Comment on "Monostable array-enhanced stochastic resonance"

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Lindner *et al.* [Phys. Rev. E **63**, 051107 (2001)] have reported multiple stochastic resonances (SRs) in an array of underdamped monostable nonlinear oscillators. This is in contrast to the single SR observed earlier in a similar but isolated oscillator. Though the idea that such an effect might occur is intuitively reasonable, the notation and the interpretation of some of the major results seem confusing. These issues are identified and some of them are clarified. In addition, comments are made on two possible extensions of the central idea of Lindner *et al.*: one of these promises to provide much more striking manifestations of multiple SR in arrays; the other significantly widens the range of systems in which multiple SRs may be observed.

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In a recent paper [1] Lindner *et al.* have discussed the occurrence of stochastic resonance (SR) in an array of coupled, underdamped, nonlinear, monostable oscillators. They conclude that SR may be manifested at several values of noise intensity, in contrast to the single value that arises in the case of an isolated oscillator [2,3] or for an array of overdamped bistable oscillators [4]. As so often happens in science, this is a case of a very interesting result that, in retrospect, seems unsurprising: we note that the coupling lifts the degeneracy of the eigenfrequencies of the individual oscillators, causing them to split and thus yield additional resonances, each of which can give rise to SR, of type (a) in the sense introduced in [2]. Some of the results for power spectra in the absence of driving are potentially important, but their interpretation, and some of the terminology, seem to us confusing. The results on noise-enhanced propagation, which probably represent the most interesting application, are also presented in an ambiguous way.

The main aim of the present Comment is to address and clarify some of the confusion and to urge the authors of [1] to remove the remaining ambiguity. In doing so, we discuss the relationship of [1] to the existing understanding of SR, in general, and in monostable underdamped nonlinear oscillators, in particular, and we point out a possible extension of the authors' central idea to a system that may be expected to exhibit a much more dramatic manifestation of array-enhanced multiple SR. We also generalize the idea of [1] to encompass a wider range of systems.

The major quantity considered in [1] (and in what follows we will mostly use the notation of [1]) is

$$R[f_D]=\frac{S[f_D]}{B[f_D]}, \tag{1}$$

where $S[f_D]$ is the power spectrum of one of the oscillators in the array in the *absence* of a periodic driving force, and $B[f_D]$ is some "smooth background" in such a spectrum: although an explicit definition of $B[f_D]$ is absent from the paper, it follows implicitly from Fig. 1 and has been confirmed by one of the authors that, when the temperature T (i.e., the noise intensity, appropriately normalized) is small or moderate, $B[f_D]$ in the range of frequencies near f_D is of the order of $S[f_D=0]$. This, in turn, is approximately proportional to T so that $B\propto T$ to a good approximation.

The authors mostly refer to the quantity R as the spectral response, and they discuss its dependence on noise intensity; f_D is called the driving frequency. We wish to point out, however, that the response to a periodic driving force is not proportional either to R or to S, so that its dependence on noise intensity may differ markedly from that of R or S. In reality, it is only the *imaginary* part of the generalized susceptibility χ that is proportional to S [5], the real part being related to it by one of the Kramers-Krönig relations [5].

Let us write this in rigorous terms. If the driving force is sinusoidal, i.e., $A_D\cos(2\pi f_D t)$, then the response of the system may be written in terms of the shift of its generalized coordinate (e.g., the coordinate of one of the array oscillators as in [1]) averaged over the statistical ensemble. In the asymptotic limit $A_D\rightarrow 0$, such a response is proportional to A_D and sinusoidal in time [5]:

$$\begin{aligned}\delta\langle q(t)\rangle &\equiv \langle q(t)\rangle_{A_D}-\langle q\rangle_{A_D=0}\\ &=A_D\mathrm{Re}[\chi(f_D)\exp(-\mathrm{i}2\pi f_D t)],\end{aligned} \tag{2}$$

where χ is the generalized (complex) susceptibility [5]. From the fluctuation-dissipation theorem [5] and one of the Kramers-Krönig relations [5], the imaginary and real parts of χ are, respectively,

$$\mathrm{Im}[\chi(f_D)]=\frac{2\pi^2 f_D}{T}S(f_D),$$

$$\mathrm{Re}[\chi(f_D)]=\frac{4\pi}{T}\mathrm{P}\int_0^\infty df\frac{f^2S(f)}{f^2-f_D^2}, \tag{3}$$

where T is temperature [equivalent to $\sigma^2/(2048 f_D\gamma)$ in the notation of [1], with $m=1$] and P denotes the Cauchy principal part. In the context of SR, these relations were first used in [6].

