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The effect of noise on strange nonchaotic attractors

I. A. Khovanov[a], N. A. Khovanova[a], P. V. E. McClintock[b*],
V. S. Anishchenko[a]

[a]Department of Physics, Saratov State University,
Astrakhanskaya St. 83, 410026 Saratov, Russia
[b]Department of Physics, Lancaster University, Lancaster LA1 4YB, UK

* *Corresponding author:* P.V.E. McClintock
Department of Physics, Lancaster University, Lancaster LA1 4YB, UK
phone: +44 1524 593073 / 593206
fax: +44 1524 844037
e-mail: p.v.e.mcclintock@lancaster.ac.uk

Abstract

The dynamical response of an underdamped Duffing oscillator to a quasiperiodic force is investigated in the presence and absence of very weak additive noise. Particular attention is focused on the effect of noise on the characteristics of strange nonchaotic attractors (SNAs). It is concluded that even extremely weak noise is sufficient to induce dynamical complexity in an SNA.

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keywords: strange nonchaotic attractor, noise in nonlinear system, Lyapunov exponent, dynamical and geometrical complexity.

1. Introduction

The defining property of chaotic motion is conventionally taken to be an exponential divergence of nearby trajectories, i.e. a sensitive dependence on initial conditions [1, 2]. Usually, it is also characterized by a strange attractor, with a complex fractal structure in phase space. Until recently the terms *strange attractor* and *chaotic attractor* were taken to be synonymous, although the two notions refer to quite different properties of the attractor: *chaotic* reflects its dynamical properties, i.e. the presence of exponential divergence of the trajectories, whereas *strange* implies the complexity of its structure, i.e. relates to geometrical properties.

It is now well established that, not only can an attractor can be strange/chaotic and nonstrange/nonchaotic (i.e. regular), but it can also be strange/nonchaotic [2, 3] and nonstrange/chaotic [4, 5, 6]. Because of the seeming oddity of the latter two combinations of features, these attractors have evoked much interest and attention. Many papers are devoted to the study of the strange nonchaotic attractor (SNA), an object that typically arises in quasiperiodically driven systems as an intermediate link between regular and chaotic attractors [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

An SNA has a geometrically complex structure (the attractor is not a finite set of points, and it is not piecewise differentiable), but it exhibits no sensitivity to initial conditions. SNAs have been studied extensively, both theoretically and experimentally. Most papers consider the process by which an SNA can be created from a regular attractor; or its disappearance in the transition to a strange chaotic attractor [9, 10, 11, 12, 13, 14, 15, 16, 17]. It was found that a typical SNA trajectory is characterized by a fluctuation of its finite-time Lyapunov exponents between negative and positive values [10, 12] but that, asymptotically as $t \to \infty$, the Lyapunov exponent is negative. Lai et al. [13, 14] compared the properties of an SNA with a chaotic attractor, and they showed that the difference relates solely to the sign of the LLE. Shuai and Wong [16] investigated a map with periodically fluctuating finite-time Lyapunov exponents. They described a different route to the creation of an SNA in this map, and discussed the effect of truncation errors on the dynamics.

In all these papers, both experimental and theoretical, it was the sign of the LLE that was used to establish the nonchaotic character of the SNA, and thus to distinguish the SNA from a chaotic attractor. At first sight, this is entirely rerasonable because a positive LLE implies that an infinitely-small perturbation will grow exponentially, indicating a sensitivity to initial conditions leading to unpredictability of the motion, mixing, and thus chaos. A negative LLE implies that an infinitely-small perturbation tends to zero as $t \to \infty$. The LLE is also used to demonstrate the robustness of the SNA to weak noise: it has been shown that weak noise does not change the sign of LLE. However it is well known [24] that, for a noisy system, the LLE is not good indicator of complexity and predictability of the motion and, in [24] another approach was offered. It examines the predictability (complexity) of the trajectory on a strong nonuniform attractor: if, following a period of relaxation, there are time intervals during which nearby trajectories diverge exponentially and can be in any region of the attractor, then such an attractor is described as complex, because during such time intervals it is impossible to predict the trajectory's behavior. Under such conditions, the attractor may have a negative LLE.

We will call the attractor *complex*, if the trajectory behavior is unpredictable in the sense described above. To avoid misunderstandings, we emploiy the term *chaotic* (*non-chaotic*) to imply a positive (negative) LLE. In other words, we will refer to the attractor as chaotic (nonchaotic), if the divergence (convergence) of trajectories predominates on average.

