# RESULTS ON THE NUMBER OF ZERO MODES OF THE WEYL-DIRAC OPERATOR

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# *Results on the number of zero modes of the Weyl-Dirac operator*

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# Abstract

For a given magnetic potential $A$ one can define the Weyl-Dirac operator $\sigma\cdot(-i\nabla - A)$ on $\mathbb{R}^3$. An $L^2$ eigenfunction of $\sigma \cdot (-i\nabla - A)$ corresponding to 0 is called a zero mode. In this thesis we will be concerned with the zero mode problem for the Weyl-Dirac operator and some related problems. The main results are (i) upper bounds for the number of zero modes of the Weyl-Dirac operator in three dimensions when scaling a given magnetic field. A similar version for the Dirac operator in two dimensions is also obtained. There are also related results to estimate the number of zero modes of the massless Dirac operator, and the dimension of the eigenspaces at threshold energies for the Dirac operator with positive mass. (ii) construction of Dirac operators on the unit ball $\mathbb{S}^2$ of $\mathbb{R}^3$ as well as the determination of their spectrum in case of "constant" magnetic fields. We also show another proof for the Aharonov-Casher theorem for $\mathbb{S}^2$ based on results about spectral properties of Dirac operators that we have obtained. (iii) a formula giving the number of zero modes of the Weyl-Dirac operator for a special magnetic field, which is the result of pullbacks from the "constant" volume form of $\mathbb{S}^2$. We also obtain a lower bound for the number of zero modes for the Weyl-Dirac operator corresponding to certain scaled magnetic fields; the magnetic fields are parallel to fibres of the Hopf fibration (pulled-back to $\mathbb{R}^3$ using inverse stereographic projection).

# Declaration

*I declare that the work in this thesis was carried out in accordance with the Regulations of Lancaster University. The work is original except where indicated by references in the text and no part of the thesis has been submitted for any other degree. The thesis has not been presented to any other University for examination either in the United Kingdom or overseas.*

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# Contents

# Chapter 1

# Introduction

In this chapter we will be concerned with the background of the zero mode problem. We will answer the question of when and in what circumstances this concept appeared. We will see the concept of zero mode in fact arose for the first time from a problem in physics. However we will also see some arguments about why this concept is interesting from a mathematical point of view. Comparison between the zero mode problems on $\mathbb{R}^3$ and on $\mathbb{R}^2$ is discussed. Apart from the first and the final section to be concerned with operators and notations to be used throughout the thesis this chapter briefly summaries most known results about the zero mode problem; that is hopefully about the zero mode problem progress in the last twenty years. It also helps us to clarify the contribution of results in the thesis, which will be its core part afterward.

## 1.1 Magnetic operators

Suppose that we have a vector field $A = (A_1, A_2, A_3)$, where here $A_j,\ j = 1, 2, 3$ are real-valued functions of $x \in \mathbb{R}^3$. Here we call $A$ the *vector potential* of a *magnetic field* $B$, where $B := \text{curl } A$. More specifically, we have $B = (B_1, B_2, B_3)$, where

$$B_1 = \partial_2 A_3 - \partial_3 A_2,\ B_2 = \partial_3 A_1 - \partial_1 A_3,\ B_3 = \partial_1 A_2 - \partial_2 A_1. \tag{1.1}$$

We also sometimes call $A$ a *magnetic potential*. Let $\nabla = (\partial_1, \partial_2, \partial_3)$ be the usual gradient operator on $\mathbb{R}^3$, we notice that

$$B = \nabla \times A.$$

Take $L^2(\mathbb{R}^3)$ and consider the the Laplace operator

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2.$$

Obviously, we cannot define the operator $-\Delta$ in the whole $L^2(\mathbb{R}^3)$. However, we may define $-\Delta$ with the domain as the usual Sobolev space $H^2(\mathbb{R}^3)$. The thus obtained operator is called the *free Schrödinger operator*, denoted by $\mathsf{H}_0$. From the physical point of view the Hilbert space $L^2(\mathbb{R}^3)$ corresponds to one particle in $\mathbb{R}^3$, and the free Schrödinger operator is the corresponding non relativistic Halmiltonian when the particle does not interact with anything.

In Physics the Hamiltonian of a quantum mechanical system is usually the sum of the free Schrödinger operator and a "multiplication" operator $V$ corresponding to the potential energy. From the mathematical point of view that means in $L^2(\mathbb{R}^3)$ we consider the operator $-\Delta + V$ or $\mathsf{H}_0 + V$; this is known as the *Schrödinger operator* in $L^2(\mathbb{R}^3)$.

Denote by $p$ the operator $-i\nabla$ in $L^2(\mathbb{R}^3)$. Then, $p$ is often called to be the *momentum operator*. In case we have a vector potential $A = (A_1, A_2, A_3)$ we may consider the operator $p - A$ or $(-i\partial_1 - A_1, -i\partial_2 - A_2, -i\partial_3 - A_3)$ in $L^2(\mathbb{R}^3)$; this is called the *magnetic momentum operator* in $L^2(\mathbb{R}^3)$. Observe that $p^2 = -\Delta$. As for the case of Schrödinger operators we may consider the operator $(p - A)^2 + V$; this is called the *magnetic Schrödinger operator*.

We denote by $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are usual *Pauli matrices*, and $i$ is the imaginary unit, $i^2 = -1$. Now, for a certain vector potential $A = (A_1, A_2, A_3)$ in $[L^2(\mathbb{R}^3)]^2$ we may consider the *Weyl-Dirac operator*, denoted by $\mathcal{D}_A$, where

$$\mathcal{D}_A := \sigma \cdot (p - A). \tag{1.2}$$

(We follow Balinsky and Evans (see [12]) in using the term *Weyl*-Dirac operator). In $[L^2(\mathbb{R}^3)]^2$ the Weyl-Dirac operator $\mathcal{D}_A$ is formally self-adjoint.

**Definition 1.1.1.** If we have a nontrivial $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in [L^2(\mathbb{R}^3)]^2$ such that $\mathcal{D}_A\psi = 0$, then $\psi$ is called the *zero mode* for $\mathcal{D}_A$.

Suppose that $B$ is a (smooth) magnetic field on $\mathbb{R}^3$; that is $B$ is a smooth vector field satisfying div $B = 0$. Then we can find a (smooth) magnetic potential $A$ with $B =$ curl $A$ (see, for instance [33], p.206). Furthermore, although $A$ is not uniquely determined, if $A$ and $A'$ are two potentials satisfying curl $A = B =$ curl $A'$, then $A' - A = \nabla\varphi$ for some smooth function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (see, for instance [38], p.106). Multiplication by $e^{i\varphi}$ defines a unitary map on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ called a *gauge transformation*. We have

$$(p - A') = e^{i\varphi}(p - A)e^{-i\varphi}. \tag{1.3}$$

Therefore

$$\mathcal{D}_{A'} = e^{i\varphi}\mathcal{D}_A e^{-i\varphi}, \tag{1.4}$$

so the Weyl-Dirac operators $\mathcal{D}_A$ and $\mathcal{D}_{A'}$ are unitarily equivalent. Thus the spectrum of a Weyl-Dirac operator, and in particular the number of corresponding zero modes, is determined entirely by the magnetic field $B = \operatorname{curl} A$. For this reason we will sometimes write, for instance, $\mathcal{D}_{B,\mathbb{R}^3}$ for $\mathcal{D}_A$.

The square of $\mathcal{D}_A$ is called the *Pauli operator* and is denoted by $\mathcal{P}_A$; that is

$$\mathcal{P}_A := \big[\sigma \cdot (p - A)\big]^2. \tag{1.5}$$

Observe that if we have two vector potentials $A$ and $A'$ in $\mathbb{R}^3$ we have

$$(\sigma \cdot A) \cdot (\sigma \cdot A') = A \cdot A' + i\sigma \cdot (A \times A').$$

Then, we may see that

$$\mathcal{P}_A = \big[\sigma \cdot (p - A)\big]^2 = (p - A)^2 - \sigma \cdot B. \tag{1.6}$$

Here in (1.6) we may put $\mathbb{I}_2$-the $2 \times 2$ identity matrix wherever it is needed.

## 1.2 Why zero modes?

In [29] Fröhlich *et al.* consider the problem of the stability of the hydrogen atom in external magnetic fields. Here the stability for a system means the finiteness of its ground state energy or the finiteness of the bottom eigenvalue for the corresponding operator from a mathematical point of view. Specifically, Fröhlich *et al.* were concerned with the problem of the one-electron atom in a magnetic field $B = (B_1, B_2, B_3)$. The Hamiltonian in this case is

$$\mathbf{H} = (p - A)^2 - \sigma \cdot B - \frac{z}{|x|}$$

with $p = -i\nabla$ as usual and $z$ is a nuclear charge number. They noted that the problem above is not very interesting if the electron spin is not included (in mathematical point of view that means removing the $\sigma \cdot B$ term). Truly, in that case we can apply Kato's inequality (see [34], Lemma A) and then obtain for any $\psi \in [C_0^\infty(\mathbb{R}^3)]^2$

$$\langle \psi,\ (p - A)^2\psi\rangle \geq \langle |\psi|,\ p^2|\psi|\rangle.$$

We also notice that

$$\langle \psi,\ |x|^{-1}\psi\rangle = \langle |\psi|,\ |x|^{-1}|\psi|\rangle.$$

Then, the finiteness of the ground state energy follows from the classical results for the similar problem in case of having no potential $A$, which was fully solved after the introduction of the Schrödinger equation during the first years of the twentieth century.

Denote by $E_0(B,\ z)$ the ground state energy for the system above. Fröhlich *et al.* saw that the problem with the electron spin included is more interesting. In [9]

Avron *et al.* showed that the ground state $E_0(B,\ z)$ of $\mathbf{H}$ is always finite, but it depends on the electron spin interacting with the magnetic field $B$ in such a way that $E_0(B,\ z) \longrightarrow -\infty$ as $B \longrightarrow \infty$. For instance, with constant magnetic field $B$ which is big enough, Avon *et al.* proved that $E_0(B,\ z)$ is approximately $-(\log B)^2$. Therefore, Fröhlich *et al.* want to prevent $B$ from spontaneously growing large and driving $E_0(B,\ z)$ towards $-\infty$. So, they raised the question of whether the stability of the system is independent of $B$ when adding the magnetic field energy (in their units) $\varepsilon \int_{\mathbb{R}^3} B^2\ dx$, where $8\pi\varepsilon\alpha^2 = 1$ for the fine structure constant $\alpha$. From a mathematical point of view the question becomes whether

$$E(B,\ z) = E_0(B,\ z) + \varepsilon \int_{\mathbb{R}^3} B^2 dx$$

is bounded below independent of $B$. To investigate that question they showed that there is a critical nuclear charge $z_c > 0$ such that there is stability for one electron atoms if $z < z_c$ and instability if $z > z_c$.

The next question for Fröhlich *et al.* is whether or not $z_c$ is finite. It turns out that the finiteness of $z_c$ depends on the existence of a nontrivial two-component spinor $\psi \in [H^1(\mathbb{R}^3)]^2$ (the usual Sobolev space) such that

$$\sigma \cdot (p - A)\psi(x) = 0 \tag{1.7}$$

with potential $A \in L^6(\mathbb{R}^3)$ satisfying $\mathrm{div}A = 0$ and $B = \nabla \times A \in L^2(\mathbb{R}^3)$. Thus, it turns out that they have to study the problem of existence of zero modes for the Weyl-Dirac operator (1.2)

$$\mathcal{D}_A = \sigma \cdot (p - A).$$

Loss (one of authours for [39]) and his collaborator Yau were the first to find examples of such spinors. They called them *the zero energy bound states*. They also sketched two general methods of constructing such specific zero modes, as has become the popular name nowadays. By extension we also use "zero mode" for the squared-integrable elements in the kernel of some other operators. We will later mention the two methods proposed by Loss and Yau and we will discuss the first zero mode of Loss and Yau as well.

