Newton's constant in $f(R, R_{\mu\nu}R^{\mu\nu}, \Box R)$ theories of gravity and constraints from BBN

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We consider corrections to the Einstein-Hilbert action, which contain both higher order and nonlocal terms. We derive an effective Newtonian gravitational constant applicable at the weak field limit and use the primordial nucleosynthesis (BBN) bound and the local gravity constraints on $G_{\rm eff}$ in order to test the viability of several cases of our general Lagrangian. We will also provide a BBN constrain on the $\Box R$ gravitational correction.

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I. INTRODUCTION

One would naturally expect corrections to the Einstein-Hilbert action of gravity at scales close to the 4-dimensional Planck scale. However, the details of these corrections in a general time dependent background are less known. Thus, one would expect a generic action of type $f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R\Box R)$ [1–3].

Of these examples, $f(R)$ theories have received much attention due to their capability to mimic the late-time acceleration, see [4–8] including the solar system constraints [9]. On the other hand the nonlocal higher derivative corrections of the type $R + \sum_i R\Box^i R$ yield an asymptotically free and ghost free nonperturbative action of gravity [10], which has played an important role in resolving the big bang singularity in the Friedmann-Robertson-Walker (FRW) universe. It also explains the observed temperature anisotropy in a noninflationary bouncing universe setup [11]. Models of this type were also studied in [12], where it was shown that it is conformally equivalent to Einstein gravity coupled to two scalar fields. Also, models with nonlocal corrections but with negative powers of the d'Alambert operator have been considered in [13], where it was shown that such a theory may lead to the unification of early time inflation with late-time cosmic acceleration.

However, in this paper we will not consider the analysis for infinite, higher derivative nonlocal corrections of the type $\sum_i R\Box^i R$, rather we will only concentrate on the $i = 1$ case, and the $\Box R$ case. The complete nonperturbative action will be dealt with separately in a future publication.

The aim of the present paper is to study the low scale and long range behavior of a generic class of Lagrangian density, $1/2 f(R, R_{\mu\nu}R^{\mu\nu}, \Box R)$, where we derive the scalar Newtonian potentials in a homogeneous and an isotropic expanding background such as in a FRW cosmology. The perturbations in the FRW background yield a Newtonian potential for a matter distribution and therefore determine an effective Newtonian constant $G_{\rm eff}$. At long ranges the linear perturbation analysis differentiates Einstein's gravity with respect to any modification through the time evolution of the gravitational constant, see [14,15], and this is one of the most important differences between Einstein and modified gravity theories.[1]

This difference can be tested by using the primordial nucleosynthesis (BBN) bounds on the gravitational constant, which are of the order of 10% [21–24]. The BBN bounds are important due to the fact that the value of the gravitational constant determines the expansion rate of the Universe and thus the relevant time scales for the production of light elements (H, He, and Li), see [24]. As a consequence, if we assume that the gravitational constant at the time of BBN is different from its value today, this means that the light element abundances will be different with respect to the standard BBN predictions. Even a weak time dependence, which gives no observable effects in Solar System experiments performed at the present epoch and at small scales, could give observable effects when translated over cosmological time scales. So, it will be interesting to analyze some special cases of our general Lagrangian and use the BBN bounds on the gravitational constant to place constraints on the parameters of these simple models.

Furthermore, we will be applying the BBN constraints to study the $\Box R$ corrections in the Einstein-Hilbert action. Previous studies of nonlocal action has concentrated on formal aspects of the validity of effective field theory [25] and particle creation [26]. It should be noted that the effective Newton's constant in nonlocal gravity and its implications to cosmology, BBN, and the Solar System have been also considered in [27] and more recently in models generalizing this in [28].

Here, we consider the alteration of classical dynamics of the Universe due to the presence of the $\Box R$ gravitational correction.

[1] The modifications in general relativity also affect structure formation [16–18] and the predictions in the cosmic microwave background radiation through radiation-matter equality [19,20]. We will study various consequences to structure formation and cosmic microwave background radiation in a separate publication.

II. BACKGROUND EQUATIONS

The action we will consider is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(R, R_{\mu\nu}R^{\mu\nu}, \Box R) + L_m\right], \tag{2.1}$$

where R is the Ricci scalar, $R_{\mu\nu}$ the Ricci tensor, $\Box$ is the d'Alembert operator $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, and L_m the matter Lagrangian. We use the metric signature $(-, +, +, +)$.