In this Letter we discuss the effect of noise on an SNA in an underdamped system and we show that even extremely weak noise is sufficient to convert it to complex attractor. We will consider the peculiarities of time evolution from nearby initial conditions in an underdamped system driven by a quasiperiodic signal. As an example, we consider the quasiperiodically driven Duffing oscillator:

$$\ddot{x} + \alpha\dot{x} + \beta(x^3 - x) = A(\sin\omega_1 t + \sin\omega_2 t) + \sqrt{D}\xi(t). \tag{1}$$

Here α and β govern the dynamics of the system: we choose $\alpha = 0.632$, $\beta = 4$. The quantities A, ω_1 and ω_2 are respectively the amplitude and frequencies of the external two-frequency forcing; $\xi(t)$ is a Gaussian white noise of intensity D. We examine the case of an irrational ratio between the frequencies: $\omega_1 = 2.1235$, $\omega_2 = \sqrt{5}$. As a control parameter we choose the amplitude A of the quasiperiodic forcing.

2. Dynamics in the absence of noise

We first analyze the behavior of the oscillator in the absence of noise. We calculate the LLE over a time 10000 $2\pi/\omega_1$ by use of a standard algorithm [18]. Fig. 1(a) shows that, with increasing signal amplitude, the LLE also grows and that it changes sign at $A \approx 0.7875$. Consequently at this moment a chaotic attractor appears. To distinguish the SNA from the quasiperiodic attractor it is necessary to use other characteristics [7, 9]: we construct the stroboscopic section in coordinates $(x(t_n), \Theta_n)$, where $\Theta_n = (n\frac{\omega_2}{\omega_1})(\text{mod } 1)$, $t_n = n2\pi/\omega_1$; then choose a point on section $(x(t_{n'}), \Theta_{n'})$, and its nearest calculated neighbor in the Θ coordinate $(x(t_{n''}), \Theta_{n''})$, such that $|\Theta_{n'} - \Theta_{n''}|$ and $|x(t_{n'}) - x(t_{n''})|$ are small; finally, we monitor the evolution of the distance between the coordinates $|x(t_{n'+m}) -$

$x(t_{n''+m})|$, with the distance in the Θ direction remaining constant. If the distance between points varies erratically with time and exhibits sharp and short-lived bursts, then it implies the presence of discontinuities on the attractor, thereby indicating its fractal structure or strangeness. Based on this method we have found that an SNA arises at $A \approx 0.519$. Thus we can observe three distinct regions: a quasiperiodic attractor for $A < 0.519$, an SNA for $0.519 < A < 0.7875$, and a chaotic attractor for $A > 0.7875$. We now consider the dynamical properties of the attractor inside these regions.

Since the system (1) is passive, its dynamics is in the main formed by the external forcing. The external force (dashed line of Fig. 2) can be considered as a periodic field with a slowly-varying amplitude. Thus the motion can be either stable or unstable, depending on the instantaneous amplitude, and it becomes unstable if the amplitude exceeds some threshold. Thus, with the growth of amplitude, the proportion of unstable motion in the total system dynamics will increase. We have calculated the local (finite-time) largest Lyapunov exponents [12, 16, 19] as

$$\Lambda(t', T) = \frac{1}{T} \ln \frac{|\Delta(t' + T)|}{|\Delta(t')|},$$

where $|\Delta(t)| = \sqrt{\delta x^2(t) + \delta \dot{x}^2(t)}$, δx and $\delta \dot{x}$ are the solutions of equation variational to (1), t' is the time at which an amplitude minimum of the applied quasiperiodic force occurs, and $T = 2\pi/\omega_1$. The finite-time LLE correspond the local LLE of stroboscopic (Poincaré) map of system (1). Note that the LLE is obtained by taking the limit $T \to \infty$. It can clearly be seen in Fig. 2 that the system trajectory divides into alternating time regions having positive and negative local exponents [10]. The same temporal behavior of local LLEs is observed for all regimes: the torus, the SNA and the chaotic attractor [20]. However, their asymptotic behavior as $t \to \infty$ differs: a torus and an SNA have negative LLEs, whereas a chaotic attractor has a positive one.