To conclude this section we would like to discuss more details about one of estimates in [39] which was again mentioned in [30]. In general for a given vector potential $A$ we do not know exactly whether we obtain any zero modes. However we may show a upper bound for the number of zero modes for an arbitrary potential $A$. The concern here is the details of the proof about the upper bound for the number of zero modes we would expect for magnetic potentials $A$. In our view we can argue as follows: first, it follows from (1.6) that

$$(p - A)^2 - \sigma \cdot B = [\sigma \cdot (p - A)]^2.$$

The operator on right-hand side is the Pauli operator $\mathcal{P}_A$. Now if $\psi$ is a zero mode, then we also have $[\sigma \cdot (p - A)]^2\psi(x) = 0$. So, $\psi$ is the ground bound state for the Pauli operator $\mathcal{P}_A$ (since the Pauli operator is positive). Then, we get $[(p-A)^2 - \sigma \cdot B]\psi = 0$. We notice that $\sigma \cdot B \leq |B|$, and it implies that

$$(p-A)^2 - |B| \leq (p-A)^2 - \sigma \cdot B$$

as an operator inequality. It follows from the min-max principle (or the result of Problem 1 in Reed & Simon, IV) that the number of non-positive bound states for the operator $(p-A)^2 - |B|$ is greater than or equal to the number of non-positive bound states for the Pauli operator (in this case that is the number of zero modes we are interested in). Now, we can apply the theorem of CLR (CLR inequality) for the magnetic Schrödinger operators $(p-A)^2 + V$ (see [8], Theorem 2.15), where $V$ is a real-valued function for $x \in \mathbb{R}^3$. Finally, we obtain an upper bound for the number of zero modes for the magnetic potential $A$ as $C \int_{\mathbb{R}^3} |B|^{\frac{3}{2}} \, dx$, where $C$ is a constant, independent of $A$.

Here, CLR stands for Cwikel, Lieb and Rosenbljum, who initiated the Cwikel-Lieb-Rosenbljum theorem for Schrödinger operators $-\Delta + V$ (see [45], Theorem XIII.12). We recall here that $p^2$ is exactly the usual Laplacian $-\Delta$. The best constant $C$ known up to now is 0.116, which was obtained by Lieb for CLR inequality in case of Schrödinger operators. It is not clear whether or not this $C$ is also the best constant for the CLR theorem for the magnetic Schrödinger operator (see [37]), and then for our upper bound here. In section 4.4 of [30] there is another justification for the upper bound we are discussing, which is based on the variational principle and is suggested by Loss and Yau in [29].

## 1.3 Some mathematical problems related to zero modes

As discussed in the previous section, the zero mode problem first arose from a problem in physics. Now we will see why we need to consider the existence of zero modes from a purely mathematical point of view. Let us consider the Pauli operator as the square of the Weyl-Dirac operator above. This operator looks like the Laplacian $-\Delta$ at least in the way that it is the square of the another ($-\Delta = p^2$). So, a natural question is whether we have a similar version of CLR inequality for the Pauli operator. That means whether the inequality

$$\#\{\text{eigenvalues } \lambda \text{ of } \mathcal{P}_A + V \text{ such that } \lambda < 0\} \leq C \|V\|_p^\alpha \tag{1.8}$$

holds for some $p$ with $1 \leq p \leq \infty$ and some positive $\alpha$, where the constant $C$ is independent of $V$. Similar questions are also raised as to whether we can obtain inequalities of the Sobolev and Hardy type for the Pauli operator (see [11]).

The existence of the zero modes obviously shows the negative answers for the latter questions. For the former or the question about the inequality of CLR type it needs some arguments as we will see below; the result is that we have a clear answer for this question that we cannot get such an estimate. Truly, we assume conversely that (1.8) held. Then, we choose a real-valued $V < 0$ such that $\|V\|_p$ is small enough that can make the right-hand side of (1.8) less than 1. So, for such $V$, the inequality (1.8) would tell us that there are no negative eigenvalues for $\mathcal{P}_A + V$ or this operator would be positive. However, we remark that if there is a zero mode $\psi$ we also have $\mathcal{P}_A\psi = 0$. It would follow that

$$\langle(\mathcal{P}_A + V)\psi,\ \psi\rangle = \int V|\psi|^2\,dx < 0$$

or the operator $\mathcal{P}_A + V$ is not positive. This would be a contradiction. Thus the existence of at least one zero mode shows that the answers for the similar versions of the CLR, Sobolev or Hardy inequality in case of the Pauli operator are negative!

In [58] T. Weidl obtained a formula for the number of negative eigenvalues of the operator $\mathcal{P}_A - \lambda V$ in two dimensions if $V \neq 0$ is non-negative and sufficiently regular for sufficiently small $\lambda > 0$. It is related to the total flux of the magnetic field which we will be concerned with in the next section. Recently, Frank *et al.* in [28] have obtained similar results for a class of magnetic fields and another class of $V'$s. They showed the asymptotic behaviour for the $j$-'additional' negative eigenvalue as $\lambda \to 0$ as well. As far as we know there is no similar result in three dimensions. Although in three dimensions we can estimate the number of zero modes, there are no general results about the exact number of zero modes for such a magnetic field like the Aharonov-Casher theorem in two dimensions (see Theorem 1.4.1 in the next section).

## 1.4 The zero mode problem in two dimensions

Contrary to the three dimensional case, for a given magnetic field we can determine exactly the number of zero modes for the *Dirac operator*

$$\sigma \cdot (p - A) := \sigma_1 \cdot (p_1 - A_1) + \sigma_2 \cdot (p_2 - A_2) \tag{1.9}$$

in two dimensions. This well-known result was initiated in [7] by Aharonov and Casher in 1979. Now their result is widely known as *the Aharonov-Casher theorem.* We will mention this result below in a simple case with a compactly-supported magnetic field; a wider class of magnetic fields for which a similar conclusion still holds can be found in [40].

In two dimensions a vector potential or magnetic potential $A = (A_1, A_2)$ is used, with real-valued components $A_1$ and $A_2$ depending on $(x_1, x_2) \in \mathbb{R}^2$. The corresponding magnetic field $B$ is then the simple scalar function given by

$$B = \partial_1 A_2 - \partial_2 A_1. \tag{1.10}$$

Then, we have the following well-known result.

**Theorem 1.4.1.** (**Aharonov-Casher theorem**) *Suppose the magnetic field $B$ is bounded and has compact support in two dimensions, and $A$ is a vector potential associated with $B$. Denote by $F$ the total flux of $B$*

$$F := \frac{1}{2\pi}\int_{\mathbb{R}^2} B(x)dx, \tag{1.11}$$

*and by $\lfloor x \rfloor$ the biggest integer which is strictly smaller than $x$ for $x > 0$ and $\lfloor 0 \rfloor = 0$. Then, the number of zero modes for the Dirac operator $\sigma \cdot (p - A)$ is $\lfloor |F| \rfloor$.*

We can summarise the proof of that result (see [7] or [57] ) as follows; for

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|B(y)dy,$$

we notice that $\Delta\phi(x) = B(x)$ since the Green's function of the Laplacian on $\mathbb{R}^2$ is $\frac{1}{2\pi}\log|x-y|$. Next, we see that

$$\phi(x) - F\log|x| = O(\frac{1}{|x|}), \text{ as } |x| \to \infty.$$

Our Dirac operator has a *gauge invariance* property as well (recall (1.4) in case of the Weyl-Dirac operator). Suppose both magnetic potentials $A = (A_1, A_2)$ and $A' = (A'_1, A'_2)$ have the same magnetic field $B$; that is $\partial_1 A_2 - \partial_2 A_1 = \partial_1 A'_2 - \partial_2 A'_1$. Then, there exits a smooth scalar valued function $\lambda$ such that $A' - A = \nabla\lambda$. Now we can check that

$$e^{i\lambda}(p-A)e^{-i\lambda} = p - A'.$$

It follows that

$$e^{i\lambda}\Big(\sigma\cdot(p-A)\Big)e^{-i\lambda} = \sigma\cdot(p-A').$$

The map $e^{i\lambda}$ is unitary and therefore the spectrum of the Dirac operators $\sigma\cdot(p-A)$ and $\sigma\cdot(p-A')$ are the same. It follows that they have the same zero modes.

Therefore, we can choose the vector potential $A$ as $(-\partial_2\phi,\ \partial_1\phi)$ and now we need to find the number of independent square integrable solutions $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ for the following equation

$$\sigma\cdot(p-A)\psi = 0.$$

That equation is equivalent to

$$\begin{pmatrix} 0 & (p_1 - A_1) - i(p_2 - A_2) \\ (p_1 - A_1) + i(p_2 - A_2) & 0 \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0,$$

or

$$(\partial_1 - i\partial_2)(e^{\phi}\psi_2) = 0 \quad \text{and} \quad (\partial_1 + i\partial_2)(e^{-\phi}\psi_1) = 0.$$

It follows from the Cauchy-Riemann equations that the function $f_2 = e^{\phi}\psi_2$ is entire analytic in $\bar{z} = x_1 - ix_2$ and $f_1 = e^{-\phi}\psi_1$ is entire analytic in $z = x_1 + ix_2$. For large $|z| = |x_1 + ix_2|$ we see that

$$|e^{\phi}\psi_2| \approx |x|^F |\psi_2| \quad \text{and} \quad |e^{-\phi}\psi_1| \approx |x|^{-F}|\psi_1|.$$

Now, assume that $F > 0$ so the entire analytic function $f_1$ is square integrable (recall that $\psi_1$ is square integrable). It follows that it must be 0, so we have $\psi_1 = 0$. Also we need $\psi_2 = f_2 e^{-\phi}$ to be square integrable. It follows that the entire analytic function $f_2$ must increase no faster than a polynomial in $\bar{z}$ of degree less than $F - 1$. Since there are just $\lfloor F \rfloor$ linearly independent polynomials of this type $(1, \bar{z}, \bar{z}^2, \ldots, \bar{z}^{\lfloor F \rfloor - 1})$, then we can obtain exactly $\lfloor F \rfloor$ zero modes for the Dirac operator $\sigma \cdot (p - A)$ as the theorem claims. The case of $F \leq 0$ can be treated similarly.

In [40] Miller obtained a similar result to that above with a wider class of magnetic fields $B$, that is for bounded magnetic fields $B$ such that $\int_{\mathbb{R}^2} |B(x)|\,|\log|x||dx < \infty$. He also showed that if the flux $F \neq 0$ is an integer there will be either $\lfloor |F| \rfloor$ or $\lfloor |F| \rfloor - 1$ zero modes (of course always $\lfloor |F| \rfloor - 1$ if $B$ is compactly supported). Erdös and Vougalter in [25] remarked that the condition of boundedness for magnetic fields $B$ can be replaced by the weaker condition in which $B \in \mathcal{K}(\mathbb{R}^2)$- Kato class; that means $B$ satisfies the following condition

$$\lim_{r \downarrow 0} \sup_{x} \int_{|x-y| \leq r} \log|x-y|^{-1} |B(y)| dy = 0.$$

Furthermore, they gave an explicit example in which the Aharonov-Casher Theorem does not hold for continuous bounded magnetic fields satisfying only that $\int_{\mathbb{R}^2} |B| dx < \infty$. However the main result in [25] is the Aharonov-Casher theorem (more or less) still holds for a big class of 'reasonable' magnetic fields which are measures with bounded total variation. Recently, in [48] the case of magnetic fields with infinite flux has been investigated.

There is another version of the Aharonov-Casher theorem, but for two dimensional compact manifolds like the unit ball $\mathbb{S}^2$ of $\mathbb{R}^3$. We can see it in [17] or more details in [24]. We will also mention it later in this thesis with our proof for this result (see Theorem 3.6.1).

