Logarithmic Sobolev inequality for the invariant measure of the periodic Korteweg-de Vries equation

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Abstract
The periodic KdV equation $u_t = u_{xxx} + \beta u u_x$ arises from a Hamiltonian system with infinite-dimensional phase space $L^2(\mathbf{T})$. Bourgain has shown that there exists a Gibbs probability measure ν on balls $\{\phi : \|\phi\|^2_{L^2} \leq N\}$ in the phase space such that the Cauchy problem for KdV is well posed on the support of ν, and ν is invariant under the KdV flow. This paper shows that ν satisfies a logarithmic Sobolev inequality. The stationary points of the Hamiltonian on spheres are found in terms of elliptic functions, and they are shown to be linearly stable. The paper also presents logarithmic Sobolev inequalities for the modified periodic KdV equation and the cubic nonlinear Schrödinger equation, for small values of N.

Résumé
L'equation KdV périodic $u_t = u_{xxx} + \beta u u_x$ résulte d'un systéme hamiltonian avec des espaces infinis phase dimensions $L^2(\mathbf{T})$. Bourgain a montré qu'il existe une mesure de probabilité de Gibbs ν sure les billes $\{\phi : \|\phi\|^2_{L^2} \leq N\}$ dans l'espace des phases telles que le probléme Cauchy pour KdV est bein posé sur le support de ν et ν est invariant sous le flux de KdV. Ce document montre que ν satisfait á une inégalité de Sobolev logarithmique. Les points fixes de l'hamiltonian sur les spheres sont trouvées en termes de fonctions elliptiques, et qu'll est demonstré qu'elles soient linéairement stable. Le document présents également les inegalités de Sobolev logarithmique pour l'equation KdV modifée et les cubes nonlinear Schrödinger, put les petit valeurs de N.
Key words Gibbs measure, concentration inequality, nonlinear Schrödinger inequality

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1 Introduction

In this paper, we are concerned with solutions of the KdV equation which are periodic in the space variable and typical in the sense that they form the support of an invariant measure on an infinite-dimensional phase space. Specifically, we consider $u : \mathbf{T} \times (0, \infty) \to \mathbf{R}$ such that $u(\,, t) \in L^2(\mathbf{T})$ for each $t > 0$, then we introduce the Hamiltonian

$$H(u) = \frac{1}{2} \int_{\mathbf{T}} \Bigl(\frac{\partial u}{\partial x}(x,t) \Bigr)^2 \frac{dx}{2\pi} - \frac{\beta}{6} \int_{\mathbf{T}} u(x,t)^3 \frac{dx}{2\pi}. \tag{1.1}$$

Here β is the reciprocal of temperature, and without loss of generality, we assume throughout that $\beta > 0$. The canonical equation of motion is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \frac{\delta H}{\delta u}, \tag{1.2}$$

which gives the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} - \beta u \frac{\partial u}{\partial x}. \tag{1.3}$$

Given a solution of (1.3) that is suitably differentiable, one can easily verify that $\int_{\mathbf{T}} u(x,t)^2 dx/(2\pi)$ and $H(u)$ are invariant with respect to time. In order to ensure that the Gibbs measure can be normalized, we work on bounded subsets of $L^2(\mathbf{T})$. Hence we introduce the particle number $N < \infty$, the ball

$$B_N = \Bigl\{ \phi \in L^2(\mathbf{T}) : \int_{\mathbf{T}} \phi(x)^2 \frac{dx}{2\pi} \leq N \Bigr\} \tag{1.4}$$

with indicator function $\mathbf{I}_{B_N}$ and the Gibbs measure

$$\nu_N^\beta(d\phi) = Z_N(\beta)^{-1} \mathbf{I}_{B_N}(\phi) e^{-H(\phi)} \prod_{e^{ix} \in \mathbf{T}} d\phi(x) \tag{1.5}$$

where the normalizing constant $Z_N(\beta)$ is so chosen as to give a probability measure.

Definition. The modified canonical ensemble is the probability space (B_N, ν_N^β) that has particle number N at inverse temperature β.

The canonical ensemble would be a probability measure on the sphere $S_N = \{\phi \in L^2(\mathbf{T}) : \|\phi\|_{L^2}^2 = N\}$, but this is technically difficult to deal with, so we prefer the modified canonical ensemble. However, in section 3 we consider the Hamiltonian on the sphere and show that the stationary points of H on S_N are given by elliptic functions.

There are various means for introducing Gibbs measures on infinite-dimensional phase spaces. In [8], Lebowitz, Rose and Speer constructed an invariant measure for the nonlinear

Schrödinger equation on the line, and investigated the stability of the ground state. Using purely probabilistic arguments, McKean and Vaninsky gave an alternative construction [14].