The connection between chaos and a positive sign of the LLE has been strictly proven only for hyperbolic and quasihyperbolic systems [21]. For the nonhyperbolic systems with which we usually deal in practice, including the Duffing system (1), there is as yet no rigorous proof of such a connection. In many papers, therefore, in addition to calculating the LLE, authors also explore sensitivity to initial conditions by use of a variety of more direct methods [4, 9, 14, 22, 23]. For example, in [14, 23] the evolution of an ensemble of initially near trajectories, characterized by the same phase of the external force, is considered. If the nearby trajectories diverge exponentially over the entire attractor and, consequently, the distance between them reaches the size of the attractor, then it testifies to an unpredictability of the trajectories' behavior, i.e. temporal disorder, and the attractor is therefore classified as complex [24]. In the opposite case the attractor is non-complex (regular). In the absence of noise a complex attractor has a positive LLE, i.e. it is chaotic. Following these remarks we consider the time evolution of an ensemble of nearby trajectories, a so-called a snapshot of slices of the attractor [14, 23]: we choose a set of points on the phase plane $x - \dot{x}$ on a circle of radius 0.00001 around an attractor point, and monitor its time evolution. To describe the evolution of the trajectory ensemble we have calculated the dispersion $S_{\dot{x}}(t)$ [14, 23], which is defined as:

$$S^2_{\dot{x}}(t) = \frac{1}{N}\sum_{i=1}^{N}\left[\dot{x}_i(t) - \langle \dot{x}_i(t)\rangle\right]^2 ,$$

where $N = 500$ is the number of points and "$\langle . \rangle$" indicates an ensemble average.

Fig. 3(a) shows that, for the chaotic attractor, $S_{\dot{x}}(t)$ periodically increases and decreases at frequency $\omega_2 - \omega_1$. In the time domain we can distinguish periodically alternating regions within which the size of the snapshot of slices of the attractor is qualitatively different (Fig. 4): depending on the sign of the local LLEs, the points either diverge exponentially over the entire attractor (positive sign) or they converge in very small regions of the phase plane (negative sign). Thus, in the quasiperiodically driven Duffing oscillator the chaotic spreading out of trajectories has a periodic alternating character. Note that similar behavior is distinguished from *typical* evolutions of nearby trajectories for chaotic attractor, e.g. in non-quasiperiodic systems, when the value S increases from initial value, reaching certain maximum, but thereafter practically does not change [22].

For the SNA, the ensemble of trajectories converges to a single trajectory, i.e. the size of the slice falls from its initial value to zero during a characteristic relaxation time (Fig. 3(b)), during which local trajectory spreading over the entire attractor occurs. It is clear that if, after this relaxation period, the trajectory of the system was again perturbed, further spreading would occur. Indeed, the sum of local LLEs over the regions with positive sign can reach $\Lambda_s \sim 10 - 15$. Consequently, any initial perturbation exceeding the value $|\delta \mathbf{x_0}| = S_A \exp(-\Lambda_s) \sim 1.e-6$ must attain the size of the attractor S_A in these regions. Note, however, that this conclusion has a value-dependent character since, as shown in [25], the evolution of a finite-size perturbation differs from that of the infinitly-small perturbation described by a local LLE. Hence conclusions about the evolution of a finite-size perturbation cannot be reached without also considering the evolution of a trajectory ensemble.

For the torus we also observe local trajectory spreading (not shown here), but the distance between trajectories does not attain the attractor size: we recall that perturbations for the torus and SNA regimes tend to zero asymptotically as $t \to \infty$.

3. Dynamics in the presence of noise

In reality, of course, we almost always deal with finite-size perturbations, because the fluctuations in real systems occur continuously. In the presence of fluctuations the size of the perturbation cannot be less than some value related to the noise intensity [24]. In other words, when we take account of fluctuations, two initially close trajectories will always differ from each other by some value; this is true even for the regular regime. Consequently, for the torus, SNA and chaotic regimes, perturbations must evolve as follows: during time intervals with positive local LLEs the perturbation increases exponentially, whether mixing or not (depending on the noise intensity); and during the time intervals with negative local LLEs the perturbation decreases to a non-zero value defined by the noise intensity.

We now consider the behaviour of (1) in the presence of noise. For the sake of definiteness, we choose a noise intensity comparable with that of the weak internal noise of an analog electronic model of (1). Such noise originates in the analog components [26] and is usually at least $\sim 1\ \mu V$ [26] (and often much larger than this). For our numerical simulation we choose $D = 0.000001$.