## 1.5 The Loss-Yau zero mode

Loss and Yau obtained the first zero mode in [39] by a 'reverse' construction; that means they chose the zero mode first, then they constructed the vector potential $A$, and lastly the corresponding magnetic field $B$. They started this way of finding examples of zero modes by supposing they had a spinor $\psi$ and a scalar valued $\lambda(x)$ which satisfied

$$(\sigma \cdot p)\psi(x) = \lambda(x)\psi(x), \tag{1.12}$$

with $\langle\psi,\ \psi\rangle(x) \neq 0$ for all $x$. Then, they put

$$A(x) = \lambda(x)\frac{\langle\psi,\ \sigma\psi\rangle}{\langle\psi,\ \psi\rangle}(x) \tag{1.13}$$

By direct calculation we can see that for $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, the matrix $\sigma\cdot\dfrac{\langle\psi,\ \sigma\psi\rangle}{\langle\psi,\ \psi\rangle}$ becomes

$$\frac{1}{|\psi_1|^2+|\psi_2|^2}\begin{pmatrix}|\psi_1|^2-|\psi_2|^2 & 2\psi_1\bar{\psi}_2\\ 2\bar{\psi}_1\psi_2 & -|\psi_1|^2+|\psi_2|^2\end{pmatrix}.$$

Now we can check that

$$\sigma\cdot\frac{\langle\psi,\ \sigma\psi\rangle}{\langle\psi,\ \psi\rangle}\psi(x) = \psi(x). \tag{1.14}$$

We notice that formula (1.14) can be verified by another way. Truly, we always have

$$(\sigma\cdot a)\cdot(\sigma\cdot b) = a\cdot b + i\sigma\cdot(a\times b).$$

Then, it follows that for any normalised spinor $\chi$

$$[\sigma\cdot\langle\chi,\ \sigma\chi\rangle]^2 = \langle\chi,\ \sigma\chi\rangle^2\mathbb{I}_2 = \mathbb{I}_2.$$

It follows that the eigenvalues of the matrix $\sigma\cdot\langle\chi,\ \sigma\chi\rangle$ are $\pm1$. On the other hand $\langle\chi,\ \sigma\cdot\langle\chi,\ \sigma\chi\rangle\chi\rangle = \langle\chi,\ \sigma\chi\rangle^2 = 1$. This shows that $\chi$ is the eigenvector for the matrix $\sigma\cdot\langle\chi,\ \sigma\chi\rangle$ with 1 as its eigenvalue. Taking $\chi = \dfrac{\psi}{\langle\psi,\psi\rangle^{\frac{1}{2}}}$ gives (1.14).

Finally, (1.12) and (1.14) give us (1.7); that is $\psi$ is a zero mode for the magnetic potential $A$.

Loss and Yau gave a specific solution to (1.12); for a constant spinor $\phi_0$ with the unit length we can directly check that

$$(\sigma\cdot p)\ \psi_{LY} = \frac{3}{1+|x|^2}\psi_{LY},\ \text{ where } \psi_{LY}(x) = \frac{1+i\sigma\cdot x}{(1+|x|^2)^{\frac{3}{2}}}\phi_0.$$

Then, for $w = \langle\phi_0,\ \sigma\phi_0\rangle$ it follows from the formula (1.13) that the corresponding vector potential for the Loss-Yau zero mode $\psi_{LY}$ is

$$\begin{aligned}A_{LY}(x) &= \frac{3}{1+|x|^2}\frac{\langle\psi,\ \sigma\psi\rangle}{\langle\psi,\ \psi\rangle}\\ &= \frac{3}{(1+|x|^2)^2}[(1-x^2)w + 2(w\cdot x)x + 2w\times x].\end{aligned}$$

The formula for the corresponding magnetic field for $\psi_{LY}(x)$ is

$$\begin{aligned}B_{LY}(x) &= \text{curl } A_{LY}\\ &= \frac{12}{(1+|x|^2)^3}[(1-x^2)w + 2(w\cdot x)x + 2w\times x].\end{aligned}$$

If we choose $\phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we will have the Loss-Yau zero mode

$$\psi_{LY} = \frac{1}{(1+|x|^2)^{\frac{3}{2}}} \begin{pmatrix} 1 + ix_3 \\ ix_1 - x_2 \end{pmatrix} \tag{1.15}$$

with the corresponding magnetic potential

$$A_{LY}(x) = \frac{3}{(1+|x|^2)^2} \begin{pmatrix} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}^T, \tag{1.16}$$

and the magnetic field

$$B_{LY}(x) = \frac{12}{(1+|x|^2)^3} \begin{pmatrix} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}^T, \tag{1.17}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

## 1.6 The zero modes of Elton

In fact Loss and Yau in [39] propose two methods for constructing zero modes. However, they give only the specific zero mode for the first method, which we met in the previous section. Motivated by their second method, Elton in [21] has constructed two other specific zero modes. The key point of this method is to skillfully choose the spinor $\psi$ such that $\langle \psi,\ \psi \rangle \neq 0,$ for all $x \in \mathbb{R}^3$ and $\mathrm{div}\langle \psi,\ \sigma\psi \rangle = 0$. Then, Loss and Yau verify that such a spinor will be the zero mode with the corresponding vector potential and magnetic field

$$A(x) = \frac{1}{\langle \psi,\ \psi \rangle} \left( \frac{1}{2} \mathrm{curl} \langle \psi,\ \sigma\psi \rangle + \mathrm{Im} \langle \psi,\ \nabla\psi \rangle \right),$$

$$B(x) = \frac{1}{2|U|^3} \left[ \sum_j U_j (\mathrm{curl} U \times \nabla U_j) - |U|^2 \Delta U + \frac{1}{2} \sum_{ijk} \varepsilon_{ijk} U_i \nabla U_j \times \nabla U_k \right],$$

where $U = \langle \psi,\ \sigma\psi \rangle$.

Elton obtains the first specific example of a zero mode by the method above by taking a function $g$ satisfying the following conditions

- $g:\ \mathbb{R} \longrightarrow \mathbb{R}$ is smooth compactly supported and non-negative
- $g(t) = (4 - t^2)^{\frac{1}{2}}$ for $t \in [-1/2,\ 1/2]$
- $\mathrm{supp}\ (g) \subset [-1,\ 1]$
- $\pm g'(t) \leq 0$ for $\pm t \geq 0$.

Then, he chooses a real-valued function $f$ on $\mathbb{R}^+$, the set of non-negative real numbers by

$$f(r) = r^{-3}\left(-\int_0^r t^4 (g^2)'(t)\ dt\right)^{\frac{1}{2}},\ \text{for all } r > 0.$$

It turns out that the spinor

$$\psi_E^1(x) = f(r)\begin{pmatrix} x_3 \\ x_1 + ix_2 \end{pmatrix} + \begin{pmatrix} ig(r) \\ 0 \end{pmatrix}$$

is a smooth zero mode with the corresponding vector potential of compact support in the unit ball.

The second zero mode, which Elton constructs in [21] has the the corresponding magnetic field, which can be written as a perturbation of a constant magnetic field. Moreover, the perturbation is smooth, supported on $\{|x_3| \leq 1\}$ and decays like $o(|x|^{-1})$ as $|x| \to \infty$. That zero mode is

$$\psi_E^2(x) = u\begin{pmatrix} (\rho^2 - 1)(-h + ig) \\ 2(x_1 + ix_2) \end{pmatrix},$$

where $u = \exp(-\dfrac{\rho^2}{2} - k);\ \rho = x_1^2 + x_2^2,\ k$ is a smooth function on $\mathbb{R}^3$ depending on $x_3$ and constant in $x_1$ and $x_2$; $g$ above, and $h$ defined as

$$h(t) = \begin{cases} -[4 - g^2(t)]^{\frac{1}{2}} \text{ for } t \leq 0 \\ [4 - g^2(t)]^{\frac{1}{2}} \text{ for } t \geq 0. \end{cases}$$

## 1.7 Some results by Adam, Muratori and Nash

Adam *et al.* in a series of papers (see [2], [3], [4], [5]) developed the results of Loss and Yau in [39] in some different directions. We notice that Adam *et al.* use the term the *Abelian Dirac operators* in three dimensions for Weyl-Dirac operators.

In [2] Adam *et al.* firstly constructed some new examples of zero modes by studying the Loss and Yau zero mode $\psi_{LY}$. They wrote $\psi_{LY} = (1 + r^2)^{-\frac{3}{2}}(1 + \mathbf{X})\phi_0$, where $\mathbf{X} = i\sigma \cdot x,\ r = |x|$, and $\phi_0$ is the constant unit spinor. Then, they tried to find other zero modes of the more general type

$$\psi^{(l)} = (1 + r^2)^{-(\frac{3}{2}+l)}\big[\big(1 + \sum_{n=1}^{l} a_n r^{2n}\big)\mathbb{I}_2 + \sum_{n=0}^{l} b_n r^{2n}\mathbf{X}\big]\phi_0, \tag{1.18}$$

where $\mathbb{I}_2$ is the $2 \times 2$ identity matrix, by using (1.12) and $\mathbf{X}^2 = -r^2\mathbb{I}_2,\ x_j\partial_j\mathbf{X} = \mathbf{X}$ and $i\sigma_j\partial_j\mathbf{X} = -3 \cdot \mathbb{I}_2$. Adam *et al.* also showed that for each integer $l \geq 0$ we always get one zero mode of the type (1.18) above!

To express $\psi_{LY}$ as $g(r)U\psi_0$, where $g(r) = \dfrac{1}{1+r^2}$ plays the role of a scalar function; and $U = \dfrac{1}{(1+r^2)^{1/2}}(1+\mathbf{X})$, an $SU(2)$ matrix (a complex-valued unitary $2\times 2$-matrix with determinant 1), Adam *et al.* found some other zero modes with the general form $\psi^{(n)} = gU^n\psi_0$, where $g$ (we have to look for) is dependent of only $r$. The key technique is still to use the remark of Loss and Yau (1.12) and calculation.

There is another way of obtaining some more zero modes by replacing $\phi_0$ with another class of spinors (rather than constant unit spinors). More specifically, Adam *et al.* used the class

$$\Phi_{l,m} = \begin{pmatrix} \sqrt{l+m+1/2}\; Y_{l,m-1/2} \\ -\sqrt{l-m+1/2}\; Y_{l,m+1/2} \end{pmatrix},$$

where $m \in [-l-1/2,\ l+1/2]$ and $Y_{l,m\pm 1/2}$ are spherical harmonics (see [31], p. 38-39 for these spherical harmonics). Then, some zero modes can be found by the ansatz

$$\psi_{l,m} = r^l(1+r^2)^{-l-3/2}(1+\mathbf{X})\Phi_{l,m}.$$

See [2] for details.

Motivated by the zero mode proposed by Elton, Adam *et al.* in [3] constructed a whole class of magnetic potentials with compact support (and therefore magnetic fields with compact support) so that the corresponding Pauli operators (then, the Weyl-Dirac operators) have zero modes. This is interesting from the physical point of view because magnetic fields with compact support are the ones which can be well treated in the computing lab. To do this they firstly remarked that the spinor $\Psi^0(x) = \dfrac{i}{r^3}\begin{pmatrix} x_3 \\ x_1+ix_2 \end{pmatrix}$ satisfies

$$(\sigma\cdot p)\Psi^0 = 0.$$

This spinor is well-behaved for large $r = |x_1^2+x_2^2+x_3^2|^{\frac{1}{2}}$. So, they constructed zero modes that are equal to $\Psi^0$ outside a ball with radius $r$ and they differ from $\Psi^0$ inside that ball. One example we want to mention here from [3] are the zero modes obtained by the ansatz in [2] above with $g(r) = \exp(-4r^2+3r^4-\dfrac{4}{3}r^6+\dfrac{1}{4}r^8)$ for $r<1$ and $g(r) = \exp(-\frac{25}{12})r^{-2}$ for $r\geq 1$. The magnetic potentials and magnetic fields in these cases have compact support in the unit ball. Please see [3] for more details.