Here we construct the measure via random Fourier series. We write $\phi(x) \sim \frac{1}{2}a_0 + \sum_{j=1}^{\infty}(a_j \cos jx + b_j \sin jx)$, and regard (a_j, b_j) as an ℓ^2 sequence of coordinates for $\phi \in L^2(\mathbf{T})$. Let $(\gamma_j)_{j=-\infty}^{\infty}$ be mutually independent standard Gaussian random variables on some probability space $(\Omega, \mathbf{P})$, and let W be the probability measure on $L^2(\mathbf{T})$ that is induced by

$$\omega \mapsto \phi_\omega(x) = \gamma_0 + \sum_{j=-\infty}^{-1} \gamma_j \frac{\sin jx}{j} + \sum_{j=1}^{\infty} \gamma_j \frac{\cos jx}{j}, \tag{1.6}$$

namely Brownian loop. Then $\Omega_N = \{\omega \in \Omega : \frac{1}{2}\sum_{j=-\infty;j\neq 0}^{\infty} \gamma_j^2/j^2 \leq N\}$ maps into B_N, and we can introduce the Gibbs measure as

$$\nu_N^\beta(d\phi) = Z_N(\beta)^{-1}\mathbf{I}_{\Omega_N}(\omega)\exp\Big(\frac{\beta}{6}\int_{\mathbf{T}} \phi_\omega(x)^3 \frac{dx}{2\pi}\Big)W(d\phi_\omega). \tag{1.7}$$

Bourgain [3] shows that there exists $Z_N(\beta) > 0$ such that ν_N^β is a Radon probability measure on the closed subset B_N of $L^2(\mathbf{T})$. Further, the Cauchy initial value problem

$$\begin{cases} u_t = -u_{xxx} - \beta u u_x \\ u(x,0) = \phi(x) \end{cases} \tag{1.8}$$

is locally well posed on the support of ν_N^β; more precisely, for each $\delta > 0$, there exists $\tau(\delta) > 0$ and a compact set K_δ such that $\nu_N^\beta(K_\delta) > 1 - \delta$ and such that for all $\phi \in K_\delta$ there exists a unique solution $u(x,t)$ to (1.7) for $t \in [0, \tau(\delta)]$. Existence of the invariant measure ν_N^β implies that the local solution extends to a global solution for almost all initial data with respect to ν_N^β. We should expect the long term behaviour of solutions to consist of a solitary travelling wave coupled with fluctuations, as described by the invariant measure. The main result of this paper is a logarithmic Sobolev inequality which shows that such a space of solutions is stable.

Definition. Suppose that $F : B_N \to \mathbf{R}$ is Gâteaux differentiable, so that for all ϕ inside B_N, there exists $\nabla F(\phi) \in L^2(\mathbf{T})$ such that

$$\langle \nabla F(\phi), \psi\rangle_{L^2} = \lim_{t\to 0+} \frac{F(\phi + t\psi) - F(\phi)}{t} \tag{1.9}$$

for all $\psi \in L^2$. Suppose further that the limit exists uniformly on $\{\psi \in L^2 : \|\psi\|_{L^2} = 1\}$; then F is Fréchet differentiable.

Let $\dot{H}^{1/2} = \{\phi(x) = \sum_{k\neq 0; k=-\infty}^{\infty} a_k e^{ikx} : \sum_{k\neq 0; k=-\infty}^{\infty} |k||a_k|^2 < \infty\}$, and let $G : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ be the operator

$$G\phi(x) = \int_{\mathbf{T}} \log \frac{1}{|e^{ix} - e^{iy}|}\, \phi(y)\, \frac{dy}{2\pi} \sim \sum_{k\neq 0; k=-\infty}^{\infty} \frac{\hat{\phi}(k)}{|k|} e^{ikx}. \tag{1.10}$$

Then

$$\langle G\psi, \phi\rangle_{\dot{H}^{1/2}} = \langle \psi, \phi\rangle_{L^2}. \tag{1.11}$$

We write $\delta F(\phi) = G(\nabla F(\phi))$, and observe that $\|\delta F(\phi)\|_{L^2} \leq \|\nabla F(\phi)\|_{L^2}$.

Definition (*Logarithmic Sobolev inequality*). Say that a probability measure ν_N on B_N satisfies the logarithmic Sobolev inequality with constant $\alpha > 0$ if

$$\int_{B_N} F(\phi)^2 \log\Bigl(F(\phi)^2 / \int_{B_N} F^2 d\nu_N\Bigr) \nu_N^{\beta}(d\phi) \leq \frac{2}{\alpha} \int_{B_N} \|\delta F(\phi)\|_{L^2}^2 \nu_N(d\phi) \tag{1.12}$$

for all Fréchet differentiable functions $F \in L^2(B_N; \nu_N^{\beta})$ such that $\|\delta F\|_{L^2} \in L^2(B_N; \nu_N)$.

Theorem 1. *For all $\beta, N > 0$ the measure ν_N^{β} satisfies the logarithmic Sobolev inequality with*

$$\alpha = 2^{-1} \exp\bigl(-C\beta^{5/2} N^{9/4}\bigr)$$

some absolute constant $C > 0$.

In section 2 we prove Theorem 1 and deduce a concentration inequality concerning Lipschitz functions on the B_N. This shows that certain random variables are tightly concentrated around their mean values, just as a Gaussian random variable is concentrated close to its mean. McKean [12] considered the Laplace operator on the infinite dimensional sphere $S^{\infty}(\sqrt{\infty})$ and showed how one can interpret this as the sum of uncoupled Ornstein–Uhlenbeck operators in infinitely many variables. In section 3, we consider stability of the stationary points of the Hamiltonian restricted to spheres. Analysis of the stationary points reduces to classical spectral theory of Lamé's equation, and we are able to identify stationary points as elliptic functions.

