smooth (nonoscillating) optimal path $q(t)$, with

$$\dot{q}_c(t) = \dot{q}(t) = V_c'(q_c, 0, 0) = U'(q). \quad (10)$$

The path $\dot{q} = U'(q)$ is the well-known solution for the MPEP in an overdamped system in thermal equilibrium. Such a system is described by (1) with $f(t)$ being white noise with a flat power spectrum of height $2k_BT$. In the present case the solution (10) arises because the noise spectrum $\Phi(\omega)$ has a broad low-frequency plateau, of width ω_0 which is parametrically larger than the relaxation rate of the system, and of height $2\tilde{D}/\omega_0^4$. Correspondingly, $R = \omega_0^4\Delta U$ [$\Delta U = U(q_b) - U(q_n)$], and the escape rate $W \propto \exp(-\omega_0^4\Delta U/\tilde{D})$.

The smooth optimal path (10) provides a minimum to the functional $\bar{\mathcal{R}}$ if the eigenvalue problem

$$\int dt' \sum_j [\delta^2\bar{\mathcal{R}}/\delta q_i(t)\delta q_j(t')]\psi_{nj}(t') = \lambda_n\psi_{ni}(t) \quad (11)$$

has only non-negative eigenvalues λ_n [here, $i, j = c, \pm$, and the derivatives are calculated for the path (10), $\psi_{ni}(\pm\infty) = 0$]. The matrix $\delta^2\bar{\mathcal{R}}/\delta q_i\delta q_j = 0$ for $i = c$, $j = \pm$ or $i = j = \pm$. Using standard arguments [16] one can show that all eigenvalues of $\delta^2\bar{\mathcal{R}}/\delta q_c\delta q_c$ are positive except for the trivial zero eigenvalue with the eigenfunction $\propto \dot{q}_c$.

Of special interest in (11) is the equation for $\psi_{n+} = \psi_{n-}$,

$$-\ddot{\psi}_{n\pm} + \frac{1}{2}(V_c'V'''_{c,-,+} + 2\Gamma^2)\psi_{n\pm} = \lambda_n\psi_{n\pm}. \quad (12)$$

Since $|V_c'|, |V'''_{c,-,+}| \sim 1/t_r$, the eigenvalues are always positive for large Γt_r. The smooth path (10) is indeed the optimal escape path in this case.

As Γt_r decreases, the lowest eigenvalue of (12) λ_0 may become equal to zero. This signals instability of the solution (10) and defines the critical Γ_{cr} where it occurs. For still smaller Γ the minimum of $\bar{\mathcal{R}}$ is provided by the solution with finite $q_\pm$. From (6), such a solution oscillates in time, with an arbitrary phase in the approximation (8), i.e., the time symmetry of the MPEP is spontaneously broken.

The activation energy R is a nonanalytic function of Γ for $\Gamma = \Gamma_{cr}$. The situation is similar to a second order phase transition, with $\bar{\mathcal{R}}$ and $\psi_{0\pm}(t)$ playing the roles of the free energy functional and the soft mode. Indeed, for small $|\Gamma - \Gamma_{cr}|$, one can expand the functional $\bar{\mathcal{R}}[q_c, q_\pm]$ (8) in deviations $\delta q_i = \sum c_n \psi_{ni}$ of the actual trajectory from the extreme path (10), with ψ_{ni} being the normalized eigenfunctions of (11) for $\Gamma = \Gamma_{cr}$. The coefficient of $|c_0|^2$ in $\delta\bar{\mathcal{R}}/\omega_0^4$ is $2(\Gamma^2 - \Gamma_{cr}^2)$. Therefore c_0 plays the role of an order parameter. For $\Gamma > \Gamma_{cr}$ the minimum of $\bar{\mathcal{R}}$ is reached for $c_0 = 0$. For $\Gamma < \Gamma_{cr}$, it is reached for finite $|c_0|$ determined by higher-order terms in the expansion of $\bar{\mathcal{R}}$. The nontrivial part of $\delta\bar{\mathcal{R}}/\omega_0^4$, after appropriate renormalization, takes the familiar form $a|c_0|^4 + 2(\Gamma^2 - \Gamma_{cr}^2)|c_0|^2$, with a being a Γ-independent constant. Therefore $R/\omega_0^4 = \Delta U - (\Gamma^2 - \Gamma_{cr}^2)^2/a$ for $\Gamma < \Gamma_{cr}$, and $d^2R/d\Gamma^2$ is discontinuous for $\Gamma = \Gamma_{cr}$.