In [4] Adam *et al.* constructed a class of magnetic potentials $A^{(l)}$ (there they called them the *gauge fields*) such that each $A^{(l)}$ will give more than one zero mode. At first they remarked that if the function $\chi$ satisfies $(\sigma\cdot p\chi)(1+i\sigma\cdot x)\phi_0 = 0$, then $\chi^n\psi_{LY}$, $n\in\mathbb{Z}$ also satisfies (1.12) for the same magnetic potential $A_{LY}$. The function $\chi$ were shown as

$$\chi = \frac{2(x_1+ix_2)}{2x_3-i(1-r^2)} = S\,\exp(i\sigma),$$

where

$$S^2 = \frac{4(r^2 - x_3^2)}{4x_3^2 + (1-r^2)^2} \qquad \text{and} \qquad \sigma = \tan^{-1}\frac{x_2}{x_1} + \tan^{-1}\frac{1-r^2}{2x_3}.$$

Adam *et al.* chose

$$\Psi_l := \frac{Y_{l,l+1/2}r^l}{(1+r^2)^{l+3/2}}(1 + i\sigma \cdot x)\phi_0$$

corresponding to the maximal quantum number $m = l + 1/2$. Then, they showed that $\Psi_{n,l} = \chi^{-n}\Psi_l,\ n = 0, 1, \ldots, l$ are zero modes for the scaled magnetic potential

$$A^l = \frac{3+2l}{(1+r^2)^2}\begin{pmatrix} 2x_1x_2 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}^T = \frac{3+2l}{3}A_{LY} \quad \text{(recall (1.16))}.$$

In [5] Adam *et al.* extended the results in [4]. They considered a much wider class of Weyl-Dirac operators which have 'multiple zero modes'. That means for one magnetic potential the dimension of the kernel of the corresponding Weyl-Dirac operator is greater than 1. As far as we know they were the first to give an explicit example of a magnetic potential for which the corresponding Weyl-Dirac operator has (at least) $l + 1$ different zero modes.

## 1.8 Erdös and Solovej's work on zero modes

The work by Erdös and Solovej in [24] published in 2001 gave a new insight into the zero mode problem. They considered this problem from a the geometrical point of view. The last two chapters, especially Chapter 3 of this thesis, will discuss some details about this work. Here we will sketch some key points of Erdös and Solovej's idea.

Denote by $\mathbb{S}^2$ the unit ball in $\mathbb{R}^3$ and by $\mathbb{S}^3$ the unit ball in $\mathbb{R}^4$. The main result in Erdös and Solovej's work in [24] is the construction of a certain class of magnetic fields on $\mathbb{R}^3$ for which the dimension of any corresponding Weyl-Dirac operator can be counted exactly. They obtained this class of magnetic fields by pulling back magnetic fields from $\mathbb{S}^2$ to $\mathbb{R}^3$ (we notice that the magnetic fields on $\mathbb{S}^2$ are actually two-forms on $\mathbb{S}^2$). Erdös and Solovej in fact used $\mathbb{S}^3$ as the *bridge* while pulling back two-forms on $\mathbb{S}^2$. More specifically, firstly they pulled back those two-forms using the Hopf map from $\mathbb{S}^3$ to $\mathbb{S}^2$. Then, they continued to pull back the obtained two-forms on $\mathbb{S}^3$ to get the class of magnetic fields on $\mathbb{R}^3$ we mentioned above using the inverse stereographic projection from $\mathbb{R}^3$ to $\mathbb{S}^3$. It turns out that the Loss-Yau zero mode as well as Adam *et al.*'s multiple zero modes can be obtained from this construction if one starts with certain multiples of $\mathrm{vol}_{\mathbb{S}^2}$, the volume form on $\mathbb{S}^2$.

To show their result Erdös and Solovej constructed Dirac operators with magnetic fields on the Riemannian manifolds $\mathbb{S}^2$ and $\mathbb{S}^3$. Therefore they needed to define the Spin$^c$ structures on compact Riemannian manifolds; these include Spin$^c$ spinor bundles and Spin$^c$ connections. Then, the magnetic field is related to the curvature of the connection. We will discuss more about these in Chapter 3.

## 1.9 Some other results on the zero mode problem

In 2001 Balinsky and Evans showed in [10] that there are only finite values of $t \in [0,\ T]$ such that the kernel of the Pauli operator $\mathcal{P}_{tA}$ is non-trivial. That means that in any compact interval of the positive real line there are only a finite numbers of values of $t$ such that we can get some zero modes from the operator $\mathcal{D}_{tA} = \sigma \cdot (p - tA)$ with scaled potential. Balinsky and Evans also showed that the set of magnetic fields $B$ such that we have no zero modes is "big"; it actually contains an open dense subset of $[L^{\frac{3}{2}}(\mathbb{R}^3)]^3$. These results explain why we are struggling a bit to construct a zero mode. A point in their proofs of these results is to describe the Pauli operator (by Theorem 1.11.4) as the form sum of the operator $\big(p - A\big)^2$ and its "small" perturbation $\sigma \cdot B$.

In 2002 Balinsky and Evans gave another proof (in [12]) for the results (in the view of the zero mode problem) they had obtained in 2001. This time the point in their proof is to write $\sigma\cdot(p-A)$ (in fact it is $+$ in place of $-$ in their paper) as the operator sum between $\sigma \cdot p$ and its "small" perturbation $\sigma \cdot A$. They also use a bound in [18] obtained by I. Daubechies to estimate the number of nonpositive eigenvalues for the operator $|p| - V$ which can be considered as the $|p|$-version of the well-known Cwikel-Lieb-Rosenbljum bound (for the operator $|p|^2 - V$). Furthermore, they obtain an estimate for the dimension of kernel of $\mathcal{D}_A$ :

$$\dim \operatorname{Ker} \mathcal{D}_A \leq C_n \int_{\mathbb{R}^n} |A|^n \, dx, \text{ for } n = 2, 3, \tag{1.19}$$

where the constant $C_n$ independent of $A$. However, in their proof they have used an operator estimate that is actually wrong. We will discuss this issue and show the complete proof for the estimate (1.19) in Theorem 2.2.10.

To study the zero mode problem from another perspective Elton in [22] constructed a class of magnetic potentials called $\mathcal{A}$, which includes all functions $A \in C^0(\mathbb{R}^3,\ \mathbb{R}^3)$ such that $A = o(|x|^{-1})$ as $|x| \to \infty$. This set $\mathcal{A}$ can be equipped with the norm $\|A\| = \|(1 + |x|)A\|_{L^\infty}$, making it a Banach space with $C_0^\infty$ as a dense subset. Elton used this set and proved some results for zero modes with magnetic potentials in this class. Some of those results are that (i) There are finitely many zero modes (maybe none) for each magnetic potential $A \in \mathcal{A}$ (ii) The set of all magnetic potentials in $\mathcal{A}$ which give no zero modes is 'big'; it is actually a dense open subset of $\mathcal{A}$ (this is similar to Balinsky and Evans' result) (iii) For any positive integer number $m$ and any open and non-empty set $\Omega$ there exists at least one $C_0^\infty-$magnetic potential $A$ with its support in $\Omega$ such that the zero mode equation $\sigma \cdot (p - A)\psi = 0$ gives us $m$ independent zero modes. Coming from struggling to construct one zero mode for one magnetic potential we are happy to see from this result that there always exists some magnetic potentials for which the corresponding zero mode equation gives us *any given* number of independent zero modes! (iv) The sets of magnetic potentials which give $m$ independent zero modes are smooth sub-manifolds of $\mathcal{A}$ with corresponding co-dimensions of $m^2$ if $m = 1$ or $m = 2$; and are contained in a smooth sub-manifold with co-dimension of $2m - 1$ if $m \geq 3$. This conclusion shows us that in spite of sparsity zero modes are actually abundant!

While most of mathematicians have been focusing on studying the relationship between the existence of zero modes and the corresponding magnetic potentials or magnetic fields, Saitō and Umeda studied the relationship between the properties of zero modes and the corresponding magnetic potentials or magnetic fields; see [49] and [50]. In those papers Saitō and Umeda were mainly interested in the massless Dirac operators $\alpha \cdot p + Q(x)$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of $4 \times 4$ Dirac matrices

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \; j = 1, 2, 3$$

with the $2 \times 2$ zero matrix $0_2$; $Q(x) = (q_{jk}(x))$ is a $4 \times 4$ Hermitian matrix-valued functions. Extending the concept for the Weyl-Dirac operator they defined the *zero modes* for the the massless Dirac operator as a non-zero bispinor $f \in [H^1(\mathbb{R}^3)]^4$ such that $(\alpha \cdot p + Q)f = 0$. In fact in some special cases the massless Dirac operator will become the operator $\alpha \cdot (p - A)$ or

$$\begin{pmatrix} 0_2 & \sigma \cdot (p - A) \\ \sigma \cdot (p - A) & 0_2 \end{pmatrix}, \text{ where } 0_2 \text{ is the } 2 \times 2 \text{ zero-matrix.}$$

Then, Saitō and Umeda can obtain some properties of zero modes, not only for the massless Dirac operators but also for the Weyl-Dirac operators. They assume that the Hermitian matrix-valued functions $q_{jk}(x),\; j, k = 1, \ldots, 4$ satisfy the conditions

$$|q_{jk}(x)| \leq C(1 + |x|^2)^{-\rho/2}, \text{ for some } \rho > 1 \text{ and constant } C \text{ positive.}$$

If this matrix-potential gives the zero mode $f(x)$ for the massless Dirac operator, then
(i) $|f(x)| \leq c_f(1 + |x^2|)$ for all $x \in \mathbb{R}^3$
(ii) the function $f$ is continuous on $\mathbb{R}^3$ and for any $\omega \in \mathbb{S}^2$ (the unit ball in $\mathbb{R}^3$)

$$\lim_{r \to \infty} r^2 f(r\omega) = -\frac{i}{4\pi}(\alpha \cdot \omega) \int_{\mathbb{R}^3} Q(y) f(y)\, dy, \text{ uniformly for } \omega.$$

For the Weyl-Dirac operator we get a more elaborate formula for the zero mode $\psi(x)$

$$\lim_{r \to \infty} r^2 \psi(r\omega) = \frac{i}{4\pi} \int_{\mathbb{R}^3} \{(\omega \cdot A(y))\mathbb{I}_2 + i\sigma \cdot (\omega \times A(y))\}\psi(y)\, dy, \text{ uniformly for } \omega.$$

## 1.10 About the thesis

We will be concerned with the zero mode problem for the Weyl-Dirac operator as well. More specifically, let $\mathcal{D}_{tA}$ be the Weyl-Dirac operator with magnetic potential $tA$, where we think of $A$ as fixed and $t > 0$ as a scale. We are principally interested in the quantity $n_A(T)$ and related ones, where

$$n_A(T) := \sum_{0 \le t \le T} \dim \operatorname{Ker} \mathcal{D}_{tA}.$$

We firstly show that (see Theorem 2.3.6) for $|A| \in L^3(\mathbb{R}^3)$ we have

$$n_A(T) \le CT^3 \|A\|_{L^3}^3. \tag{1.20}$$

Indeed, it is obvious that the estimate (1.20) is stronger than the one by Balinsky and Evans in [12]. Their estimate works for the number of zero modes of $\mathcal{D}_{tA}$ for *each* $t \in [0, T]$. Ours works for the *total* of zero modes of $\mathcal{D}_{tA}$ which we may obtain for *all* $t$, $t \in [0, T]$. A similar estimate for $n_A(T)$ in two dimensions is also obtained with some additional changes in the proof (see Theorem 2.4.1).