In section 4, we consider the Gibbs measure associated with the modified periodic KdV equation, and obtain a logarithmic Sobolev inequality when $N\beta$ is small and positive. We apply a similar analysis to the periodic cubic Schrödinger equation in section 5.

2 Logarithmic Sobolev inequalities

As in Parseval's identity, there is a unitary map $\ell^2 \to L^2(\mathbf{T})$

$$(a_0; a_n, b_n)_{n=1}^{\infty} \mapsto a_0 + \sum_{k=1}^{\infty} \sqrt{2}(a_k \cos kx + b_k \sin kx); \tag{2.1}$$

under this correspondence, $F : B_N \to \mathbf{R}$ may be identified with $f : \Omega_N \to \mathbf{R}$ and ∇F corresponds to $(\frac{\partial f}{\partial a_j}, \frac{\partial f}{\partial b_{j+1}})_{j=0}^{\infty}$. To see this, we consider $\psi(x) = \sum_{k=1}^{\infty} \sqrt{2}(c_k \cos kx + d_k \sin kx)$ and observe that

$$\langle \nabla f, (c_k, d_k)_{k=1}^{\infty} \rangle_{\ell^2} = \sum_{k=1}^{\infty} \Big(\frac{\partial f}{\partial a_k} c_k + \frac{\partial f}{\partial b_k} d_k \Big) \tag{2.2}$$

while

$$\langle \nabla F, \psi \rangle_{L^2} = \int (\nabla F) \bar{\psi} \frac{dx}{2\pi}, \tag{2.3}$$

and we can recover the Fourier coefficients of ∇F. Further, the Gibbs measure ν_N^β may be expressed in terms of the Fourier components as

$$Z_N(\beta)^{-1} \exp\Big(\frac{\beta}{6} \int_{\mathbf{T}} \Big(\big(a_0 + \sqrt{2} \sum_{j=1}^{\infty} (a_j \cos jx + b_j \sin jx)\big)^3 \frac{dx}{2\pi} - a_0^2 - \sum_{j=1}^{\infty} j^2 (a_j^2 + b_j^2) \Big)$$

$$\times \mathbf{I}_{[0,N]} \Big(a_0^2 + \sum_{j=1}^{\infty} (a_j^2 + b_j^2) \Big) \frac{da_0}{\sqrt{2\pi}} \prod_{j=1}^{\infty} \frac{j^2 da_j db_j}{2\pi}. \tag{2.4}$$

For notational simplicity, we write $a_{-j} = b_j$ for $j = 1, 2, \dots$, and assume that $0 < \beta < \sqrt{3}/(4\pi\sqrt{N})$. We introduce the potential

$$V(a,b) = \frac{a_0^2}{2} + \frac{1}{2} \sum_{j=1}^{\infty} j^2 (a_j^2 + b_j^2) - \frac{\beta 2\sqrt{2}}{6} \int_{\mathbf{T}} \Big(\frac{a_0}{\sqrt{2}} + \sum_{j=1}^{\infty} (a_j \cos jx + b_j \sin jx) \Big)^3 \frac{dx}{2\pi}. \tag{2.5}$$

Lemma 2. *Suppose that $0 < \beta\sqrt{N} \leq \sqrt{3}/(32\pi)$ and that $F \in L^2(B_N; \nu_N^\beta)$ has ∇F defined on B_N with $\|\nabla F\|_{L^2} \in L^2(B_N; \nu_N^\beta)$. Then V is uniformly convex on B_N and has a unique minimum at the origin. Moreover, ν_N^β satisfies the logarithmic Sobolev inequality (1.12) with $\alpha = 1/2$.*

Proof. First we scale the variables to $x_j = j a_j$ and $y_j = j b_j$, so that

$$\Omega_N = \{(a_0; a_j, b_j) \in \ell^2 : a_0^2 + \sum_{j=1}^{\infty} (a_j^2 + b_j^2) \leq N\} \tag{2.6}$$

is transformed to the ellipsoid

$$E_N = \{(x_0; x_j, y_j) : x_0^2 + \sum_{j=1}^{\infty} (x_j^2 + y_j^2)/j^2 \leq N\}.$$

Let $G : E_N \to \Omega_N$ be the diagonal map $G : ((x_k, y_k))_{k=1}^\infty \mapsto ((x_k/k, y_k/k))_{k=1}^\infty$, with left inverse $D : ((a_k, b_k))_{k=1}^\infty = ((ka_k, kb_k))_{k=1}^\infty$ so that $DG = I$. We then introduce $W : E_N \to \mathbf{R}$ by $W(x) = V(G(x))$.