Away from Γ_{cr}, the prefactor in the escape rate is determined by the eigenvalues λ_n (cf. [16]). It blows up as Γ approaches Γ_{cr}.

We note an interesting similarity between the above time-symmetry-breaking transition and breaking of the *spatial* symmetry of the MPEP in two-variable white-noise-driven systems that was first found and investigated by Maier and Stein [17]. As in (8), the Lagrangian for the optimal fluctuational paths analyzed in Ref. [17] was even in one of the variables. In a way, a one-variable QMN-driven system provides a natural realization of the model [17].

A solution of the variational problem (8) that is qualitatively different from (10) can be obtained in the limit of small Γt_r [8]. It has the form $q_\pm(t) = q_+^{(0)} \exp(-\Gamma|t|)$ for $|t|/t_r \gg (\Gamma t_r)^{-1/2}$, with $V_c'(q_c, q_\pm) = 0$. This solution corresponds to the vibration center q_c following the increase of $q_\pm$ adiabatically and staying at the minimum of the average potential V for given $q_\pm$ until this minimum merges with the barrier top (see Fig. 1). This occurs for some $q_+ = q_- = q_+^{(0)}$ such that $V_c' = V_{cc}'' = 0$. From (8)

$$R = 4\omega_0^4 \Gamma |q_+^{(0)}|^2. \tag{13}$$

The above solution corresponds to the "barrier preparation" for the vibration center q_c by the slowly varying vibration amplitude $q_\pm$ [8] and gives an activation energy proportional to the small parameter Γ. However, it applies only if the potential minimum of V from which the system escapes merges with the barrier top with increasing $q_\pm$. It is clear from Fig. 1 that, in the case of an asymmetric double-well potential, this only happens for one of the wells. For the other well, the barrier for q_c remains finite for all $q_\pm$. Therefore the value of R, although less than $\omega_0^4 \Delta U$, is not proportional to Γ. This means that, even for a weakly asymmetric potential, activation energies of escape from different wells are *parametrically different*. Similarly, for a periodic potential of the ratchet type (see [5]), the activation energies for diffusion in opposite directions differ, leading to strongly directed diffusion for low noise intensities [15].

To test these predictions, we have performed analog and digital simulations of a QMN-driven system. The techniques are described in [18]. The activation energies were extracted from measured escape rates as functions of noise intensity. We chose the simplest system: a biased double-well Duffing oscillator, with potential

$$U(q) = -\frac{1}{2} q^2 + \frac{1}{4} q^4 - Aq. \tag{14}$$

For $A = 0$ the two wells are symmetrical. The relaxation time $t_r = 1/2$. For large Γt_r, when (10) applies, the escape activation energy $R/\omega_0^4 = 1/4$ whereas, from (13), $R/\omega_0^4 = 2\Gamma/3$ for $\Gamma t_r \ll 1$. The critical value Γ_{cr}, where time symmetry of the MPEP (10) breaks down, can be obtained from Eq. (12) using the explicit form of the optimal path (10) $q_c(t) = \mp \exp(-t/2)(2\cosh t)^{-1/2}$ [the signs $\mp$ correspond to escape from the left and right wells of $U(q)$, respectively]. The soft mode $\psi_{0\pm}(t) = C(\cosh t)^{-1/2}$, and $\Gamma_{cr} = 1/2$.

A sharp change of R as a function of Γ for $\Gamma = 1/2$ is observed in the experimental data (plusses) in Fig. 2. As expected for a symmetry-breaking transition, $R = \text{const}$ for $\Gamma > 1/2$, whereas $\delta R \propto (\Gamma - \Gamma_{cr})^2$ for small positive $\Gamma_{cr} - \Gamma$. The data are in excellent agreement with the results (crosses) obtained by solving the Euler equations for the reduced functional $\bar{\mathcal{R}}$ (8). They are also close to results (squares) obtained from the Euler equations for the functional R (4) [8] using the numerical method of Einchcomb and McKane [10(b)]. At criticality the extreme paths of $\bar{\mathcal{R}}$, which emanate from ($|q_c| = 1, q_\pm = 0$), are focused at the barrier top $q_\pm = q_c = 0$ (see inset of Fig. 2), which gives rise to blowing up of the prefactor in the escape rate. Similar focusing was observed and carefully analyzed at a spatial symmetry-breaking transition in a white-noise-driven system [17].