Our proof for the estimate for (1.20) may work for some other cases and we then obtain estimates on $n_Q(T)$ with

$$n_Q(T) := \sum_{0 \le t \le T} \dim \operatorname{Ker} \mathsf{T}_t,$$

where $\mathsf{T}_t$ is the massless Dirac operator with scaled potential, $\mathsf{T}_t := \alpha \cdot p + tQ$ (see Theorem 2.5.1); and estimates on $n_A(T, \pm m)$ with

$$n_A(T, \pm m) := \sum_{0 \le t \le T} \dim E_{\pm m}(\mathsf{H}_{tA}),$$

where $E_{\pm m}(\mathsf{H}_{tA})$ is the eigenspace of the Dirac operator with positive mass at the threshold energy $\pm m$ (see Theorem 2.6.1). These estimates are also better versions of recent ones in [13] and in [51].

We will spend almost a chapter (Chapter 3) discussing the paper [24] of Erdös and Solovej and our remaining main results in this thesis (in Chapters 3 and 4) build on ideas in [24]. In Chapter 3 we show explicitly the construction of the Dirac operator on the sphere $\mathbb{S}^2$. The spectrum of the Dirac operator in $\mathbb{S}^2$ in a special case (corresponding to a constant field) is obtained explicitly (see Theorem 3.5.2). Furthermore, another proof for the Aharonov-Casher theorem on $\mathbb{S}^2$ is justified as a byproduct of the work on $\mathbb{S}^2$ (see Theorem 3.6.1). More specifically, in Chapter 3 we will firstly collect some general concepts in differential geometry used in the thesis. Then, we introduce and construct the Spin$^c$ structures on $\mathbb{S}^2$. The second part deals with the Dirac operator defined on $\mathbb{S}^2$ based on structures in the first part. Some properties and spectral details for the Dirac operators defined on $\mathbb{S}^2$ will be obtained as well. We also study the Laplacian built on sections of a line bundle, and its relation to the Dirac operators. This allows us to determine the spectrum explicitly for a class of Dirac operators on $\mathbb{S}^2$ with specific magnetic fields $\frac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$. In this case we show not only all eigenvalues, but also their multiplicity. This is one of main results in Chapter 3.

In the final chapter, Chapter 4, we first obtain an explicit formula for $n_{B_0}(T)$ for the Weyl-Dirac operator with a certain magnetic field $B_0$ (see Theorem 4.3.1). Here, the magnetic field $B_0$ is the result of pullbacks from the "constant" volume form of $\mathbb{S}^2$, and is in fact, a scaled version of $B_{LY}$, the magnetic field corresponding to the first zero mode constructed by Loss and Yau. The final main result in the thesis (see Theorem 4.5.1) gives a lower bound for $n_B(T)$ for the Weyl-Dirac operator with scaled magnetic field $tB$, where $B$ is the result of pullbacks from an arbitrary two-form on $\mathbb{S}^2$. This bound is indeed a strengthened version of an estimate in [24].

## 1.11 Notation and Background

Although in Chapter 2 and Chapter 3 the mathematical concepts used in these chapters will be mentioned we would like to list here the background and notation used throughout the thesis.

### 1.11.1 Notation

First, as usual $\mathbb{N}$ is the set of natural numbers $\{1, 2, \dots\}$, while $\mathbb{N}_0$ is the set of natural numbers and 0; that is $\{0, 1, 2, \dots\}$. Notation $\mathbb{Z}$ is for the set of integer numbers $\{0, \pm 1, \pm 2, \dots\}$. The set of real numbers is denoted by $\mathbb{R}$ and $\mathbb{C}$ is the set of complex numbers. The imaginary unit is denoted by $i$. Notation $\mathbb{I}_2$ is for $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbb{I}_4$ is for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $0_2$ is for $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The notation Ker is for the kernel (of an operator) and dim is for the dimension (of a vector space). We also use Spec for the spectrum (of an operator), Dom for the domain while Ran for the range (of an operator). Furthermore, we use # as the number of elements (of a set). The notation $\rightharpoonup$ is for the weak convergence and $\hookrightarrow$ is for the continuous embedding (between spaces).

We use the notation $C_0^\infty$ for the set of smooth functions with compact support, and $C^\infty$ for the set of smooth functions. In addition, $C^0$ denotes the space of continuous functions. If we use $\|.\|_H$ we mean the norm defined on the Banach space $H$.

The notation $\partial^\alpha f$ means $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}_0$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For $1 \leq q < \infty$,

$$L^q(\mathbb{R}^n) := \left\{ f : \int_{\mathbb{R}^n} |f(x)|^q < \infty \right\}$$

with norm for $f \in L^q(\mathbb{R}^n)$ given by

$$\|f\|_{L^q} := \left( \int_{\mathbb{R}^n} |f(x)|^q \right)^{\frac{1}{q}}.$$

We also set

$$L^\infty(\mathbb{R}^n) = \left\{ f : \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty \right\}$$

with norm for $f \in L^\infty$ given by

$$\|f\|_{L^\infty} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Denote by $\mathsf{S}$ (for each positive integer $n$) the Schwartz class: the set of all smooth functions $f$ (defined on $\mathbb{R}^n$) such that

$$\sup_x |x^\beta \partial^\alpha f(x)| < \infty, \text{ for all } \alpha,\ \beta.$$

We observe that $\mathsf{S}$ is a dense subspace for $L^2$, and $C_0^\infty \subset \mathsf{S}$.

Given the Schwartz class $\mathsf{S}$ (for functions defined on $\mathbb{R}^n$) and $f \in \mathsf{S}$ we define

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(x) e^{-i\omega x} dx.$$

We may check that $(\mathcal{F}\partial^\alpha f)(\omega) = (i\omega)^\alpha (\mathcal{F}f)(\omega)$ and $\mathcal{F}(x^\alpha f)(\omega) = i^{|\alpha|} \partial^\alpha \mathcal{F}(\omega)$ for every $f \in \mathsf{S}$ and multi-index $\alpha$. Obviously, this is well defined and in fact it is a bijection from $\mathsf{S}$ to itself with its inverse as

$$(\mathcal{F}^{-1} f)(x) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} (\mathcal{F}f)(\omega) e^{ix\omega} d\omega.$$

There is a fact that $\langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle$ for all $f, g \in \mathsf{S}$. Then, it follows from the density of $\mathsf{S}$ in $L^2(\mathbb{R}^n)$ that we may extend $\mathcal{F}$ to all functions in $L^2(\mathbb{R}^n)$. We will obtain *the Fourier transform* for functions in $L^2(\mathbb{R}^n)$. We sometimes use $\hat{f}$ instead of $\mathcal{F}f$ for convenience. We summarise some key properties of the Fourier transform in the following result

**Theorem 1.11.1.** (See [43], for instance) *The Fourier transform $\mathcal{F}$ is a unitary operator $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. That means $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I$, the identity.*

We use $H^k(\mathbb{R}^n),\ k \in \mathbb{Z},\ k \geq 0$ to denote the Sobolev space

$$\{f:\ \partial^\alpha f \in L^2(\mathbb{R}^n),\ |\alpha| \leq k\};$$

with norm for $f \in H^k(\mathbb{R}^n)$ given by

$$\|f\|_{H^k} = \left(\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^2}^2\right)^{\frac{1}{2}}.$$

When we want to emphasise that we may use a more general class of Sobolev spaces, we take the Sobolev space $H^s$ as the space

$$\left\{f \in \mathsf{S}':\ \int (1+|\omega|^2)^s |\mathcal{F}f(\omega)|^2 d\omega < \infty\right\},$$

where $\mathcal{F}f$ is the Fourier transform of $f$. The norm for $f \in H^s$ is defined as

$$\|f\|_{H^s} = \left(\int (1+|\omega|^2)^s |\mathcal{F}f(\omega)|^2 d\omega\right)^{\frac{1}{2}}.$$

### 1.11.2 Operators

We briefly summarise some concepts on linear operators used in the thesis; material here comes from [42], [15], [43], [45], [20] and [36].

Let $\mathcal{T} \in \mathcal{B}(H)$ denote the space of bounded linear operators acting on a Hilbert space $H$. For a compact operator $\mathcal{T} \in \mathcal{B}(H)$ we set $|\mathcal{T}| = \sqrt{\mathcal{T}^*\mathcal{T}}$ as usual. The eigenvalues of the operator $|\mathcal{T}|$ are called the *singular values* of $\mathcal{T}$. By the *multiplicity* of an eigenvalue $\lambda$ for the operator $\mathcal{T}$ we mean the *geometric multiplicity* of $\lambda$, or the dimension of the subspace $\{x \in H:\ (\mathcal{T}-\lambda)x = 0\}$. In general this multiplicity may be slightly different from the *algebraic multiplicity* of $\lambda$ for $\mathcal{T}$, which is the dimension of the subspace $\{x \in H:\ (\mathcal{T}-\lambda)^n x = 0 \quad \text{for some positive interger } n\}$. However for self-adjoint operators these two concepts of multiplicity are the same.

If $\mathcal{T} \in \mathcal{B}(H)$ is compact we may enumerate its eigenvalues (including multiplicity) as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0.$$

If in addition $\mathcal{T}$ is self-adjoint its eigenvalues are real so we can enumerate them (including multiplicity) as

$$\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq 0 \geq \cdots \geq \lambda_2^- \geq \lambda_1^-.$$

The singular values of $\mathcal{T}$ will be enumerated (including multiplicity) as

$$\mu_1 \geq \mu_2 \geq \cdots \geq 0.$$

When we need to emphasise the dependence on $\mathcal{T}$ we also write $\lambda_j(\mathcal{T}), \lambda_j^{\pm}(\mathcal{T})$ and $\mu_j(\mathcal{T})$.

Take a compact operator $\mathcal{T} \in \mathcal{B}(H)$. If

$$\left(\sum_{j\geq 1} \mu_j^q\right)^{\frac{1}{q}} < \infty$$

for some $1 \leq q < \infty$, we say that $\mathcal{T}$ belongs to the *Schatten class* $\mathbf{S}_q$. In fact one may show that $\mathbf{S}_q$ along with

$$\|\mathcal{T}\|_{\mathbf{S}_q} := \left(\sum_{j\geq 1} \mu_j^q\right)^{\frac{1}{q}}$$

is a Banach space. We will also use the notation $\mathbf{S}_q$ to refer to this Banach space. We observe that $\mathbf{S}_{q_1} \subseteq \mathbf{S}_{q_2}$ if $q_1 < q_2$. One often denotes by $\mathbf{S}_\infty$ the class of compact operators. In case $q = 2$ the Schatten space $\mathbf{S}_2$ is in fact a Hilbert space and elements in $\mathbf{S}_2$ are called *Hilbert-Schmidt operators*. Similarly, elements in $\mathbf{S}_1$ are called *trace class operators*. See [42] for more details.

By a *positive operator* $\mathcal{T} \in \mathcal{B}(H)$ we mean that $\mathcal{T}$ is non-negative; that is $\mathcal{T}$ is self-adjoint and $\langle \mathcal{T}x,\ x\rangle \geq 0$, for all $x \in H$, where $\langle\ \cdot, \cdot\ \rangle$ denotes the inner product in $H$. Then, we may write $\mathcal{T} \geq 0$. We observe that a self-adjoint operator is positive if and only if its spectrum is non-negative. In addition, we will write $\mathcal{T}_1 \geq \mathcal{T}_2$ or $\mathcal{T}_2 \leq \mathcal{T}_1$ for $\mathcal{T}_1,\ \mathcal{T}_2 \in \mathcal{B}(H)$ if $\mathcal{T}_1 - \mathcal{T}_2 \geq 0$.