To verify Bakry and Emery's criterion [1] for the logarithmic Sobolev inequality, we need to show that the Hessian matrix of W satisfies

$$\mathrm{Hess}\, W \geq \frac{1}{2} I \tag{2.7}$$

and hence that $\omega_N^\beta = \zeta_N(\beta)^{-1} \exp(-W(x))\, dx$ satisfies the logarithmic Sobolev inequality

$$\int_{E_N} g(x)^2 \log\Big(g(x)^2 / \int g^2 d\omega_N^\beta\Big) \leq 4 \int_{E_N} \|\nabla g\|_{\ell^2}^2 d\omega_N^\beta. \tag{2.8}$$

Now G induces ν_N^β from ω_N^β; so with $g = f \circ G$ we have $\nabla g = ((\nabla f) \circ G)(\nabla G)$ where ∇G is represented by the diagonal matrix $(1/k)_{k=1}^\infty$. The condition (2.7) is equivalent to

$$\mathrm{Hess}\, V = \begin{bmatrix} \frac{\partial^2 V}{\partial a_j \partial a_k} & \frac{\partial^2 V}{\partial a_j \partial b_k} \\ \frac{\partial^2 V}{\partial a_j \partial b_k} & \frac{\partial^2 V}{\partial b_j \partial b_k} \end{bmatrix} \geq \frac{1}{2} D^2 \tag{2.9}$$

Let D be the diagonal matrix (j) with respect to the Fourier basis and let

$$\begin{aligned} v_{jk} &= \frac{\partial^2}{\partial a_j \partial a_k} \frac{2\sqrt{2}}{6} \int_{\mathbf{T}} \Big(\frac{a_0}{\sqrt{2}} + \sum_{\ell=1}^\infty (a_\ell \cos \ell x + b_\ell \sin \ell x) \Big)^3 \frac{dx}{2\pi} \\ &= 2\sqrt{2} \int_{\mathbf{T}} \cos jx \cos kx \Big(\frac{a_0}{\sqrt{2}} + \sum_{\ell=1}^\infty (a_\ell \cos \ell x + b_\ell \sin \ell x) \Big) \frac{dx}{2\pi} \end{aligned} \tag{2.10}$$

The matrix that represents $\frac{\partial^2 V}{\partial a_j \partial a_k}$ is

$$D^2 - \beta[v_{jk}] = \frac{7}{8} D^2 + \frac{1}{8} D \Big(I - 8\beta \big[\frac{v_{jk}}{jk} \big] \Big) D \tag{2.11}$$

where $D^2 \geq I$ and by the Cauchy–Schwarz inequality

$$\begin{aligned} &\sum_{j,k=1}^\infty \frac{v_{jk} \xi_j \eta_k}{jk} \\ &\quad = 2\sqrt{2} \int_{\mathbf{T}} \sum_{j=1}^\infty \frac{\xi_j \cos jx}{j} \sum_{k=1}^\infty \frac{\eta_k \cos kx}{k} \Big(\frac{a_0}{\sqrt{2}} + \sum_{\ell+1}^\infty \big(a_\ell \cos \ell x + b_\ell \sin \ell x \big) \Big) \frac{dx}{2\pi} \end{aligned} \tag{2.12}$$

where by the Cauchy–Schwarz inequality

$$\Big(\sum_{j=1}^\infty \frac{\xi_j \cos jx}{j} \Big)^2 \leq \sum_{j=1}^\infty \frac{1}{j^2} \sum_{j=1}^\infty \xi_j^2 \leq \frac{\pi^2}{6} \sum_{j=1}^\infty \xi_j^2 \tag{2.13}$$

and

$$\int_{\mathbf{T}}\Big(\frac{a_0}{\sqrt{2}}+\sum_{\ell+1}^{\infty}\big(a_\ell\cos\ell x+b_\ell\sin\ell x\big)\Big)^2\frac{dx}{2\pi}\leq N. \tag{2.14}$$

Hence we have

$$D^2-\beta[v_{jk}]\geq (1/2)D^2; \tag{2.15}$$

similar estimates apply to the sine terms when we consider $\frac{\partial^2 V}{\partial b_j\partial b_k}$, and to the mixed sine and cosine term which arise in $\frac{\partial^2 V}{\partial a_j\partial b_k}$. Hence W is uniformly convex and thus satisfies Bakry and Emery's criterion, so ω_N^β satisfies the logarithmic Sobolev inequality (2.8), and hence ν_N^β satisfies (2.6)

□

Proof of Theorem 1. We need to extend the logarithmic Sobolev inequality to a typical pair $N,\beta>0$, possibly at the expense of a worse constant. So we choose $K>4\beta\sqrt{N}+1$, and split $\phi\in L^2(\mathbf{T})$ into the tail $\phi_K(x)=\sum_{k:|k|\geq K}a_ke^{ikx}$ and the head $h_K(x)=\sum_{k:|k|\leq K}a_ke^{ikx}$ of the series. We note that for $\phi\in B_N$, the components satisfy $\|h_K\|_\infty\leq (2K+1)^{1/2}N^{1/2}$ and $\int\phi_K(x)^2dx/(2\pi)\leq N$; hence by some simple estimates

$$\begin{aligned}\Big|\int_{\mathbf{T}}(\phi(x)^3-\phi_K(x)^3)\frac{dx}{2\pi}\Big| &= \Big|\int_{\mathbf{T}}(3\phi_K(x)^2h_K(x)+3\phi_K(x)h_K(x)^2+h_K(x)^3)\frac{dx}{2\pi}\Big|\\ &\leq 7\beta(2K+1)^{3/2}N^{3/2}.\end{aligned} \tag{2.16}$$