If one characterizes the response by the intensity I_δ (square) of the δ spike at the driving frequency f_D in the power spectrum of the driven system, which is one of the most important characteristics of the response (cf. [3,6,7]), then it is easily shown [6] that

$$I_\delta = \frac{1}{4} A_D^2 |\chi(f_D)|^2. \tag{4}$$

The authors of [1] infer a "distinct SR" in their system based on the fact that the quantity R may drastically increase as noise intensity changes from zero and passes through certain optimal values. This occurs because the spectrum in the absence of driving, $S[f]$, has pronounced peaks whose maxima shift in frequency as noise intensity grows. But, in reality, things are not so simple because, as pointed out above, the response to an input signal is not simply proportional to R.

The system's response to a weak periodic driving force is completely specified by the complex susceptibility $\chi(f_D)$. And Eqs. (2) and (3) show that it is only the imaginary part of $\chi(f_D)$ that can be maximized through "tuning" by noise; in the range of small temperatures, it differs from R only by a temperature-independent factor. The real part of $\chi(f_D)$ behaves differently, and must also be taken into account. A simple analysis of Eq. (3) for an isolated Duffing oscillator (cf. [3]) shows that $\mathrm{Im}[\chi(f_D)]$ at the temperature $T_m \equiv T_m(f_D)$ yielding its maximum, and $\mathrm{Re}[\chi(f_D)]$ at the same T are of the same order as $\mathrm{Re}[\chi(f_D)]$ at $T=0$. As in [1], we are assuming here that the deviation of f_D from the natural frequency f_0 greatly exceeds the friction parameter γ, while being less or of the same order as f_0: $\gamma \ll f_D - f_0 \lesssim f_0$; and we also allow for the explicit expression [8] for the spectrum $S[f]$ in the Duffing oscillator in the relevant range of frequencies and temperatures. Consequently, as T varies, the maximum possible ratio of $|\chi(f_D)|$ to its value at $T=0$ is not much larger than unity (cf. [2], where it was ≈ 2.5 for an oscillator similar to the oscillator used in [1]). This is in contrast to the analogous ratio for $R[f_D]$ (and, similarly, for $\mathrm{Im}[\chi(f_D)]$), which is typically larger by a few orders of magnitude (cf. Fig. 3 in [1]). The situation for arrays should be similar. This inference could readily be checked by performing an explicit calculation of $|\chi|$ using Eq. (3) and the spectra $S[f]$ presented in [1], and/or backed up by a digital or analog simulation of the array in the presence of a periodic driving force.

Lindner *et al.* [1] comment that R is "faithful to the squared stochastic amplification factor" (SAF) introduced in [2], but we must point out that this is not quite right, for the reasons discussed above. The SAF is the ratio between the amplitudes of the response in the presence and absence of noise which, in the asymptotic limit of vanishing driving amplitude, is equal to the ratio between the absolute values of the corresponding generalized susceptibilities or, equivalently, between the square roots of the intensities I_δ of the δ spike. It follows from our analysis above that neither the SAF, nor its square, behave in just the same way as R.

In [1] R is sometimes referred to (e.g., in captions to Figs. 4 and 5) as the *signal-to-noise ratio* (SNR). But R is not an SNR—at least not in the conventionally accepted sense of being the ratio of a signal at the output (i.e., the difference between the outputs in the presence and absence of periodic driving) to the noise at the output (i.e., the output in the absence of periodic driving). For example, the SNR of [7], which is still widely used, relates to I_δ and $S[f_D]$:

$$(\mathrm{SNR}) = \frac{I_\delta}{S[f_D]}. \tag{5}$$

Although there is no single universally accepted definition of the SNR, it is in our view confusing to refer to R as an SNR, given that R refers to quantities both of which are measured in the *absence* of driving. Moreover, as shown in [3], the conventional SNR (5) exhibits local maxima as a function of noise only for special classes of systems (see below); but the symmetrical monostable Duffing oscillator considered in [1] does not belong to such a class. So the allusion by Lindner *et al.* to multiple SNR maxima in their system is potentially misleading.