Varying the action with respect to the metric $g^{\mu\nu}$ we obtain the field equations as [29,30]

$$\begin{aligned}&\left(F + \Box\frac{\partial f}{\partial\Box R}\right)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + 2f_{,RR}g^{\lambda\kappa}R_{\nu\kappa}R_{\mu\lambda}\\ &\quad - (\nabla_{(\mu}R)\left(\nabla_{\nu)}\frac{\partial f}{\partial\Box R}\right) + \frac{1}{2}g_{\mu\nu}\left[R^{;\sigma}\frac{\partial f}{\partial\Box R}\right]_{;\sigma}\\ &\quad + [g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\left(F + \Box\frac{\partial f}{\partial\Box R}\right) + \Box(f_{,RR}R_{\mu\nu})\\ &+ g_{\mu\nu}\nabla_\alpha\nabla_\beta(f_{,RR}R^{\alpha\beta}) - 2\nabla_\alpha\nabla_\beta(f_{,RR}R^{\alpha}{}_{(\mu}\delta^{\beta}{}_{\nu)}) = T_{\mu\nu},\end{aligned} \tag{2.2}$$

where $F = \frac{\partial f}{\partial R}$ and $f_{,RR} = \frac{\partial f}{\partial(R_{\mu\nu}R^{\mu\nu})}$. We have also defined the energy-momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \tag{2.3}$$

and the parentheses next to indices mean symmetrization, e.g. $A_{(ij)} = 1/2(A_{ij} + A_{ji})$. Note that by using the field Eqs. (2.2) it is easy to see that it has the correct limits, i.e. in the case when the Lagrangian is given by $f = R + \xi\Box R$ then we get general relativity, as the $\Box R$ term can be written as a total divergence. Also, for a conformally flat metric and a Lagrangian given by [31] $f = R + \xi(3R_{\mu\nu}R^{\mu\nu} - R^2)$ the field equations give general relativity at the background level, but not at the perturbations level, as the metric is no longer conformally flat.

In a flat FRW metric with a scale factor $a(t)$, we obtain the zero-order (background) equations

$$\begin{aligned}&\frac{f}{2} - 3F(H^2 + \dot H) + 3\dot F H - 9H^3\left(2f_{,RR}H - 2\dot f_{,RR}\right.\\ &\quad\left. - \frac{\partial\dot f}{\partial\Box R}\right) + 36\dot H H^2\left(\frac{\partial f}{\partial\Box R} + \frac{1}{2}f_{,RR}\right) - 6\frac{\partial\ddot f}{\partial\Box R}H^2\\ &\quad + 3H\ddot H\left(7\frac{\partial f}{\partial\Box R} + 4f_{,RR}\right) + 12\dot H H\left(\dot f_{,RR} - \frac{\partial\dot f}{\partial\Box R}\right)\\ &\quad - 3\frac{d^3\frac{\partial f}{\partial\Box R}}{dt^3}H + 12\dot H^2\left(\frac{\partial f}{\partial\Box R} - f_{,RR}\right) - 3\ddot H\frac{\partial\dot f}{\partial\Box R}\\ &\quad + 3\dot H\frac{\partial\ddot f}{\partial\Box R} + 3\frac{\partial f}{\partial\Box R}\frac{d^3H}{dt^3} = \rho,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&-2F\dot H - \ddot F - 3\left(2\ddot f_{,RR} + \frac{\partial\ddot f}{\partial\Box R}\right)H^2 + \left(\dot F - 12\ddot H f_{,RR}\right.\\ &\quad\left. - 24\dot H\dot f_{,RR} - 21\dot H\frac{\partial\dot f}{\partial\Box R} + 2\frac{d^3\frac{\partial f}{\partial\Box R}}{dt^3}\right)H + 6\dot f_{,RR}H^3\\ &\quad - 4\ddot f_{,RR}\dot H - \ddot H\left(8\dot f_{,RR} + 3\frac{\partial\dot f}{\partial\Box R}\right) - 24\dot H^2 f_{,RR}\\ &\quad + 8\dot H\frac{\partial\ddot f}{\partial\Box R} - 4f_{,RR}\frac{d^3H}{dt^3} + \frac{d^4\frac{\partial f}{\partial\Box R}}{dt^4} = \rho,\end{aligned} \tag{2.5}$$

where the dot $(\dot{})$ denotes a derivative with respect to time, e.g. $\dot f_{,RR} \equiv \frac{\partial f_{,RR}}{\partial t}$ and $f_{,RR} = \frac{\partial f}{\partial(R_{\mu\nu}R^{\mu\nu})}$.