We replace the original potential V by

$$V_K(a,b)=\frac{a_0^2}{2}+\frac{1}{2}\sum_{j=1}^{\infty}j^2(a_j^2+b_j^2)-\frac{\beta 2\sqrt{2}}{6}\int_{\mathbf{T}}\Big(\sum_{j=K+1}^{\infty}(a_j\cos jx+b_j\sin jx)\Big)^3\frac{dx}{2\pi}, \tag{2.17}$$

which is a bounded perturbation of V on B_N and satisfies

$$\|V-V_K\|_\infty\leq 7\beta(2K+1)^{3/2}N^{3/2}. \tag{2.18}$$

The matrix $[v_{jk}]_{j,k:|j|,|k|\geq K}$ that arises from V_K via (2.5) involves only high frequency components and satisfies

$$\beta^2\Big\|\Big[\frac{v_{jk}}{jk}\Big]\Big\|^2_{c^2}\leq 4\beta^2N\Big(\sum_{k=K}^{\infty}\frac{1}{k^2}\Big)^2\leq\frac{4\beta^2N}{(K-1)^2}\leq\frac{1}{4} \tag{2.19}$$

by the choice of K. By Lemma 2, V_K is uniformly convex and the corresponding Gibbs measure satisfies a logarithmic Sobolev inequality with constant independent of N and β.

Since V is a bounded perturbation of V_K, the Holley–Stroock lemma [7] shows that the Gibbs measure associated with V also satisfies the logarithmic Sobolev inequality

$$\int_{B_N} F(\phi)^2 \log\Bigl(F(\phi)^2/\int_{B_N} F^2 d\nu_N^\beta\Bigr)\nu_N^\beta(d\phi) \leq 4\exp(C\beta^{5/2}N^{9/4})\int_{B_N} \|\delta F(\phi)\|_{L^2}^2 \nu_N^\beta(d\phi). \tag{2.20}$$

for come universal constant C.

□

Corollary 2. *Let $F : B_N \to \mathbf{R}$ be a Lipschitz function such that $|F(\phi)-F(\psi)| \leq \|\phi-\psi\|_{L^2}$ for all $\phi, \psi \in B_N$; suppose further that $\int_{B_N} F(\phi)\nu_N^\beta(d\phi) = 0$. Then*

$$\int_{B_N} \exp(tF(\phi))\nu_N^\beta(d\phi) \leq \exp\Bigl(e^{C\beta^{5/2}N^{9/4}}t^2\Bigr) \qquad (t \in \mathbf{R}). \tag{2.21}$$

On the probability space (B_N, ν_N^β), the random variable F has mean zero and takes values that are tightly concentrated about its mean value.

Proof. Let P_n be the orthogonal projection onto $\mathrm{span}\{e_j : 1 \leq j \leq n\}$, where (e_j) is some orthonormal basis of $L^2(\mathbf{T})$. Then $F \circ P_n$ is Lipschitz continuous on a finite-dimensional subspace, and hence Féchet differentiable almost everywhere by Rademacher's theorem. We observe that

$$\|\delta F(P_n\phi)\|_{L^2} \leq \|\nabla F(P_n\phi)\|_{L^2} \leq 1 \tag{2.22}$$

since F and P_n are Lipschitz. Since $F \circ P_n \to F$ uniformly on compact sets as $n \to \infty$, it suffices by Fatou's lemma to prove (2.21) for $F \circ P_n$ and then let $n \to \infty$.

The inequality then follows from Theorem 1 by the general theory of functional inequalities, as in [16]; here we give a brief argument. Let

$$J(t) = \int_{B_N} e^{tF(\phi)}\nu_N^\beta(d\phi) \tag{2.23}$$

which defines an analytic function of t such that $J(0) = 1$ and $J'(0) = 0$. Further, the logarithmic Sobolev inequality gives

$$\begin{aligned} tJ'(t) &= \int_{B_N} tF(\phi)e^{tF(\phi)}\nu_N^\beta(d\phi) \\ &\leq J(t)\log J(t) + \frac{t^2}{2\alpha}J(t) \qquad (t > 0) \end{aligned} \tag{2.24}$$

which integrates to the inequality

$$J(t) \leq \exp\Bigl(\frac{t^2}{2\alpha}\Bigr). \tag{2.25}$$

□

Remark. We leave it as an open problem to determine whether ν_N^β satisfies (1.9) with a constant independent of N for given β.

3 Stationary points of the Hamiltonian on spheres

The Hamiltonian $H(\phi)$ is unbounded above and below for $\phi \in L^2(\mathbf{T})$; however, we can consider the minimal energy constrained to the spheres in $L^2(\mathbf{T})$:

$$E_N = \inf\Big\{ H(\phi) : \int_{\mathbf{T}} \phi(x)^2 \frac{dx}{2\pi} = N \Big\}. \tag{3.1}$$

Korteweg and de Vries introduced a travelling wave solution $u(x,t) = v(x-ct)$ of $u_t + u_{xxx} + \beta u u_x = 0$ which is periodic and is commonly known as the cnoidal wave. We recover this solution below. Subject to some reservations, Drazin [5] showed that the cnoidal wave is linearly stable with respect to any infinitesimal perturbation.