In addition to bounded operators we will also need to consider unbounded operators. If a linear operator $\mathcal{T}:\ H \longrightarrow H$ is bounded, then there is a constant $C \geq 0$ such that

$$\|\mathcal{T}x\| \leq C\|x\|, \quad \text{for all } x \in H.$$

An *unbounded operator* $\mathcal{T}$ is a linear map defined on a domain $\mathrm{Dom}(\mathcal{T}) \subseteq H$ such that there is a sequence $\{x_j\},\ x_j \in \mathrm{Dom}(\mathcal{T}),\ \|x_j\| = 1,\ j = 1, 2, \ldots$ and $\|\mathcal{T}x_j\| \longrightarrow \infty$ as $j \to \infty$. Normally, $\mathrm{Dom}(\mathcal{T})$ is a dense linear subspace of $H$. Now we will give an example. Let $\mathcal{T}$ be the operator defined on the subspace $\mathsf{S}$ of $L^2(\mathbb{R})$, such that $\mathcal{T}f(x) = -f''(x) + x^2 f(x)$ for $f \in \mathsf{S}$. We may show that if

$$f_j = (2^j j!)^{-\frac{1}{2}}(-1)^j \pi^{-\frac{1}{4}} e^{\frac{1}{2}x^2} \frac{d^j}{dx^j}(e^{-x^2}),$$

then $f_j \in \mathsf{S},\ \|f_j\| = 1$ and $\mathcal{T}f_j = 2j + 1$ for $j = 1, 2, \ldots$ so $\|\mathcal{T}f_j\| = 2j + 1 \to \infty$ as $j$ goes to $\infty$. (In fact $\{f_j\}$ is an orthonormal base for $L^2(\mathbb{R})$). We call $f_j'$s *Hermite functions*. See [42] for details.

Suppose we have an unbounded operator $\mathcal{T}$ defined on $\mathrm{Dom}(\mathcal{T})$. We say $\mathcal{T}$ is *closed* if whenever $x_j \in \mathrm{Dom}(\mathcal{T}),\ x_j \to x$ and $\mathcal{T}x_j \to y$, then $x \in \mathrm{Dom}(\mathcal{T})$ and $\mathcal{T}x = y$. We will call an operator $\mathcal{T}'$ an *extension* of $\mathcal{T}$ if $\mathrm{Dom}(\mathcal{T}) \subseteq \mathrm{Dom}(\mathcal{T}')$ and $\mathcal{T}x = \mathcal{T}'x$ for all $x \in \mathrm{Dom}(\mathcal{T})$. In addition, we will say $\mathcal{T}$ is *closable* if $\mathcal{T}$ has a closed extension. Then, every closable operator has a smallest closed extension, called its *closure*, which is often denoted by $\overline{\mathcal{T}}$. If $\mathcal{T}$ is closed a *core* for $\mathcal{T}$ is a subset of $\mathrm{Dom}(\mathcal{T})$ such that the closure of $\mathcal{T}$ restricted to this set is exactly $\mathcal{T}$.

Next we consider the *adjoint* of an unbounded operator $\mathcal{T}$. We will denote by $\mathrm{Dom}(\mathcal{T}^*)$ the set of $y \in H$ for which there is a $z \in H$ such that for all $x \in \mathrm{Dom}(\mathcal{T})$ we have

$$\langle \mathcal{T}x,\ y\rangle = \langle x,\ z\rangle.$$

For each $y \in \mathrm{Dom}(\mathcal{T}^*)$ we set $\mathcal{T}^*y = z$ and this $\mathcal{T}^*$ is called the *adjoint* of $\mathcal{T}$. It is the case that not all unbounded operators $\mathcal{T}$ have an adjoint (as an unbounded operator). However if $\mathcal{T}$ is closable there always exists $\mathcal{T}^*$. In fact we may show that $\mathcal{T}$ is closable if and only if $\mathrm{Dom}(\mathcal{T}^*)$ is dense in $H$. In that case we have $\overline{\mathcal{T}}^* = \mathcal{T}^*$. There is a difference between symmetric (or Hermitian) and self-adjoint operators for unbounded operators, which is obviously not the case for bounded operators. More specifically, an unbounded operator $\mathcal{T}$ is called *symmetric* if $\mathcal{T}^*$ is an extension of $\mathcal{T}$. We may see that $\mathcal{T}$ is symmetric if

$$\langle \mathcal{T}x,\ y\rangle = \langle x,\ \mathcal{T}y\rangle, \text{ for all } x,\ y \in \mathrm{Dom}(\mathcal{T}).$$

If we want $\mathcal{T}$ to be *self-adjoint*, we need $\mathcal{T}$ not only to be symmetric, but also to satisfy $\mathrm{Dom}(\mathcal{T}^*) = \mathrm{Dom}(\mathcal{T})$. To prove a symmetric operator $\mathcal{T}$ is self-adjoint we need only show that $\mathcal{T}$ is closed and the kernel of $\mathcal{T} \pm i$ is trivial or the range of $\mathcal{T} \pm i$ is exactly $H$.

Suppose $\mathcal{T}$ is symmetric. We may see from above that $\mathcal{T}$ is closable. In case its closure $\overline{\mathcal{T}}$ is self-adjoint, the symmetric operator $\mathcal{T}$ is called *essentially self-adjoint*. In fact we may prove that symmetric operator $\mathcal{T}$ is essentially self-adjoint if it has only one self-adjoint extension. To prove a symmetric operator is essentially self-adjoint we need only show that the kernel of $\mathcal{T}^* \pm i$ is trivial or the range of $\mathcal{T} \pm i$ is dense in $H$. To use all remarks above we may justify the following result, which allows us to consider operator sums.

**Theorem 1.11.2.** (Kato-Rellich theorem (see [43], Theorem X.12)) *Suppose that $\mathcal{T}_1$ is self-adjoint and $\mathcal{T}_2$ is symmetric with $\mathrm{Dom}(\mathcal{T}_1) \subseteq \mathrm{Dom}(\mathcal{T}_2)$. Furthermore, suppose that there exist $a,\ b$ with $a < 1$ such that*

$$\|\mathcal{T}_2 x\| \leq a\|\mathcal{T}_1 x\| + b\|x\|,$$

*for all $x \in \mathrm{Dom}(\mathcal{T}_1)$. Then the operator $\mathcal{T}_1 + \mathcal{T}_2$ is self-adjoint on $\mathrm{Dom}(\mathcal{T}_1)$ and essentially self-adjoint on any core of $\mathcal{T}_1$.*

The operator $\mathcal{T}_2$ in the Kato-Rellich Theorem may be seen as a *small perturbation* of $\mathcal{T}_1$.

Let $\mathcal{T}$ be an unbounded operator on $H$. We say that a complex number $\rho$ is in the *resolvent set* of $\mathcal{T}$ if $\mathcal{T} - \rho I$ is a bijection from $\mathrm{Dom}(\mathcal{T})$ onto $H$ with a bounded inverse, where $I$ is the identity. The *spectrum* of $\mathcal{T}$, denoted by $\mathrm{Spec}(\mathcal{T})$ is the set of complex numbers which are not in the resolvent set of $\mathcal{T}$. Any eigenvalue of $\mathcal{T}$ is obviously in $\mathrm{Spec}(\mathcal{T})$. The *discrete spectrum* of $\mathcal{T}$ is the set of isolated eigenvalues with finite multiplicity, while the *essential spectrum* of $\mathcal{T}$, denoted by $\mathrm{Spec}_{ess}(\mathcal{T})$ is the set $\mathrm{Spec}(\mathcal{T})$ excluding the discrete spectrum. As we know the spectrum set of a bounded operator is bounded. However that is not the case for unbounded operators.

There is another method to define the self-adjoint extension of some certain kinds of unbounded operators; that is through the *quadratic form*. A quadratic form is a map $q: \quad \mathcal{Q}(q) \times \mathcal{Q}(q) \longrightarrow \mathbb{C}$, where $\mathcal{Q}(q)$ is a dense linear subset of $H$ called the *form domain*, such that $q(x, \cdot)$ is linear and $q(\cdot, y)$ is conjugate linear for all $x, y \in \mathcal{Q}(q)$. We will briefly summarise how to relate a quadratic form and an unbounded operator. Firstly, we observe that the definition of a positive operator extends to unbounded operators; that is $\mathcal{T}$ is positive, denoted by $\mathcal{T} \geq 0$, if $\mathcal{T}$ is symmetric and $\langle \mathcal{T}x,\ x\rangle \geq 0$ for all $x \in \mathrm{Dom}(\mathcal{T})$. For a positive operator $\mathcal{T}$ we may define an inner product $\langle x,\ y\rangle_{\mathcal{T}}$ on $\mathrm{Dom}(\mathcal{T})$ by

$$\langle x,\ y\rangle_{\mathcal{T}} = \langle \mathcal{T}x,\ y\rangle + \langle x, y\rangle.$$

If we denote by $\mathcal{Q}(\mathcal{T})$ the completion of $\mathrm{Dom}(\mathcal{T})$ with respect to the norm $\|.\|_{\mathcal{T}}$ induced by the inner product above, then $\mathrm{Dom}(\mathcal{T}) \subseteq \mathcal{Q}(\mathcal{T}) \subset H$. Truly, we observe that if $\{x_j\}$ is a Cauchy sequence in $\mathrm{Dom}(\mathcal{T})$, it will also be a Cauchy sequence in $H$ because $\|x\| \leq \|x\|_{\mathcal{T}}$. It follows that we may identify the limit in $\mathcal{Q}(\mathcal{T})$ with the limit we have in $H$. Therefore, the quadratic form associated with $\mathcal{T}$, denoted by $q_{\mathcal{T}}$ can be extended to every $x \in \mathcal{Q}(\mathcal{T})$ by setting

$$q_{\mathcal{T}}(x) = \langle x,\ x\rangle_{\mathcal{T}} - \|x\|^2.$$

We will call by $\mathcal{Q}(\mathcal{T})$ the form domain of $\mathcal{T}$. Now we can say that consideration of quadratic forms leads to a useful way of defining a self-adjoint operator if we start with a *semi-bounded* symmetric operator by looking at the following result.

**Theorem 1.11.3.** (Friedrichs extension) *Let $\mathcal{T}$ be a semi-bounded symmetric operator; that is, suppose there exists $\gamma \in \mathbb{R}$ such that*

$$q_{\mathcal{T}}(x) = \langle \mathcal{T}x,\ x\rangle \geq \gamma\|x\|^2, \quad \text{for all } x \in \mathrm{Dom}(\mathcal{T}).$$

*Then there is a self-adjoint extension $\mathcal{T}'$ of $\mathcal{T}$ which is also bounded below by $\gamma$ and which satisfies* $\mathrm{Dom}(\mathcal{T}') \subseteq \mathcal{Q}(\mathcal{T})$. *Moreover, $\mathcal{T}'$ is the only self-adjoint extension of $\mathcal{T}$ with domain contained in $\mathcal{Q}(\mathcal{T})$.*

The converse of this result is also important; given a quadratic form $q$, is there a corresponding operator $\mathcal{T}$ such that $q = q_{\mathcal{T}}$? The answer is yes under certain conditions, although we will not discuss this further here (see [42] for more details).

Now we will consider the quadratic form version of the Kato-Rellich theorem; this is called the *KLMN theorem*, and is due to Kato, Lions, Lax, Milgram and Nelson (see Theorem X.17, [43]). The theorem will allow us to consider the *form sum* of operators.

**Theorem 1.11.4.** *Let $\mathcal{T}_1$ be a positive self-adjoint operator and $q_{\mathcal{T}_2}$ be a quadratic form associated with a symmetric operator $\mathcal{T}_2$ which is defined on $\mathcal{Q}(\mathcal{T}_1)$. If there are real numbers $a < 1$ and $b$ such that*

$$|q_{\mathcal{T}_2}(x)| \leq a q_{\mathcal{T}_1}(x) + b\langle x,\ x\rangle, \quad \textit{for all } x \in \mathcal{Q}(\mathcal{T}_1),$$

*then there exists a unique self-adjoint operator $\mathcal{T}$ with $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{T}_1)$ such that $\mathcal{T}$ is associated with the form $q_{\mathcal{T}_1} + q_{\mathcal{T}_2}$.*

In this case we also call $\mathcal{T}_2$ a *small perturbation* of $\mathcal{T}_1$.