III. PERTURBATION EQUATIONS

We will consider the following perturbed metric with scalar metric perturbations Φ and Ψ in a longitudinal gauge:

$$ds^2 = -(1 + 2\Phi)dt^2 + a(t)^2(1 - 2\Psi)\delta_{ij}dx^i dx^j. \tag{3.1}$$

The energy-momentum tensor of the nonrelativistic matter is decomposed as $T^0_0 = -(\rho_m + \delta\rho_m)$ and $T^0_\alpha = -\rho_m v_{m,\alpha}$, where v_m is a velocity potential. The Fourier transformed perturbation equations for the continuity equations are given by

$$-\frac{\rho_m v_m k^2}{a} - \delta\dot\rho_m - 3H\delta\rho_m + 3\dot\Psi\rho_m = 0, \tag{3.2}$$

$$\Phi\rho_m - a(H\rho_m v_m + \rho_m\dot u_m) = 0. \tag{3.3}$$

Following the approach of Refs. [32,33], we use a subhorizon approximation under which the leading terms correspond to those containing k^2 and $\delta\rho_m$. Terms that are of the form $H^2\Phi$ or $\ddot\Phi$ are considered negligible relative to terms like $(k^2/a^2)\Phi$ for modes well inside the Hubble radius $(k^2 \gg a^2H^2)$. Under this approximation, the Fourier transformed perturbation equations, coming from the $(\mu, \nu) = (0, 0)$ and $(1, 2)$ terms of the field Eqs. (2.2), are given by

$$\begin{aligned}&-\delta\rho_m - \frac{k^4}{a^4}\left(\delta\frac{\partial f}{\partial\Box R} - 2f_{,RR}(\Phi - \Psi)\right)\\ &\qquad + \frac{k^2}{a^2}(\delta F - 2F\Psi) = 0,\end{aligned} \tag{3.4}$$

$$\frac{\delta F}{F} + \Phi - \Psi + \frac{k^2}{a^2}\frac{1}{F}\left(-\delta\frac{\partial f}{\partial\Box R} + (\Phi - 3\Psi)f_{,RR}\right) = 0. \tag{3.5}$$

While in general relativity in the case of a matter fluid with no anisotropic stress the two potentials Φ and Ψ are equal, as can be seen from Eq. (3.5), this is not the case for modified gravity theories as the gravity sector alone induces an anisotropic stress and creates the inequality of Φ and Ψ, see for example [32,33]. Next, we define the gauge invariant matter density perturbation δ_m as

$$\delta_m \equiv \frac{\delta\rho_m}{\rho_m} + 3Hv, \tag{3.6}$$

where

$$v = av_m. \tag{3.7}$$

Under this approximation Eqs. (3.2), (3.3), and (3.6) yield

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2\Phi}{a^2} \simeq 0. \tag{3.8}$$

Next, we write δF and $\delta \frac{\partial f}{\partial \Box R}$ as

$$\delta F = \frac{\partial F}{\partial R}\delta R + \frac{\partial F}{\partial(R_{\mu\nu}R^{\mu\nu})}\delta(R_{\mu\nu}R^{\mu\nu}) + \frac{\partial F}{\partial(\Box R)}\delta(\Box R), \tag{3.9}$$

$$\delta\frac{\partial f}{\partial\Box R} = \frac{\partial F}{\partial\Box R}\delta R + \frac{\partial\frac{\partial f}{\partial\Box R}}{\partial(R_{\mu\nu}R^{\mu\nu})}\delta(R_{\mu\nu}R^{\mu\nu}) + \frac{\partial^2 f}{\partial(\Box R)^2}\delta(\Box R), \tag{3.10}$$

where δR, under the subhorizon approximation, is given by

$$\delta R \simeq -2\frac{k^2}{a^2}(2\Psi - \Phi), \tag{3.11}$$

while $\delta(R_{\mu\nu}R^{\mu\nu}) \sim 0$ and $\delta(\Box R)$ is given by

$$\delta(\Box R) = -\frac{2k^4}{a^4}(\Phi - 2\Psi). \tag{3.12}$$

Making these substitutions and using the subhorizon approximation in Eqs. (3.4) and (3.5) we get