We first address the question: does this weak noise change the properties of the attractors? To answer the question we have calculated (using the technique of [7]) the information dimension d_i and scaling coefficient β of the spectral distribution function $N(\sigma) \sim \sigma^{-\beta}$ for different regimes of the attractor, both in the presence and in the absence of noise. The results are shown in the Table 1. The dimension was calculated for the Poincare section in coordinates $(x(t_n), \theta_n = (n\frac{\omega_2}{\omega_1})(\mathrm{mod}\ 1))$, $t_n = n2\pi/\omega_1$. The coordinate $x(t_n)$ was used for calculating the scaling of the spectral distribution. It can be seen in the Table 1 that, for the noise-free case, there is no obvious qualitative difference between measures of the SNA ($A = 0.6$, $A = 0.7$) and measures of the chaotic attractor ($A = 0.8$, $A = 0.9$). The behavior of the LLE (Fig. 1(a)) as a function of the forcing amplitude A in the presence of noise is found to be practically coincident with that of the noise free case. Because the presence of weak noise does not change these measures, we may infer that it does not induce new regimes. Thus, the typical scenario is that: the fractal structure is simply smoothed by the noise at small scales $O(D)$, while remaining fractal on larger scales; and the LLE differs from its unperturbated value a quantity $O(D)$.

For the classification of attractor complexity we may again use the behavior of $S_{\dot{x}}(t)$: an attractor can be described as complex if the fluctuations of $S_{\dot{x}}(t)$ with respect to the zero level are comparable with the size of the attractor on the phase plane. To characterize the attractor's dynamical complexity as a function of A we consider the behavior of the maximal dispersion S_{max}. It was calculated over a time interval of 500 $2\pi/\omega_1$, following a relaxation time of 1500 $2\pi/\omega_1$. As shown in Fig. 1(b) for the noise-free case the maximal value is distinguishable from zero only in the region where the sign of the LLE is positive. Consequently, the behavior of $S_{\dot{x}}(t)$ fully correlates with the behavior of the LLE. In the chaotic regime the fluctuations in $S_{\dot{x}}(t)$ are comparable in size with the attractor on the phase plane $x - \dot{x}$; correspondingly, the attractor is dynamically complex. In the presence of noise, however, S_{max} differs from zero for all values of forcing amplitude and becomes comparable in size with the attractor for $A > 0.519$ (Fig. 1(b)), i.e. over the regions of co-existence of the SNA and chaotic attractor in the noise-free case. Consequently, for $A > 0.519$ in the presence of noise, the evolution of the ensemble of trajectories is similar to the evolution of a chaotic attractor (Fig. 3(c)) and we observe a complex and unpredictable behavior in (1). Note that for the noisy torus locally increasing and decreasing values of S are also observed, but maximal value of S (which is determined by the noise intensity) does not reach the size of attractor: see Fig. 1(a). Note also that we do not observe a diffusive divergence of trajectories in the torus regime because we are considering an ensemble of trajectories with the same phase of the eternal force.

4. Discussion

It is thus clear that, in the quasiperiodically driven Duffing oscillator, extremely weak noise is sufficient to convert an SNA into an attractor that is strange in the sense of having a fractal structure, and complex in the sense that there is unpredictability of trajectory behavior during some time intervals, but which has a *negative* LLE. This implies that use of the LLE as an indicator of the velocity of perturbation growth yields misleading information about the dynamical properties of the attractors. We have therefore also calculated the complexity measure K_σ, which was introduced [24] for the analysis of chaotic systems in the presence of fluctuations. It defines the velocity of divergence of nearby trajectories and is calculated for two trajectories with different noisy realizations. For regular motion, $K_\sigma = 0$, but K_σ differs from zero in the complex regime. In the noise-free case the value of K_σ is positive for a chaotic attractor and zero for a regular attractor. It can be seen (Fig. 5) that, for (1) the complexity differs from zero in the regions where an SNA and chaos exist in the noise-free system, and that $K_\sigma = 0$ in the region of torus. This reinforces our conclusion that, for $A > 0.519$ in the presence of noise, the motion is complex.

We have evidence to show that this conclusion is not restricted to the Duffing system (1): we have investigated [20] the effect of noise on the SNA-displaying nonlinear oscillators and maps described in [8, 9, 13, 15]. The results have turned out to be qualitatively similar to those described above for (1). Thus, in each case, an extremely small noise intensity is sufficient to convert the SNA into a complex attractor, but without altering either its geometrical properties, or the scaling of the spectral function, or the sign of the LLE. It is important to note that the role of noise here (in contrast to some other well-known cases [27]) is trivial: the noise simply defines the finite (nonzero) size of the perturbation or, in other words, the finite uncertainty in the definition of initial conditions.