Suppose that $\mathcal{T}_1$ is self-adjoint. We will say that $\mathcal{T}_2$ is *relatively compact* with respect to $\mathcal{T}_1$ if $\mathrm{Dom}(\mathcal{T}_1) \subseteq \mathrm{Dom}(\mathcal{T}_2)$ and the operator $\mathcal{T}_2(\mathcal{T}_1 + i)^{-1}$ is compact. In fact we may replace $i$ here by any complex number in the resolvent set of $\mathcal{T}_1$. We may show that $\mathcal{T}_2$ is relatively compact with respect to $\mathcal{T}_1$ if whenever we have a sequence $\{x_j\} \subset \mathrm{Dom}(\mathcal{T}_1) \subseteq \mathrm{Dom}(\mathcal{T}_2)$ such that $\|\mathcal{T}_1 x_j\| + \|x_j\| \leq C$, for some $C \geq 0$, then we may choose a subsequence $\{x_{j_k}\}$ such that $\{\mathcal{T}_2 x_{j_k}\}$ is convergent. We also have the following (for example, [45], p.113); if $\mathcal{T}_1$ is self-adjoint and $\mathcal{T}_2$ is relatively compact with respect to $\mathcal{T}_1$, then the operator sum $\mathcal{T}_1 + \mathcal{T}_2$ defined on $\mathrm{Dom}(\mathcal{T}_1)$ is closed. Moreover the operator sum has the same essential spectrum as $\mathcal{T}_1$. If we require $\mathcal{T}_2$ to be symmetric, then the operator sum is self-adjoint as well.

# Chapter 2

# Upper bounds for the number of zero modes of the Weyl-Dirac operator

## 2.1 Overview

In [12] Balinsky and Evans studied zero modes of the Weyl-Dirac operator on $\mathbb{R}^n$, $n = 2, 3$, where the potential was assumed to be in $L^q$ with $q = 3$ when $n = 3$ and $q = 2$ when $n = 2$. They gave an upper bound on the multiplicity of zero modes in the form

$$\dim \operatorname{Ker} \mathcal{D}_A \leq C_n \int_{\mathbb{R}^n} |A|^n \, dx, \tag{2.1}$$

with the constant $C_n$ independent of $A$. This result is reviewed in Section 2.2, where we present an argument that circumvents an error in [12].

Balinsky and Evans also considered the question of how common zero mode producing potentials are; one result was to show that if we scale a given potential $A$ by $t \in \mathbb{R}$ then the operator $\mathcal{D}_{tA}$ has zero modes for only a discrete set of values of $t$, in particular, given $T > 0$ there is only a finite number of $t \in [0, T]$ such that $\dim \operatorname{Ker} \mathcal{D}_{tA} > 0$. Thus the quantity

$$n_A(T) := \sum_{0 \leq t \leq T} \dim \operatorname{Ker} \mathcal{D}_{tA},$$

is finite. Information about the behaviour of $n_A(T)$ as $T$ varies clearly tells us something about how common zero mode producing potentials are. Furthermore this quantity will be less sensitive than $\dim \operatorname{Ker} \mathcal{D}_{tA}$, to perturbations in $t$ (or in $A$). In Section 2.3 and Section 2.4 we obtain bounds for $n_A(T)$ of the form (Theorem 2.3.6 and Theorem 2.4.1)

$$n_A(T) \leq C T^n \int_{\mathbb{R}^n} |A|^n \, dx, \quad n = 2, 3,$$

which clearly strengthens the Balinsky and Evans bound on multiplicity (see (2.1) above).

We will see that we must change our approach in the proof of Theorem 2.3.6 to obtain the justification for the two dimensional case. We will need some "wiser" skills to overcome difficulty which arises; that is the result of Cwikel does not work for $q = 2$ although $|x|^{-1}$ is in $L^2_w(\mathbb{R}^2)$. However we can overcome that difficulty by using the weak Weyl inequality (see [47], p.85)) as well as relations between the eigenvalues and the singular values for compact operators to obtain the similar bound for $n_A(T)$ in two dimensions (Theorem 2.4.1).

Finally in Section 2.5 and Section 2.6 we will apply one of the main results in this chapter (That is Theorem 2.3.6 in Section 2.3) to show stronger estimates than the ones in [13] for massless Dirac operators and in [51] for Dirac operators with positive mass at the threshold energies. In fact these follow directly from Theorem 2.3.6

## 2.2 An estimate on the kernel of $\mathcal{D}_A$

### 2.2.1 Set up

Denote by $\mathcal{H}_n,\ n = 2, 3$ the Hilbert space $[L^2(\mathbb{R}^n)]^2$ with the standard scalar product and its induced norm. Let $p$ be the momentum operator $-i\nabla$ with core $[C_0^\infty(\mathbb{R}^n)]^2$. It is well known that $p$ is (componentwise) essentially self-adjoint in $[C_0^\infty(\mathbb{R}^n)]^2$ (see [57], p. 113). We can then extend $p$ uniquely to be a self-adjoint operator in $\mathcal{H}_n$ which we still denote by $p$ with $\mathrm{Dom}(p) = \left[H^1(\mathbb{R}^2)\right]^2$.

Suppose that $f \in \left[H^1(\mathbb{R}^n)\right]^2, n = 2, 3$. It follows from the properties of the Fourier $f(x) \longmapsto \mathcal{F}(f)(\omega)$ that

$$pf(x) = \mathcal{F}^{-1}(\omega\mathcal{F}(f))(x).$$

Therefore, the momentum operator $p$ with the domain $\mathrm{Dom}(p) = \left[H^1(\mathbb{R}^3)\right]^2$, is unitarily equivalent to the multiplication operator

$$(\mathcal{F}p\mathcal{F}^{-1})f(x) = xf(x),\ \ \mathrm{Dom}(x) = \{f \in \mathcal{H}_n : \quad xf \in \mathcal{H}_n\}.$$

Then, since the Fourier transform $\mathcal{F}$ is unitary and the spectrum of the multiplication operator is very easy to determine we come up with the following result

**Theorem 2.2.1.** *The momentum operator $p$ is componentwise self-adjoint in $\mathcal{H}_n,\ n = 2, 3$ with $\mathrm{Dom}(p_j) = \left[H^1(\mathbb{R}^n)\right]^2$, and its spectrum is given by*

$$\mathrm{Spec}(p_j) = \mathrm{Spec}_{ess}(p_j) = \mathbb{R}, \quad j = 1, \ldots, n.$$

As usual, $|p| := (p^*p)^{\frac{1}{2}} = (p^2)^{\frac{1}{2}}$ is a positive self-adjoint operator in $\mathcal{H}_n$. We can define the operator $|p|^{\frac{1}{2}}$ (or $\sqrt{|p|}$ hereafter) to be a positive self-adjoint operator in $\mathcal{H}_n$ as well. We note that if $\psi \in \mathcal{H}_n$, then in $\mathcal{H}_n$ :

$$\langle |p|\psi,\, \psi\rangle_{\mathcal{H}_n} = \int_{\mathbb{R}^n} |\xi|\, |\hat{\psi}(\xi)|^2 \, d\xi = 0 \Longleftrightarrow \hat{\psi} = 0 \Longleftrightarrow \psi = 0,$$

using the isometric property of the Fourier transform from $\mathcal{H}_n$ to itself. Therefore, if $\sqrt{|p|}\psi = 0$ for some $\psi \in \mathcal{H}_n$, then $\| \sqrt{|p|}\psi\|_{\mathcal{H}_n} = 0 \Longleftrightarrow \langle |p|\psi,\, \psi\rangle = 0 \Longleftrightarrow \psi = 0\,(\text{in } \mathcal{H}_n) \Longrightarrow \operatorname{Ker} \sqrt{|p|} = \{0\}$. It follows that we can define the operator $\dfrac{1}{\sqrt{|p|}}$ on $\operatorname{Ran}(\sqrt{|p|})$, the range of $\sqrt{|p|}$. For $s = 1\,, \dfrac{1}{2}$ and $s = -\dfrac{1}{2}$, we define $\mathrm{D}^s(\mathbb{R}^n)$ to be the completion of $\operatorname{Dom}(|p|^s)$ (the domain of $|p|^s$) with respect to the norm

$$\|u\|_{\mathrm{D}^s(\mathbb{R}^n)} = \||p|^s u\|_{\mathcal{H}_n}.$$

Observe that in case $s = 1$ we have

$$\mathrm{D}^1(\mathbb{R}^n) \bigcap \mathcal{H}_n = \left[H^1(\mathbb{R}^n)\right]^2.$$

Obviously, $[C_0^\infty(\mathbb{R}^n)]^2$ is dense in $\mathrm{D}^s(\mathbb{R}^n)$ for $s = 1$ and $\dfrac{1}{2}$. For $\psi \in \operatorname{Dom}(\sqrt{|p|}) = \operatorname{Ran}\left(\dfrac{1}{\sqrt{|p|}}\right)$ we have

$$\left\|\frac{1}{\sqrt{|p|}}\psi\right\|_{\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)} = \left\|\sqrt{|p|}\frac{1}{\sqrt{|p|}}\psi\right\|_{\mathcal{H}_n} = \|\psi\|_{\mathcal{H}_n}.$$

Since $\sqrt{|p|}$ is self adjoint, then

$$\operatorname{Ker}(\sqrt{|p|}) \oplus [\overline{\operatorname{Ran}(\sqrt{|p|})}]^\perp = \mathcal{H}_n.$$

So we have that $\operatorname{Dom}(\sqrt{|p|})$ and $\operatorname{Ran}(\sqrt{|p|})$ are dense in $\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$ and $\mathcal{H}_n$ respectively. It follows that we can extend $\dfrac{1}{\sqrt{|p|}}$ to a unitary map from $\mathcal{H}_n$ onto $\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$. Similarly, $\dfrac{1}{\sqrt{|p|}}$ can be extended to a unitary map from $\mathrm{D}^{-\frac{1}{2}}(\mathbb{R}^n)$ onto $\mathcal{H}_n$. The notation $\dfrac{1}{\sqrt{|p|}}$ should be understood correspondingly in a specific context.

We recall here notations of $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ the triple of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and by $A$ the vector potential $(A_1, A_2, A_3)$, where $A_j$ is the measurable and real-valued function on $\mathbb{R}^3$ for $j = 1, 2$ and 3. To study zero modes of the Weyl-Dirac operator $\mathcal{D}_A = \sigma \cdot (p - A)$, we will use the idea in [12]; namely we will consider $\mathcal{D}_A$ as the sum of operators $\sigma \cdot p$ and $-\sigma \cdot A$. Similarly to Theorem 2.2.1 we have the following.

**Theorem 2.2.2.** *The operator $\mathcal{D}_0 = \sigma \cdot p$ is self-adjoint in $\mathcal{H}_n$, $n = 1, 2$ with* $\mathrm{Dom}(\mathcal{D}_0) = \mathrm{D}^1(\mathbb{R}^n) \bigcap \mathcal{H}_n$, *and its spectrum is given by*

$$\mathrm{Spec}(\mathcal{D}_0) = \mathrm{Spec}_{ess}(\mathcal{D}_0) = \mathbb{R}.$$

In order to treat $\sigma \cdot A$ as a small purturbation of $\mathcal{D}_0$ we need the following continuous embeddings to achieve that end. These results can be seen in [36], Theorem 8.4, Theorem 8.3 and Theorem 8.5, respectively. They are:

1. If $n \geq 2$, then
$$\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n) \hookrightarrow [L^{\frac{2n}{n-1}}(\mathbb{R}^n)]^2, \tag{2.2}$$

2. We have
$$\mathrm{D}^1(\mathbb{R}^3) \hookrightarrow [L^6(\mathbb{R}^3)]^2, \tag{2.3}$$

3. For $q \geq 2$, we have
$$[H^1(\mathbb{R}^2)]^2 \hookrightarrow [L^q(\mathbb{R}^2)]^2. \tag{2.4}$$

We also need the following inequality which can been found in [35], p.304; that is there exists a constant $C_n$ such that

$$\int_{\mathbb{R}^n} \frac{1}{|x|} |u(x)|^2 \, dx \leq C_n \, \|u\|^2_{\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)}, \quad \text{for } f \in \left[C_0^\infty(\mathbb{R}^n)\right]^2. \tag{2.5}$$

We remark that the dual $\left(\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)\right)^*$ of $\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$ (with respect to the extension of the standard $L^2$ pairing) is $\mathrm{D}^{-\frac{1}{2}}(\mathbb{R}^n)$. As usual for a vector valued function $A = (A_1, \ldots, A_n)$ the notation $\|A\|_{L^q}$ means the norm of $|A| := \left(\sum_{j=1}^n |A_j|^2\right)^{\frac{1}{2}}$ in $L^q(\mathbb{R}^n)$, $q \geq 1$.