$$\begin{aligned} -\delta\rho_m - 2F\frac{k^2}{a^2}\Psi + 2\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^8}{a^8}(\Phi - 2\Psi) \\ + 2\frac{k^4}{a^4}(F_{,R}(\Phi - 2\Psi) + f_{,RR}(\Phi - \Psi)) \\ - 4\frac{\partial F}{\partial\Box R}\frac{k^6}{a^6}(\Phi - 2\Psi) = 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} F(\Phi - \Psi) + \frac{k^2}{a^2}(\Phi - 3\Psi)f_{,RR} \\ - 2(\Phi - 2\Psi)\cdot\left(-\frac{k^2}{a^2}F_{,R} + 2\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} - \frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}\right) = 0. \end{aligned} \tag{3.14}$$

The next step is to express Φ and Ψ in terms of δ_m. This can be done by solving the system of Eqs. (3.13) and (3.14) for Φ and Ψ. Doing so we find

$$\Phi = -\frac{a^2}{k^2}\frac{\rho\delta_m}{2(F - \frac{k^2}{a^2}f_{,RR})} \times \frac{F + \frac{k^2}{a^2}(3f_{,RR} + 4F_{,R}) - 8\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} + 4\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}}{F + \frac{k^2}{a^2}(2f_{,RR} + 3F_{,R}) - 6\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} + 3\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}}, \tag{3.15}$$

$$\Psi = -\frac{a^2}{k^2}\frac{\rho\delta_m}{2(F - \frac{k^2}{a^2}f_{,RR})} \times \frac{F + \frac{k^2}{a^2}(f_{,RR} + 2F_{,R}) - 4\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} + 2\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}}{F + \frac{k^2}{a^2}(2f_{,RR} + 3F_{,R}) - 6\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} + 3\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}}. \tag{3.16}$$

From Eq. (3.15) we can define a Poisson equation in the Fourier space and attribute the extra terms that appear on the right-hand side to an effective gravitational constant $G_{\rm eff}$. Doing so, we get the gravitational potential

$$\Phi = -4\pi G_{\rm eff}\frac{a^2}{k^2}\delta_m\rho_m, \tag{3.17}$$

where $G_{\rm eff}$ is defined as

$$G_{\rm eff} = \frac{1}{8\pi}\frac{1}{F - \frac{k^2}{a^2}f_{,RR}} \cdot \frac{F + \frac{k^2}{a^2}(3f_{,RR} + 4F_{,R}) - 8\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} + 4\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}}{F + \frac{k^2}{a^2}(2f_{,RR} + 3F_{,R}) - 6\frac{\partial F}{\partial\Box R}\frac{k^4}{a^4} + 3\frac{\partial^2 f}{\partial(\Box R)^2}\frac{k^6}{a^6}}. \tag{3.18}$$

Note that the inclusion of the term $R\Box R$ has a negative contribution to $G_{\rm eff}$. For a certain choice of parameters it might be possible to make $G_{\rm eff}$ vanishingly small, thereby modifying the Newtonian gravity on large temporal and spatial scales.

Since the corrections from different forms of the modifications, i.e. terms like R^2, $R_{\mu\nu}R^{\mu\nu}$, $\Box R$, etc. enter with different powers of the k^2 it is interesting to check which hierarchies exist between the various coefficients in order for them to be equally important at some interesting scales. This can be very helpful to understand the relative importance of the various modifications at different regimes. However, this is possible only for some simple cases and when the Lagrangian f is completely specified. In the general case, it is not easy to tell whether a term of an arbitrary function, for example, of $R\Box R$ is more important than some other term, as any of the derivatives of f, i.e. F, $F_{,R}$, etc. may contain terms like $\Box R$.

On the other hand, by studying some simple cases, like the ones mentioned in the Examples section, we can draw some interesting conclusions. For example, as can be seen from Eq. (3.18) for very small or very large scales $\frac{k}{a}$ the terms containing $\Box R$ are not as important as terms involving F and $f_{,RR}$. However, on intermediate scales the $\Box R$ terms can affect the behavior of G_{eff} and actually enters with a negative sign, which means that it may drive G_{eff} to zero or an unphysical singularity.

Let us now study the Eq. (3.18) of matter perturbations

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\mathrm{eff}}\rho_m\delta_m \simeq 0. \tag{3.19}$$

Note that the above expression will modify the large scale structure behavior on small scales as well as on large scales through higher order modifications. We will study these interesting possibilities in future publications.