The same results for measures S and K_σ are obtained if we examine the deterministic (without noise) behavior, just forbidding the distance between trajectories of ensemble to be less then some value ϵ. Thus, the observed effects of noise are defined by the properties of the SNA and are displayed only if the noise amplitude is larger then some threshold value ϵ.

5. Conclusion

In summary, we are apparently driven to the conclusion that, in the presence of extremely weak noise, an SNA retains its negative LLE but exhibits complex unpredictable behavior and a fractal structure – just like a chaotic attractor – even though weak noise changes neither the SNA structure nor the relationship between the regions with divergence and convergence of trajectories. It is therefore impossible to classify the complexity of signals from such systems usefully in terms of their time-averaged LLEs. Rather, it is necessary to use local LLEs, a measure of complexity [24], or a direct method such as maximal velocity dispersion of an ensemble of trajectories, in order to examine the unpredictability of motion. Thus, one needs to know both the averaged LLE or distribution of local LLEs [28], and the temporal behavior of local LLEs, to evaluate the region with instability.

Finally, it is interesting to note that, in pioneering [4] and early papers on the subject, the appearance of an SNA is attributed to unstable sets. Moreover, J. Stark [29] has shown theoretically that an SNA has non-stable orbits, giving rise to its fractal structure. Non-stable orbits must also cause unstable dynamical behavior. If we compare the dependences on A of the fractal dimension D_F and the measure of complexity K_σ (see Fig. 5), it is evident that they are qualitatively similar. Thus we can suggest that the SNA appears to have as much dynamical complexity in the presence of weak noise as it has strangeness in the noise-free case. This interesting question will be considered in more detail elsewhere.

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Figure captions

Figure 1. The dependences of (a) the largest Lyapunov exponent and (b) the maximal velocity dispersion $S_{\max}$ on the forcing amplitude A, both with $D = 1.E - 6$ (pluses) and without $D = 0$ (circles) weak additive noise. In (b), results for $D = 1.E - 5$ and $D = 1.E - 7$ are shown by the dotted and dashed lines respectively.

Figure 2. Time realizations of the applied force (dashed line) and the system response (bold solid line) for $A = 0.8$. The local Lyapunov exponent is shown by the circles.

Figure 3. Time evolution of the velocity dispersion $S_{\dot{x}}(t)$ for: (a) $A = 0.8$, $D = 0$; (b) $A = 0.7$, $D = 0$; (c) $A = 0.7$, $D = 0.000001$. The envelope of the external force $\sin(\omega_1 t - \omega_2 t)$ is shown by the dashed lines in (a) and (b).

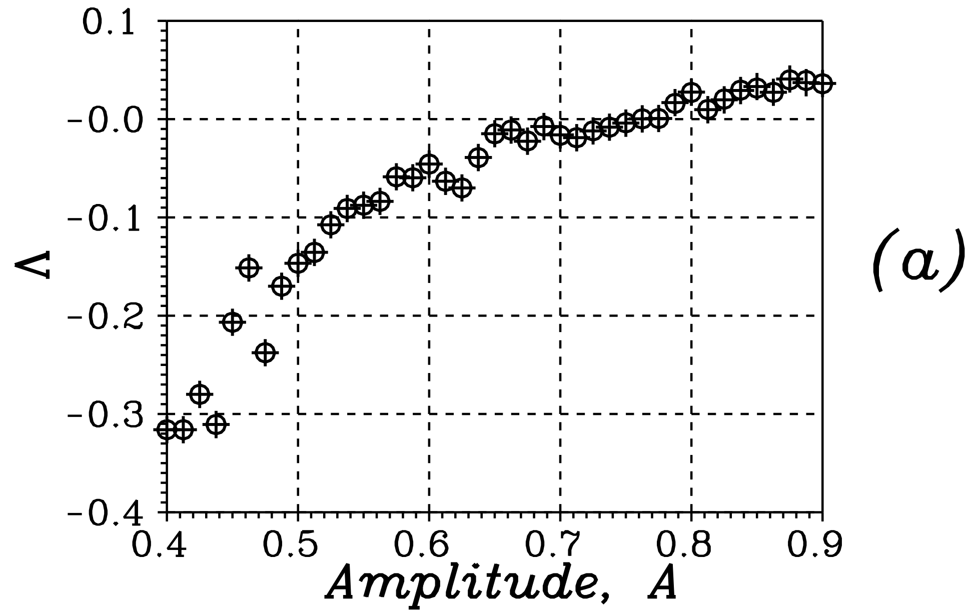
Figure 4. The stroboscopic section (point data) of Duffing system for $A = 0.8$. The location of ensemble of initially nearby trajectories after relaxation time for two time moment $t = 1000\frac{2\pi}{\omega_1}$ (in region of negative local LLE) and $t = 1010\frac{2\pi}{\omega_1}$ (in region of positive local LLE) are shown by squares and triangles respectively.