For $A \neq 0$ the potential (14) is asymmetric. Breaking of the time symmetry of the MPEP occurs in both wells, but for different Γ. For the well at positive q in Fig. 1 (well 1), $\Gamma_{cr} - 1/2 = 2A/\pi$, to first order in A. This well is deeper for $A > 0$ [clearly, the escape activation energy $R_1(A) = R_2(-A)$]. This means that the fluctuational barrier preparation in the deeper well starts for larger Γ. Therefore, unexpectedly, the ratio $r = R_1/R_2$ is a *nonmonotonic* function of Γ.

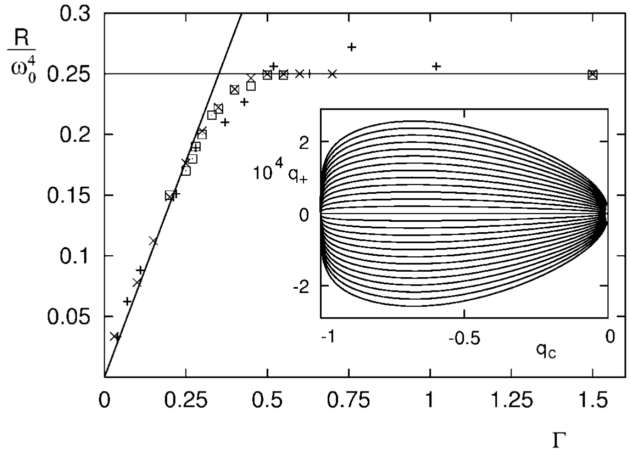

FIG. 2. The activation energy of escape R for a symmetric Duffing oscillator as a function of the noise bandwidth Γ, showing discontinuity of $d^2R/d\Gamma^2$ at the symmetry-breaking transition for $\Gamma_{cr} = 0.5$. Analog experimental data (+) for $\omega_0 = 10$ are compared to numerical results ($\times$) for the reduced Lagrangian (8) and appropriately extended numerical results (squares) [10(b)] for the full variational problem. Lines show the asymptotics. Inset: Trajectories for the Lagrangian (8) that start from $q_c = -1, q_+ = q_- = 0$ for $t \to -\infty$ and are focused into the barrier top $q_c = q_\pm = 0$ for $\Gamma = \Gamma_{cr}$.

For large Γ both R_1 and R_2 are independent of Γ, and $r = \text{const} > 1$ for $A > 0$. As Γ decreases, R_1 starts decreasing first, and r *decreases*. However, with further decrease in Γ, r starts sharply increasing, as seen in Fig. 3. This happens because, for small Γ, it is only for the shallow well that the barrier of the average potential V can disappear, as seen in Fig. 1. Therefore $R_2 \propto \Gamma$ for small Γ, and r becomes large; see Fig. 3. It is seen from Fig. 1 that the MPEPs in different wells have already become very different even when Γ has fallen only slightly below Γ_{cr}.

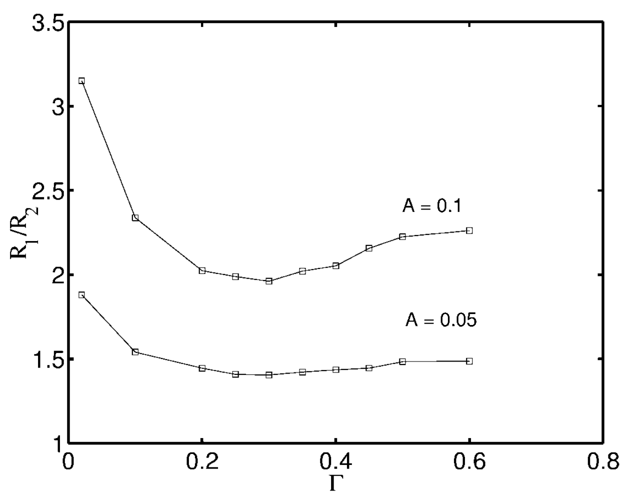

FIG. 3. The ratio of the activation energies of escape $r = R_1/R_2$ from the wells 1 and 2 of the biased Duffing oscillator (14).

In conclusion, for bistable systems driven by a quasi-monochromatic noise the *activation energies* of escape from different wells are *parametrically* different, as are also the activation energies for diffusion in opposite directions in a periodic potential. This allows the use of QMN for highly selective species separation. The transition from most probable escape along a smooth path, to that along an oscillating path, is a symmetry-breaking transition. The second derivative of the activation energy with respect to the bandwidth of the noise is discontinuous at the transition.

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