IV. EXAMPLES

In this section, we will consider several examples for the very general Lagrangian of the action (2.1) in order to demonstrate how our results can be applied to a vast group of possible theories.

A. $f(R)$ gravity

As a first example we will consider $f(R)$ theories, for which we have to set $f(R, R_{\mu\nu}R^{\mu\nu}, \Box R) = f(R)$. Then Eq. (3.18) yields

$$G_{\mathrm{eff}} = \frac{1}{8\pi F}\frac{1 + 4\frac{k^2}{a^2 R}m}{1 + 3\frac{k^2}{a^2 R}m}, \tag{4.1}$$

where

$$m \equiv \frac{RF_{,R}}{F}$$

being in agreement with the standard results from $f(R)$ gravity [32].

B. $f(R, R_{\mu\nu}R^{\mu\nu})$ gravity

A second example will be the Lagrangian

$$f(R, R_{\mu\nu}R^{\mu\nu}, \Box R) = R + \sum_{n=0}^{\infty}\xi_n(R_{\mu\nu}R^{\mu\nu})^n. \tag{4.2}$$

In this case, Eq. (3.18) gives

$$G_{\mathrm{eff}}(a) = \frac{1}{8\pi}\cdot\frac{1}{1 - \frac{k^2}{a^2}\sum_{n=0}^{\infty}n\xi_n(R_{\mu\nu}R^{\mu\nu})^{n-1}} \cdot\frac{1 + 3\frac{k^2}{a^2}\sum_{n=0}^{\infty}n\xi_n(R_{\mu\nu}R^{\mu\nu})^{n-1}}{1 + 2\frac{k^2}{a^2}\sum_{n=0}^{\infty}n\xi_n(R_{\mu\nu}R^{\mu\nu})^{n-1}}. \tag{4.3}$$

If we keep only the first order term of the sum, corresponding to the Lagrangian $R + \xi R_{\mu\nu}R^{\mu\nu}$, where ξ is a constant, then G_{eff} is

$$G_{\mathrm{eff}}(a) = \frac{1}{8\pi}\cdot\frac{1 + 3\frac{k^2}{a^2}\xi}{(1 - \frac{k^2}{a^2}\xi)(1 + 2\frac{k^2}{a^2}\xi)}. \tag{4.4}$$

Next we will use the BBN constraints on the variation of the gravitational constant to constrain the parameter ξ. The effect of the variation of G_{eff} can be constrained from BBN to be of the order of 10%, see, for example, Ref. [21], which gives $\frac{G_{\mathrm{BBN}}}{G_0} = 1.09 \pm^{0.22}_{0.19}$. It is possible to use Eq. (4.4) to find analytically the best and the 1σ values of ξ according to BBN

$$\xi_{\mathrm{BBN}} = \frac{a^2_{\mathrm{BBN}}(-3 + \frac{G_{\mathrm{BBN}}}{G_0} \pm \sqrt{9 - 14\frac{G_{\mathrm{BBN}}}{G_0} + 9(\frac{G_{\mathrm{BBN}}}{G_0})^2})}{4k^2\frac{G_{\mathrm{BBN}}}{G_0}}. \tag{4.5}$$

In Fig. 1 (left) we show the plot of G_{eff}, given by Eq. (4.3), for the values of the parameter ξ, which correspond to the central and 1σ values allowed by the BBN bounds for a value of $k = 0.002\ \mathrm{Mpc}^{-1}$. However, since the k mode is actually unknown and can only be rather arbitrarily chosen,