Figure 5. The measure of complexity K_σ as a function of amplitude A both with (pluses, for $D = 1.E - 6$) and without (circles, for $D = 0$) weak additive noise. The dependence of the fractal dimension D_F on the forcing amplitude A is shown by the "$\times$"s.

Tables

	d_i		β	
Amplitude A	$D = 0$	$D = 1.e - 6$	$D = 0$	$D = 1.e - 6$
0.4	1.01	1.01	$\ln \sigma$	$\ln \sigma$
0.5	1.08	1.08	$\ln \sigma$	$\ln \sigma$
0.6	1.36	1.36	1.16	1.16
0.7	1.43	1.43	1.5	1.5
0.8	1.53	1.53	1.57	1.57
0.9	1.54	1.54	1.46	1.46

Table 1: The information dimension d_i and the scaling of the spectrum β for different values of amplitude A in the absence and in the presence of noise.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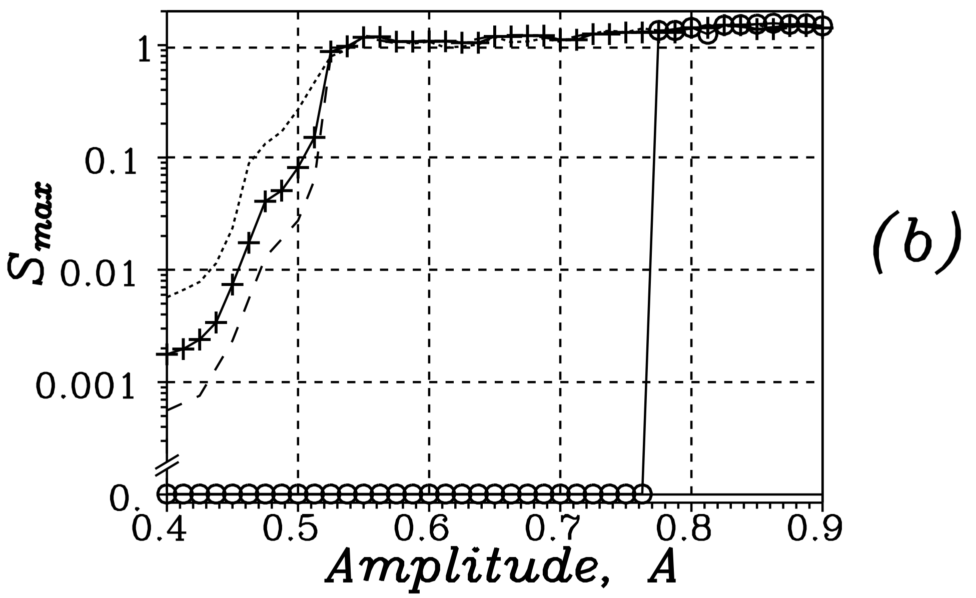
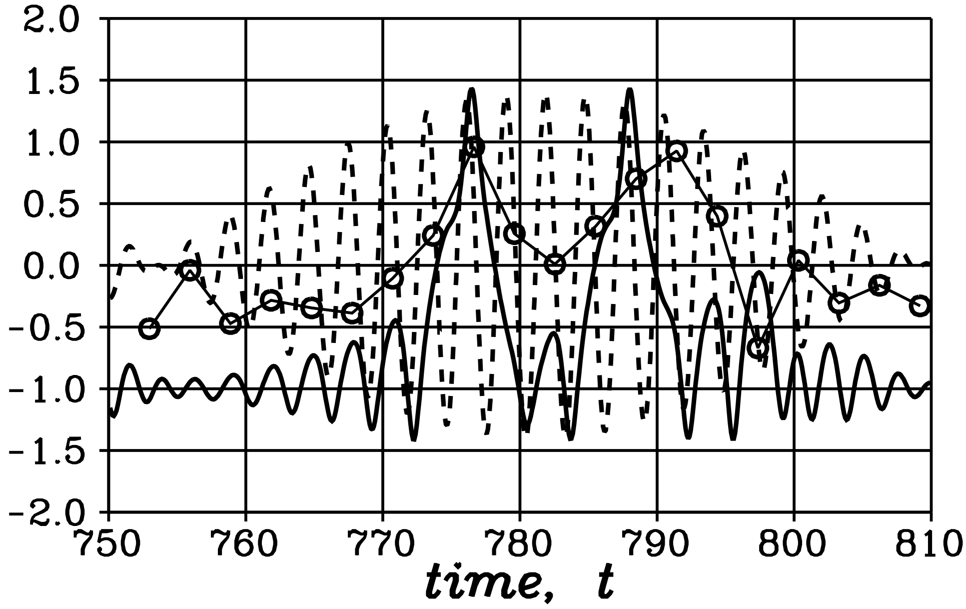
-0.0
-0.1
-0.2
-0.3
-0.4
Λ
0.4
0.5
0.6
0.7
0.8
0.9
Amplitude, A
(a)