Section V of [1] treats the interesting and potentially important question of how a signal propagates in the presence of noise if the array of underdamped oscillators is driven periodically along one side only. Unlike Secs. III and IV, therefore, it deals with the effect of a real periodic driving force rather than a virtual one. The signal propagation is discussed in terms of an SNR which evidently differs from the R defined by Eq. (1) in the absence of driving; on the other hand, if we generalize Eq. (1) to include the possibility of a periodic driving force, R will diverge to infinity at $f = f_D$ on account of the δ function in the power spectrum at the driving frequency [3,6,7], as can be seen from Eq. (2). It is therefore unclear to us just what is meant by the SNR in Sec. V and, in particular, what is being plotted in Fig. 7. The general trend of the results looks interesting, however, and could with advantage be clarified by the authors of [1]. Their research will then become more useful to other scientists.

We should perhaps mention two other minor inconsistencies in the paper that are liable to mar its understandability. The first relates to confusion between the frequency (reciprocal of the period) and the angular or cyclic frequency [9] (frequency multiplied by 2π): it is obvious from Figs. 1–5 and from Eq. (4) that f means the frequency [cf. also f_D in Eq. (1)]; but, it follows from the formulas for f_0 and f_1, just below Eq. (2) and two lines below Eq. (3), respectively, that f in Eqs. (2) and (3) means the angular frequency. Another inconsistency (or, possibly, a misprint) occurs in the formula for the angular frequency of the antisymmetric mode f_1: the multiplier of κ should be 2 rather than 3.

We end with two forward-looking comments. First, we suggest that it would probably be fruitful to apply the central idea of [1] to an array of zero-dispersion (ZD) [3,10,11,12] oscillators. Unlike conventional oscillators (e.g., the Duffing oscillator as in [1]), the dependence of the frequency of eigenoscillation of a ZD oscillator on its energy, $\omega(E)$, possesses one or more extrema, i.e., there are one or more energies at which $d\omega/dE=0$. This property provides a *very* strong enhancement of resonant behavior in the vicinity of the extrema. Manifestations of SR phenomena are therefore much stronger than in conventional oscillators, and the maximum noise-induced increase of the response (signal) can be far larger (cf. [2]). Moreover, in contrast to the conventional case, not only may the signal itself increase with noise intensity, but even the SNR, conventionally defined by (5), may undergo significant noise-induced growth [3,12]. Given that an array of zero-dispersion oscillators provides for the possibility of their synchronization, manifestations of SR in an array of ZD oscillators may be expected to be even stronger than in a single ZD oscillator (such a hypothesis was suggested first in [12]). It also seems very likely that, as in [1], the coupling will further increase the number of noise intensities at which the SNR exhibits local maxima [13]. There is a wide variety of ZD oscillators. They may be either monostable [e.g., the tilted Duffing oscillator [3,11] or a SQUID (superconducting quantum interference device) loop with a large inductance [12]] or multistable (e.g., a SQUID loop with a small inductance [12]).

Finally, we comment on a different generalization of the central idea of Lindner *et al.* [1]. If their aim is to increase the number of ranges of noise intensity where weak SR occurs [i.e., where the signal, but not the SNR (5), possesses a local maximum as a function of noise intensity], then one may seek this phenomenon in any nonlinear dynamical system of high but finite dimension. In effect, Lindner *et al.* increased the number of eigenmodes through the coupling in the array: together with the shift of the maxima in the power spectrum as noise varies, this provided for *multiple* SRs. But, quite generally, any nonlinear dynamical system of high dimension possesses many eigenmodes; the maxima in its power spectrum may be sensitive to noise intensity, and there is thus the possibility of multiple SRs as in [1].

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[13] Note that even a single ZD oscillator may possess more than one local maximum in SNR (cf. [12]), provided that $\omega(E)$ possesses more than one extremum and that the extrema are sufficiently separated in frequency.