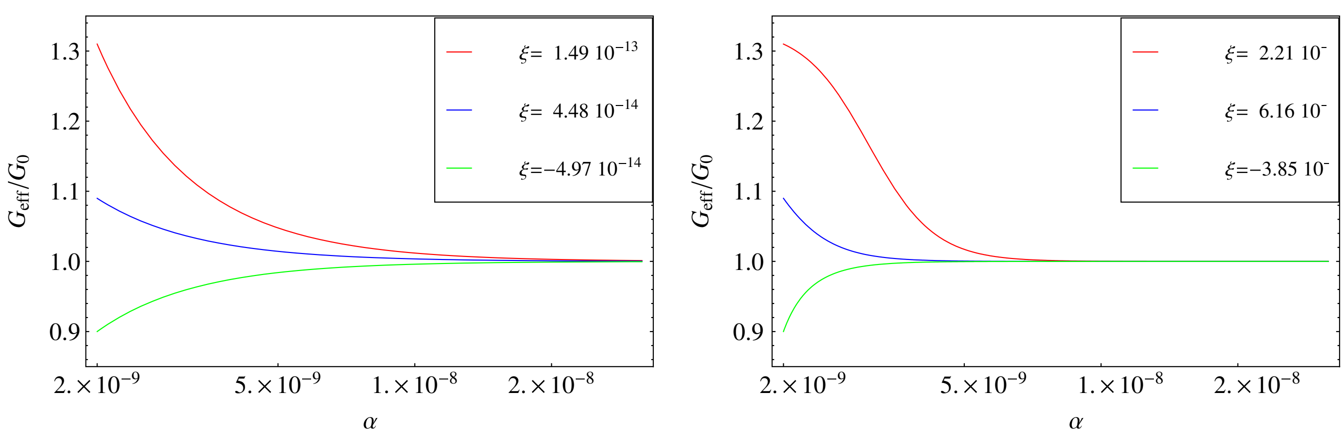

FIG. 1 (color online). Plots of G_{eff} as a function of the scale factor a, given by Eq. (4.4) for example B (left) and by Eq. (4.9) for example C (right). The values of the parameter ξ (in units of Mpc^2 and Mpc^6 respectively) used correspond to the best (blue line) and 1σ values (green and red lines) allowed by the BBN bounds and are shown in each legend, respectively.

we have also plotted the value of ξ versus k in Fig. 2 and in Table I we show ξ for various values of the scale k. It is interesting to note that in this case there are actually two allowed values of ξ by the BBN constraints, however, only one is shown in Table I for each k as the other results in completely unphysical behavior for $G_{\rm eff}$. Finally, we also consider the k mode corresponding to the horizon size at the BBN as the relevant scale. Since the horizon at the BBN is approximately $\sim 10^{-4}\ h^{-1}$ Mpc, see, for example, Ref. [34], this corresponds to a scale $k_{\rm BBN} \sim \frac{a_{\rm BBN}}{\lambda_{\rm hor}} \sim 10^{-5}\ {\rm Mpc}^{-1}$ and the corresponding constraints are shown in Table I.

It is possible to get more robust bounds on our models by considering local gravity constraints following the approach of Ref. [32]. In this case we demand that strong modifications of gravity should not be observed on scales up to $\lambda_k \sim a/k$, where in solar system experiments the scale λ_k corresponds to a value around $\lambda_k = 1$ AU. Therefore, taking into consideration Eq. (4.4), we demand that $\frac{k^2}{a^2}|\xi| \ll 1$, which gives the following constraint:

$$|\xi| \ll \lambda_k^2 \sim 10^{-23}\ {\rm Mpc}^2. \tag{4.6}$$

While this is more robust than the ones found by using the BBN constraint, the latter are not excluded as $\xi_{\rm BBN}$ has a larger 1σ error region, so the two constraints overlap with each other.

C. $f(R, \Box R)$ gravity

As an example we will consider the case where the Lagrangian contains terms of the form $\Box R$. However, to keep the analysis simple we will consider only the first order term of such corrections, and in this case the Lagrangian will be given by

$$f(R, R_{\mu\nu}R^{\mu\nu}, \Box R) = R + \xi\Box R. \tag{4.7}$$

In this case the extra term $\Box R$ can be rewritten as a total divergence and, as expected, does not contribute at all in the field equations. This can also be seen from the Friedmann Eq. (2.4), which in this special case simplifies to the usual Friedmann equation of Einstein gravity.

TABLE I. The parameter ξ using the BBN constrain for various values of the scale k. The first entry corresponds to the scale of the solar system experiments $\lambda_k \sim 1$ AU or $k_{\rm sol} \sim 2 10^{11}\ {\rm Mpc}^{-1}$, while the last ($k \sim 10^{-5}\ {\rm Mpc}^{-1}$) corresponds to the scale of the horizon during the BBN.