1
0.1
0.01
0.001
0.
S_{max}
0.4
0.5
0.6
0.7
0.8
0.9
Amplitude, A
(b)



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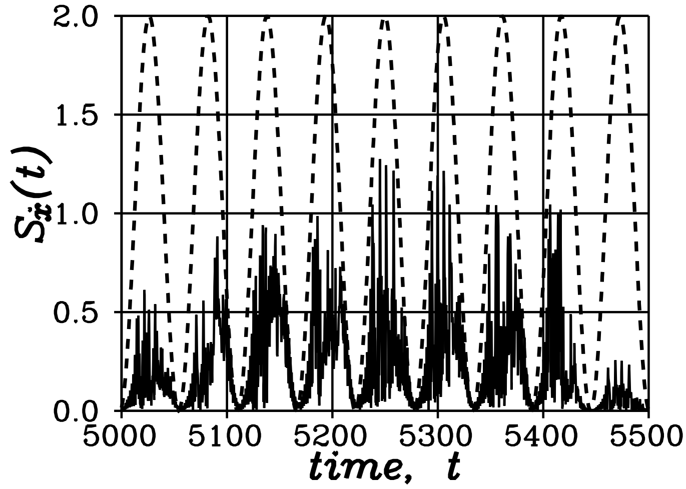
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
750
760
770
780
790
800
810
time, t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1.5
1.0
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0.0
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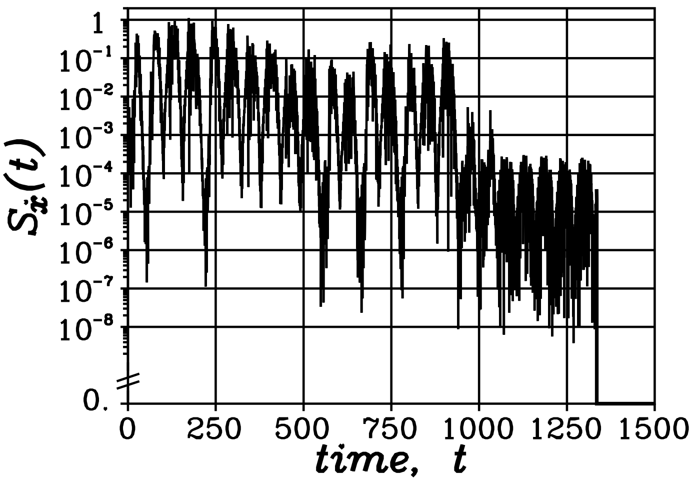
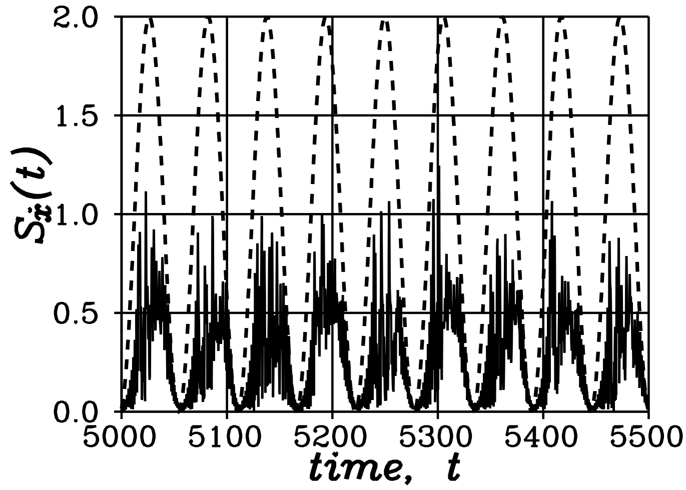
$S_x(t)$
time, t

(a)


1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0.
0 250 500 750 1000 1250 1500
$S_x(t)$
time, t