We recall Jacobi's sinus amplitudinus of modulus k is $\mathrm{sn}(x \mid k) = \sin\psi$ where

$$x = \int_0^\psi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{3.2}$$

For $0<k<1$, let $K(k)$ be the complete elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}; \tag{3.3}$$

next let $K'(k) = K(\sqrt{1-k^2})$; then $\mathrm{sn}(z \mid k)^2$ has real period K and complex period $2iK'$.

For $\ell > 0$, the standard form of Lamé's equation is

$$\Big(-\frac{d^2}{dz^2} + \ell(\ell+1)k^2 \mathrm{sn}(z \mid k)^2 \Big) \Phi(z) = \mu \Phi(z). \tag{3.4}$$

The spectrum of (3.4) is determined by a sequence $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$, which is infinite except for $\ell = 1, 2, \dots$. Typically $\sigma_B = \cup_{j=0}^\infty [\lambda_{2j}, \lambda_{2j+1}]$ gives the Bloch spectrum, so that for $\mu \in \sigma_B$ there exists a bounded solution to (3.4); whereas for $\mu \in (-\infty, \lambda_0) \cup \bigcup_{j=0}^\infty (\lambda_{2j+1}, \lambda_{2j+2})$ all nontrivial solutions of (3.4) are unbounded, and we say that μ belongs to an interval of instability. In the special case of $\ell = 1, 2, \dots$, there are only $\ell + 1$ intervals of instability, namely $(\lambda_{2j+1}, \lambda_{2j+2})$ $(j = 0, \dots, j-1)$ and $(-\infty, \lambda_0)$. See [11].

Theorem 3. *(i) Let $\phi \in C^2(\mathbf{T}; \mathbf{R})$ be a stationary point for the energy*

$$H_\lambda(\phi) = \frac{1}{2}\int_{\mathbf{T}} \phi'(x)^2 \frac{dx}{2\pi} - \frac{\beta}{6}\int_{\mathbf{T}} \phi(x)^3 \frac{dx}{2\pi} - \frac{\lambda}{2}\int_{\mathbf{T}} \phi(x)^2 \frac{dx}{2\pi}. \tag{3.5}$$

Then ϕ satisfies the differential equation

$$\phi''(x) + \frac{\beta}{2}\phi(x)^2 + \lambda\phi(x) = 0 \tag{3.6}$$

so either ϕ is constant or an elliptic function.

(ii) For $\beta > 0$, the energy H_λ on E_N has a local minimum at $\phi = -\sqrt{N}$.

(iii) Let ϕ be the elliptic function

$$\phi(x) = f_1 - (f_1 - f_2)\Big[\mathrm{sn}\Big(\sqrt{\frac{\beta(f_1 - f_3)}{12}}(x_1 - x)\Big|\sqrt{\frac{f_1 - f_2}{f_1 - f_3}}\Big)\Big]^2 \tag{3.7}$$

for suitable real constants $f_3 < f_2 < f_1$ and $\phi(x_1) = f_1$. Let $(-\infty, \lambda_0]$ be the zeroth order interval of instability of Lamé's equation

$$y''(x) + \beta\phi(x)y(x) + \lambda y(x) = 0. \tag{3.8}$$

Then for $\lambda < \lambda_0$ the energy H_λ has a local minimum at ϕ; whereas for $\lambda > \lambda_0$ the stationary point is neither a local maximum nor a local minimum.

Proof. (i) We suppose that $\beta > 0$. One can easily expand $H_\lambda(\phi+t\psi)$ as a cubic polynomial in t and examine the conditions that ensure that $t = 0$ gives a local minimum. The equation $\frac{\delta H_\lambda}{\delta\phi} = 0$ reduces to the differential equation (3.6) which has constant solutions $\phi = 0$ (which does not belong to E_N) and $\phi = -2\lambda/\beta$, and a non-constant solution satisfying

$$\frac{1}{2}\phi'(x)^2 + \frac{\beta}{6}\phi(x)^3 + \frac{\lambda}{2}\phi(x)^2 = C \tag{3.9}$$

with C some constant. This equation has periodic solutions if and only if $\beta^2 < 3\lambda^3/(2C)$; equivalently, for such constants there exist real roots $f_3 < f_2 < f_1$ such that

$$-\frac{\beta}{6}\phi^3 - \frac{\lambda}{2}\phi^2 + C = -\frac{\beta}{6}(\phi - f_1)(\phi - f_2)(\phi - f_3). \tag{3.10}$$