$k({\rm Mpc}^{-1})$	$\xi({\rm Mpc}^2)$ (case B)	$\xi({\rm Mpc}^6)$ (case C)
$2 \cdot 10^{11}$	$\lvert\xi\rvert \ll 10^{-23}$	$\lvert\xi\rvert \ll 10^{-68}$
$1 \cdot 10^{-1}$	$1.79 * 10^{-17} {}^{+4.17*10^{-17}}_{-3.78*10^{-17}}$	$3.95 * 10^{-48} {}^{+1.38*10^{-46}}_{-6.41*10^{-48}}$
$2 \cdot 10^{-3}$	$4.48 * 10^{-14} {}^{+1.04*10^{-13}}_{-9.46*10^{-14}}$	$6.16 * 10^{-38} {}^{+2.15*10^{-36}}_{-1.00*10^{-37}}$
$3 \cdot 10^{-4}$	$1.99 * 10^{-12} {}^{+4.63*10^{-12}}_{-4.20*10^{-12}}$	$5.41 * 10^{-33} {}^{+1.89*10^{-31}}_{-8.79*10^{-33}}$
$1 \cdot 10^{-5}$	$1.79 * 10^{-9} {}^{+4.17*10^{-9}}_{-3.78*10^{-9}}$	$3.95 * 10^{-24} {}^{+1.38*10^{-22}}_{-6.41*10^{-24}}$

The next most interesting case in this family of theories is the Lagrangian

$$f(R, R_{\mu\nu}R^{\mu\nu}, \Box R) = R + \xi(\Box R)^2. \tag{4.8}$$

Then, $G_{\rm eff}$ is given by

$$G_{\rm eff}(a) = \frac{1}{8\pi}\frac{1 + 8\frac{k^6}{a^6}\xi}{1 + 6\frac{k^6}{a^6}\xi}. \tag{4.9}$$

All other cases involving terms $(\Box R)^n$ with $n > 2$ give complicated functions that also involve $\Box R$ and thus are difficult to calculate.

As in the previous example, it is possible to use Eq. (4.9) to find analytically the best and the 1σ values of ξ according to BBN

$$\xi_{\rm BBN} = -\frac{a_{\rm BBN}^6(-1 + \frac{G_{\rm BBN}}{G_0})}{2k^6(-4 + 3\frac{G_{\rm BBN}}{G_0})}. \tag{4.10}$$

In Fig. 1 (right) we show the plot of $G_{\rm eff}$, given by Eq. (4.9), for the values of the parameter ξ, which correspond to the central and 1σ values allowed by the BBN bounds for a

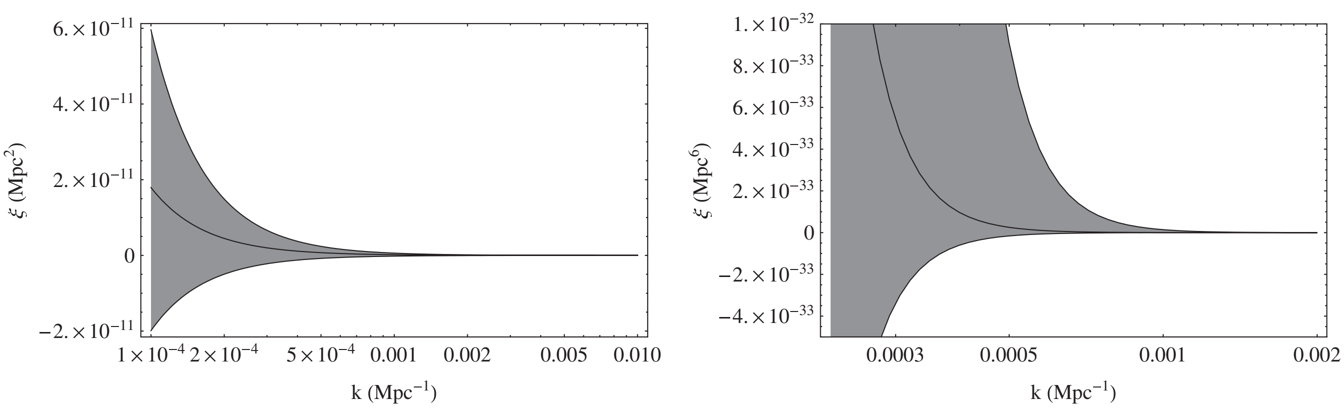

FIG. 2. Plots of ξ as a function of the scale k, given by Eq. (4.5), for example, B (left) and by Eq. (4.10), for example, C (right). The grey areas correspond to the 1σ error bars. The scale that corresponds to the horizon at the BBN is $k_{\rm BBN} \sim 10^{-5}\ {\rm Mpc}^{-1}$ and is situated outside of the range of the plots.

value of $k = 0.002\ \mathrm{Mpc}^{-1}$. However, since the k mode is actually unknown and can only be rather arbitrarily chosen, we have also plotted the value of ξ versus k in Fig. 2 (right), and in Table I we show ξ for various values of the scale k. Finally, as in the previous case we will also consider the k mode corresponding to the horizon size at the BBN, and the corresponding constraints are shown in Table I.