(b)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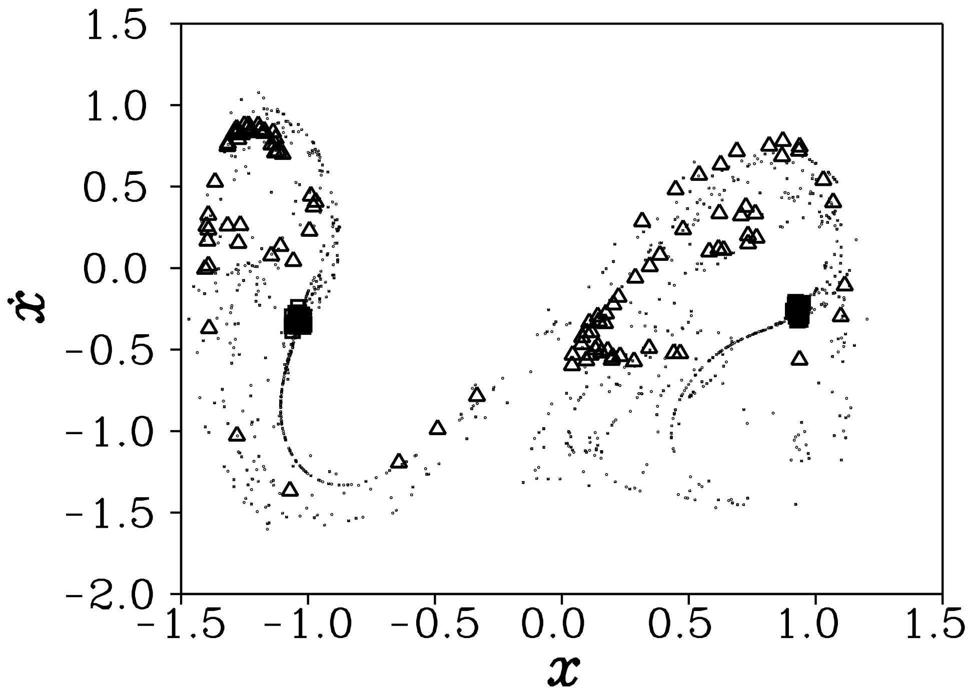
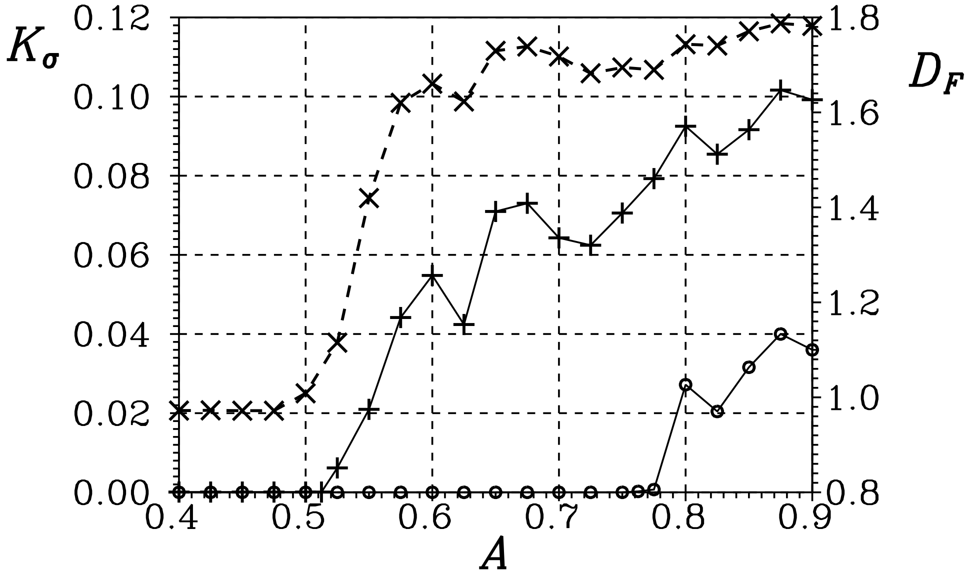
1.5
1.0
0.5
0.0
5000 5100 5200 5300 5400 5500
$S_x(t)$
time, t

(c)



1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
$\dot{x}$
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x



K_σ
0.12
0.10
0.08
0.06
0.04
0.02
0.00
1.8
1.6
1.4
1.2
1.0
0.8
D_F
0.4
0.5
0.6
0.7
0.8
0.9
A