To find these roots, we introduce $z = 1/\phi$, which satisfies the cubic $z^3 - \lambda z/(2C) - \beta/(6C) = 0$ with discriminant

$$D = \frac{-\lambda^3}{216C^3} + \frac{\beta^2}{144C^2}; \tag{3.11}$$

so by writing $re^{i\theta} = \beta/(12C) + i\sqrt{-D}$, we have

$$\frac{1}{f_1} = 2r^{1/3}\cos\frac{\theta}{3}, \quad \frac{1}{f_2} = 2r^{1/3}\cos\frac{\theta + 2\pi}{3}, \quad \frac{1}{f_3} = 2r^{1/3}\cos\frac{\theta + 4\pi}{3}, \tag{3.12}$$

for some choice of the polar angle. To convert to the standard form (3.4) of Lamé's equation, we introduce

$$k=\sqrt{\frac{f_1-f_2}{f_1-f_3}},\qquad k^2\ell(\ell+1)=\beta(f_1-f_3),\quad \gamma=\cos\frac{\theta}{3},\tag{3.13}$$

where $0<k<1$ and $\ell>0$, and by some trigonometry deduce that

$$k^2=\frac{2\gamma\sqrt{1-\gamma^2}}{\sqrt{3}/2-\sqrt{3}\gamma^2-\gamma\sqrt{1-\gamma^2}},\tag{3.14}$$

which is an algebraic function of the parameters. Using the definition (3.2), one can show that the solution of (3.9) is given by

$$\phi(x)=f_1-(f_1-f_2)\Big[\mathrm{sn}\Big(\sqrt{\frac{\beta(f_1-f_3)}{12}}(x_1-x)\Big|k\Big)\Big]^2;\tag{3.15}$$

where ϕ is 2π-periodic provided that

$$2\pi\sqrt{\frac{\beta(f_1-f_3)}{12}}=2K(k).\tag{3.16}$$

(ii) For any stationary point ϕ, we have

$$H_\lambda(\phi+t\psi)=H_\lambda(\phi)+\frac{t^2}{2}\int_{\mathbf{T}}\big(\psi'(x)^2-(\lambda+\beta\phi(x))\psi(x)^2\big)\frac{dx}{2\pi}-\frac{\beta t^3}{6}\int_{\mathbf{T}}\psi(x)^3\frac{dx}{2\pi}.\tag{3.17}$$

First we deal with the constant stationary points. Note that $\phi=-\sqrt{N}$ gives a local minimum for H_λ with $\lambda=\beta\sqrt{N}/2$ and $H_\lambda(-\sqrt{N})=-\beta N^{3/2}/12$. The other constant solution $\phi=\sqrt{N}$ has $\lambda=-\beta\sqrt{N}$ and $H_\lambda(\sqrt{N})=\beta N^{3/2}/12$.

(iii) Next we take ϕ from (3.9), so that

$$H_\lambda(\phi)=-\frac{\beta}{3}\int_0^{2\pi}\phi(t)^3\frac{dt}{2\pi}-\lambda\int_0^{2\pi}\phi(t)^2\frac{dt}{2\pi}+C.\tag{3.18}$$

To determine when this is a local minimum, we consider a nontrivial solution the equation

$$y''(x)+\beta\phi(x)y(x)+\lambda y(x)=0,\tag{3.19}$$

which is the linearization of (3.6), and re recognise this as a form of Lamé's equation.

First suppose that y has only finitely many zeros on $\mathbf{R}$; then by Hamel's theorem [10]

$$\int_{\mathbf{T}}\big(\psi'(x)^2-(\lambda+\beta\phi(x))\psi(x)^2\big)\frac{dx}{2\pi}\geq 0\tag{3.20}$$

holds for all 2π periodic and continuously differentiable functions ψ. Let λ_0 be the supremum of such λ. For $\lambda < \lambda_0$ we have strict inequality for in (3.20) all nonzero ψ, so H_λ has a local minimum at ϕ.

Suppose contrariwise that y has infinitely many zeros; so by Hamel's theorem there exists ψ such that

$$\int_{\mathbf{T}} \big(\psi'(x)^2 - (\lambda + \beta\phi(x))\psi(x)^2\big)\frac{dx}{2\pi} < 0; \tag{3.21}$$

hence H_λ does not have a local minimum at ϕ. In particular, this happens when $\lambda > \lambda_0$.

□

4 The modified periodic KdV equation

Lebowitz, Rose and Speer [8,9] considered Hamiltonians

$$H_p(\phi) = \frac{1}{2}\int_{\mathbf{T}} \phi'(x)^2 \frac{dx}{2\pi} - \frac{\beta}{p(p-1)}\int_{\mathbf{T}} |\phi(x)|^p \frac{dx}{2\pi} \tag{4.1}$$

and showed that there exists a Gibbs measure with potential $H_p(\phi)$ on $\{\phi \in L^2(\mathbf{T}) : \|\phi\|^2_{L^2} \leq N\}$ as in (1.6) for $\beta, N > 0$ and $2 \leq p < 6$, but not for $p > 6$. When $\beta < 0$, the Hamiltonian is non focusing and the problem of normalizing the probability measures is easy to address.

Their analysis of the case $p = 4$ showed that for small $\beta > 0$, the constant solution was stable; whereas for large $\beta > 0$, the soliton solution was stable. In this section we strengthen their result by showing that for $p = 4$ the modified canonical ensemble satisfies a logarithmic Sobolev inequality when βN is small. The case $p = 4$ corresponds to the modified KdV equation

$$\frac{\partial\phi}{\partial t} + \frac{\partial^3\phi}{\partial\theta^3} + \beta\phi^2\frac{\partial\phi}{\partial\theta} = 0, \tag{4.2}$$

so the result suggests that at low temperatures, solutions of the mKdV equation are most likely to occur near to the ground state.