Using the local gravity constraints for this example and taking into consideration Eq. (4.9), we demand that $\frac{k^6}{a^6}|\xi| \ll 1$, which gives the following constraint:

$$|\xi| \ll \lambda_k^6 \sim 10^{-68}\ \mathrm{Mpc}^6. \qquad (4.11)$$

Again, this is more robust than the ones found by using the BBN constraint, the latter are not excluded as ξ_{BBN} has a larger 1σ error region, so the two constraints overlap with each other.

Another very interesting case of this class of theories is to consider terms of the form $R\Box R$, instead of just $\Box R$. These terms correspond to the first order correction of a Lagrangian of the form $R + \sum_{n=0}^{\infty} c_n R\Box^n R$, which were shown in Ref. [10] to give rise to a ghost and asymptotically free theory of gravity. Thus, keeping only the first order correction the Lagrangian is

$$f(R, R_{\mu\nu}R^{\mu\nu}, \Box R) = R + \xi R\Box R, \qquad (4.12)$$

and G_{eff} is given by

$$G_{\mathrm{eff}}(a) = \frac{1 + \xi\Box R - 8\frac{k^4}{a^4}\xi}{8\pi(1 + \xi\Box R)(1 + \xi\Box R - 6\frac{k^4}{a^4}\xi)}. \qquad (4.13)$$

As can be seen by Eq. (4.13), G_{eff} also depends $\Box R$ instead of just the scale factor a like in the previous cases. Unfortunately, we were unable to find either an analytical solution, as the Friedmann Eq. (2.4) in this case is a very complex fourth order differential equation, or a numerical one as we do not have enough initial conditions. Thus, we were unable to provide a constraint for ξ using the BBN bounds or plot G_{eff} as a function of the scale factor a.

V. CONCLUSIONS

Our analysis covers modified gravity models with a generic class of Lagrangian density with higher order and terms of the form $1/2f(R, R_{\mu\nu}R^{\mu\nu}, \Box R)$. Using the fact that at long ranges the linear perturbation analysis differentiates Einsteins gravity with respect to any modification, through the time evolution of the gravitational constant, we derived the matter density perturbation equation and the effective gravitational "constant" G_{eff} for the action (2.1).

We also used the BBN bounds on the gravitational constant, which are of the order of 10%, in order to test the difference between Einstein and modified gravity theories. The reason why the BBN bounds can be used to test modified gravity theories is that the value of the gravitational constant determines the expansion rate of the Universe and thus the relevant time scales for the production of light elements (H, He, and Li). This fact allowed us to test several cases of our general Lagrangian and constrain their parameters. Furthermore, we applied the BBN constraints to study the $\Box R$ correction in the Einstein-Hilbert action.

However, the fact that the values we found for the parameter ξ are actually larger than one would expect, it means that the energy scale at which these correction terms, e.g. $R_{\mu\nu}R^{\mu\nu}$, are introduced is quite low. For instance, one could write the corresponding term in the Lagrangian as $\frac{1}{M^2}R_{\mu\nu}R^{\mu\nu}$, where $\frac{1}{M^2}$ is the parameter ξ. Now, one would naively expect M to be of the order of Planck scale or even higher, but in our case the value of M is much smaller than that.

This can be explained by the fact that presently the BBN bounds have quite a large error themselves, which means that the constraints we derived are not very strong. This can be seen by the fact that the error on the derived parameter ξ is quite large, and this fact even allows for a zero value of ξ. Also, the primordial nucleosynthesis is quite a complex phenomenon and while its essence can be captured by a single data point, it is certain that a complete analysis, i.e. one that would also include the integration of the background equations from deep in the radiation era up to today and the use of the proper nuclear reaction rates, would most certainly provide stringent constraints.

We have also implemented local gravity constraints, following the approach of Ref. [32]. As expected, the new constraints are more robust than the ones found by using the BBN constraint; however, the latter are not excluded as ξ_{BBN} has a larger 1σ error region, so the two constraints overlap with each other.

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