With this Hamiltonian and $\beta > 0$, we introduce the Gibbs measure

$$\nu_N^\beta(d\phi) = Z_N(\beta)^{-1}\mathbf{I}_{B_N}(\phi)e^{-H_4(\phi)}\prod_{e^{ix}\in\mathbf{T}} d\phi(x). \tag{4.3}$$

Theorem 4. *There exists $C > 0$ such that for $0 < \beta N < C$ the Gibbs measure ν_N^β satisfies the logarithmic Sobolev inequality with $\alpha = 1/2$.*

Proof. This is similar to the proof of Lemma 2. To simplify notation, we consider the Hamiltonian

$$H(a_n) = \sum_{n=1}^\infty n^2 a_n^2 - \frac{\beta}{12}\int_{\mathbf{T}}\Big(\sum_{n=1}^\infty a_n\cos n\theta\Big)^4\frac{d\theta}{2\pi} \tag{4.5}$$

on $B_N = \{(a_n) \in \ell^2(\mathbf{R}) : \sum_{n=1}^{\infty} a_n^2 \leq N\}$ which has essentially the same properties as the true Hamiltonian in the Fourier components. The corresponding Hessian matrix has entries

$$\frac{\partial^2 H}{\partial a_j \partial a_\ell} = \delta_{j\ell} j^2 - \beta \int_{\mathbf{T}} \Bigl(\sum_{n=1}^{\infty} a_n \cos n\theta\Bigr)^2 \cos j\theta \cos \ell\theta \frac{d\theta}{2\pi}, \tag{4.6}$$

and we deduce that

$$\sum_{j,\ell=1}^{\infty} \frac{\partial^2 H}{\partial a_j \partial a_\ell} \frac{\xi_j \xi_\ell}{j\ell} = \sum_{j=1}^{\infty} \xi_j^2 - \beta \int_{\mathbf{T}} \Bigl(\sum_{j=1}^{\infty} a_n \cos n\theta\Bigr)^2 \sum_{j=1}^{\infty} \frac{\xi_j \cos j\theta}{j} \sum_{\ell=1}^{\infty} \frac{\xi_\ell \cos \ell\theta}{\ell} \frac{d\theta}{2\pi}, \tag{4.7}$$

where by the Cauchy–Schwarz inequality

$$\Bigl(\sum_{j=1}^{\infty} \frac{\xi_j \cos j\theta}{j}\Bigr)^2 \leq \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{j=1}^{\infty} \xi_j^2; \tag{4.8}$$

hence

$$\sum_{j,\ell=1}^{\infty} \frac{\partial^2 H}{\partial a_j \partial a_\ell} \frac{\xi_j \xi_\ell}{j\ell} \geq \frac{1}{2} \sum_{j=1}^{\infty} \xi_j^2 \tag{4.9}$$

and Bakry and Emery's condition is satisfied. The logarithmic Sobolev inequality follows.

□

Remark. The author has not succeeded in proving a logarithmic Sobolev inequality for $p = 4$ and $N\beta$ large due to the lack of a suitable substitute for (2.14).

5. Periodic solutions of the cubic nonlinear Schrödinger equation

The Hamiltonian

$$H = \frac{1}{2} \int_{\mathbf{T}} \bigl((Q')^2 + (P')^2\bigr)\, dx + \frac{\beta}{4} \int_{\mathbf{T}} \bigl(Q^2 + P^2\bigr)^2\, dx \tag{5.1}$$

gives the canonical equations of motion

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} Q \\ P \end{bmatrix} = -\frac{\partial^2}{\partial x^2} \begin{bmatrix} Q \\ P \end{bmatrix} + \beta(Q^2 + P^2) \begin{bmatrix} Q \\ P \end{bmatrix} \tag{5.2}$$

which give the cubic Schrödinger equation for $u = P + iQ$

$$-iu_t = -u_{xx} + \beta |u|^2 u \tag{5.3}$$

over the circle. Again we take $\beta > 0$, which gives the focusing case. The appropriate number operator is represented by

$$N = \frac{1}{2} \int_{\mathbf{T}} \bigl(P^2 + Q^2\bigr) dx,$$

which is invariant under the flow, and we introduce $B_N = \{u = P + iQ : \int_{\mathbf{T}} (P^2 + Q^2) dx \leq N\}$. As discussed in [15], one can normalize the measure

$$\nu_N^\beta(du) = Z_N(\beta)^{-1} \exp\big(-H(u)\big) \mathbf{I}_{B_N}(u) \prod_{x \in \mathbf{T}} du(x) \tag{5.4}$$

so that it gives a probability measure on B_N. Bourgain has shown that ν_N^β is invariant under the flow associated with the cubic Schrödinger equation; see [4, p.124]

Theorem 5. *There exists $C > 0$ such that for all $0 < \beta N < 1/2$ the Gibbs measure ν_N^β satisfies the logarithmic Sobolev inequality with $\alpha = 1/2$.*

Proof. This follows the proof of Lemma 2 and Theorem 4 closely, hence is omitted.

□

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