### 2.2.2 $\mathcal{D}_A$ as an operator sum

We will need some results relating to $|A|$ so that we can consider the Weyl-Dirac operator $\mathcal{D}_A$ as the sum of operators $\sigma \cdot p$ and $-\sigma \cdot A$. The following two results from [12] will help us.

**Lemma 2.2.3.** *For $n = 2, 3$ assume that $|A| \in L^n(\mathbb{R}^n)$. Then the operator $\dfrac{1}{\sqrt{|p|}}(\sigma \cdot A)\dfrac{1}{\sqrt{|p|}} : \mathcal{H}_n \longrightarrow \mathcal{H}_n$ is compact. Moreover, for all $\varphi \in \mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$, we have*

$$\|\left(\sigma \cdot A\right)\varphi\|_{\mathrm{D}^{-\frac{1}{2}}(\mathbb{R}^n)} \leq \gamma_n^2 \|A\|_{L^n(\mathbb{R}^n)} \cdot \|\varphi\|_{\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)},$$

*where $\gamma_n$ is the norm of the embedding* (2.2).

*Remark* 2.2.4. The result of this lemma can be justified neatly by appying the result of Cwikel which we will meet later in the next two sections. We would like here to be faithful with the arguments in [12].

To prove this lemma, we notice that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^n(\mathbb{R}^n)$, so for a given $\varepsilon > 0$ there is $a_1 \in C_0^\infty(\Omega_\varepsilon)$, where $\Omega_\varepsilon$ is some ball in $\mathbb{R}^n$ such that $\| \, |A| - a_1\|_{L^n(\mathbb{R}^n)} < \varepsilon$. Setting $a_2 := \; |A| - a_1$, and we can write $|A| = a_1 + a_2$, where $\|a_1\| \leq K_\varepsilon$ and $\|a_2\| \leq \varepsilon$, for a given $\varepsilon$. Moreover, we also remark that $\dfrac{1}{\sqrt{|p|}}$ is a unitary map from $\mathcal{H}_n$ onto $\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$ and $\dfrac{1}{\sqrt{|p|}}$ is a unitary map from $\mathrm{D}^{-\frac{1}{2}}(\mathbb{R}^n)$ onto $\mathcal{H}_n$. So, to justify the lemma, we only need to prove that if $\varphi_n \rightharpoonup 0$ (weakly converges to 0) in $\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$, then $\big(\sigma \cdot A\big)\varphi_n \longrightarrow 0$ in $\mathrm{D}^{-\frac{1}{2}}(\mathbb{R}^n)$. The proof is now straightforward; see [12] for details.

The result of Lemma 2.2.3 gives us

**Lemma 2.2.5.** *Suppose that the following conditions are satisfied:*

- *when* $n = 3$, *then* $|A|$ *is a function in* $L^3(\mathbb{R}^3)$,
- *when* $n = 2$, *then* $|A|$ *is a function in* $L^r(\mathbb{R}^2)$ *for some* $r > 2$.

*Then for any given* $\varepsilon > 0$, *there is a constant* $C_\varepsilon$ *(depending on* $\varepsilon$*) such that for all functions* $\varphi \in \mathrm{D}^1(\mathbb{R}^n) \bigcap \mathcal{H}_n$, *we have*

$$\|\big(\sigma \cdot A\big)\varphi\|^2 \leq \varepsilon^2 \|\varphi\|^2_{\mathrm{D}^1(\mathbb{R}^n)} + C_\varepsilon \|\varphi\|^2.$$

*Remark* 2.2.6. It follows from the result of this lemma that $\sigma \cdot A$ is a small purturbation of $\sigma \cdot p$.

The assumptions in Lemma 2.2.5 guarantee that the embeddings (2.2) and (2.4) are applicable. Please refer to [12] for details.

Finally, by the Kato-Rellich theorem (Theorem 1.11.2) we obtain from the result of Lemma 2.2.5 that

**Theorem 2.2.7.** *Let* $A$ *be the potential which satisfies the conditions in Lemma 2.2.5. Then, the operator* $\mathcal{D}_A$ *is well defined as the operator sum of* $\mathcal{D}_0$ *and* $-\sigma \cdot A$*; that is* $-\sigma \cdot A$ *is a small perturbation of* $\mathcal{D}_0$.

The corollaries of Theorem 2.2.7 are $\mathcal{D}_A$ is self-adjoint and $\mathrm{Dom}(\mathcal{D}_A) = \mathrm{Dom}(\mathcal{D}_0) = \mathrm{D}^1(\mathbb{R}^n) \bigcap \mathcal{H}_n$.

### 2.2.3 Upper bounds for $\mathrm{Ker}(\mathcal{D}_A)$

Now we can proceed to consider one of the main result of this section; that is we will show a proof for upper bounds for $\dim \mathrm{Ker}\, \mathcal{D}_A$.

Firstly, we observe that if $\varphi \in \mathrm{Dom}(\mathcal{D}_A) = \mathrm{D}^1(\mathbb{R}^n) \bigcap \mathcal{H}_n$, then the function $\psi := \sqrt{|p|}\varphi$ will satisfy $\psi \in \mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n) \bigcap \mathcal{H}_n$; clearly we have $\psi \in \mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$. While, we also have $\psi \in \mathcal{H}_n$ since

$$\|\psi\|^2 = \langle \sqrt{|p|}\varphi,\ \sqrt{|p|}\varphi \rangle = \langle |p|\varphi,\ \varphi \rangle \leq \|\ |p|\varphi\|\ \|\varphi\|,$$

so

$$\|\psi\|^2 \leq \frac{1}{2}\big(\|p\varphi\|^2 + \|\varphi\|^2\big) \leq \frac{1}{2}\big(\|\varphi\|^2_{\mathrm{D}^1(\mathbb{R}^n)} + \|\varphi\|^2\big).$$

Now we define the operators

$$E := \frac{1}{\sqrt{|p|}}\big(\sigma \cdot p\big)\frac{1}{\sqrt{|p|}} :\ \mathcal{H}_n \longrightarrow \mathcal{H}_n,$$

and

$$K := \frac{1}{\sqrt{|p|}}\big(\sigma \cdot A\big)\frac{1}{\sqrt{|p|}} :\ \mathcal{H}_n \longrightarrow \mathcal{H}_n.$$

We remark that $E$ is self-adjoint and $E^2 = I$. So, if $\varphi \in \mathrm{Ker}\, \mathcal{D}_A$, then $\big(\sigma \cdot p\big)\frac{1}{\sqrt{|p|}}\psi = \big(\sigma \cdot A\big)\frac{1}{\sqrt{|p|}}\psi$. It follows that $(E - K)\psi = 0$ and hence $(I - EK)\psi = 0$. Moreover if $\varphi_1, \varphi_2 \in \mathrm{Ker}\, \mathcal{D}_A$ and linear independent, then it follows from $\mathrm{Ker}\sqrt{|p|} = \{0\}$ that $\psi_1, \psi_2$ are independent, where $\psi_j = \sqrt{|p|}\varphi_j,\ j = 1, 2$. So we get $\dim \mathrm{Ker}\, \mathcal{D}_A \leq \dim \mathrm{Ker}(I - EK)$. To use this relationship and the compactness of $EK$ (which follows from the result of Lemma 2.2.3), Balinsky and Evans show that for the one-parameter family of potentials $tA$ there are at most a finite set of values of $t$ in $[0, T]$ for any $T > 0$ such that $\dim \mathrm{Ker}\, \mathcal{D}_{tA} \neq 0$. Although the result below is weaker than the later one in this thesis (see Theorem 2.3.6) we still want to show it here with our proof. The reason is in the proof of Balinsky and Evans they used an incorrect inequality which we also discuss later. Let

$$L := \frac{1}{\sqrt{|p|}}|A|\frac{1}{\sqrt{|p|}} :\ \mathcal{H}_n \longrightarrow \mathcal{H}_n.$$

Then $L$ is positive and self-adjoint.

**Lemma 2.2.8.** *We have $\mp K \leq L$.*

*Proof.* We will prove that $\sigma \cdot A \leq |A|$ in $[C_0^\infty(\mathbb{R}^n)]^2$; then, using the density of $[C_0^\infty(\mathbb{R}^n)]^2$ in $\mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n)$, we obtain $K \leq L$. The inequality $-K \leq L$ follows from a similar argument.

There is a fact that the self-adjoint $2 \times 2$ matrix $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \geq 0$ if and only if $a, c$ and $ac - |b|^2$ are all non-negative. Here we have

$$|A| - \sigma \cdot A = \begin{pmatrix} |A| - A_3 & -(A_1 - iA_2) \\ -(A_1 + iA_2) & |A| + A_3 \end{pmatrix}.$$

Then, $|A| - \sigma \cdot A \geq 0$ follows by applying the result above with

$$a = |A| - A_3 \geq 0, \; c = |A| + A_3 \geq 0 \quad \text{and } b = -(A_1 - iA_2)$$

for which $ac - |b|^2 = 0$. ■

*Remark* 2.2.9. At first I proved $\sigma \cdot A \leq |A|$ by checking directly. The proof was quite long compared to the proof above. I would like to thank Dr G. Jameson for reminding me of the fact in the proof so I can have the version above.

Next we can state and prove the following result.

**Theorem 2.2.10.** *Let $|A|$ be in $L^3(\mathbb{R}^3)$ when $n = 3$, and in $L^2(\mathbb{R}^2) \bigcap L^r(\mathbb{R}^2)$ for some $r > 2$ when $n = 2$. Then, we have*

$$\dim \operatorname{Ker} \mathcal{D}_A \leq C_n \|A\|_{L^n}^n,$$

*where $C_n$ is a constant which is independent of $A$.*

*Proof.* Firstly, we see that if $\varphi \in \mathrm{Dom}(\mathcal{D}_A)$, then the above arguments give $(E - K)\psi = 0$, where

$$\psi = \sqrt{|p|}\varphi \in \mathrm{D}^{\frac{1}{2}}(\mathbb{R}^n) \bigcap \mathcal{H}_n \quad \text{and} \quad K = \frac{1}{\sqrt{|p|}}(\sigma \cdot A)\frac{1}{\sqrt{|p|}}.$$

It follows that $\dim \operatorname{Ker} \mathcal{D}_A \leq \dim \operatorname{Ker}(I - EK)$. Set $S = EK$, so we already have $\dim \operatorname{Ker} \mathcal{D}_A \leq \dim \operatorname{Ker}(I - S)$. We remark that if $\psi \in \operatorname{Ker}(I - S)$, then $\psi = S\psi$. Next,

$$\langle |S|^2\psi, \; \psi\rangle = \langle S^*S\psi, \; \psi\rangle = \langle S\psi, \; S\psi\rangle = \langle \psi, \psi\rangle = \|\psi\|^2.$$

Therefore,

$$\operatorname{Ker}(I - S) \subseteq \mathrm{K}_{|S|^2}(1),$$

where $\mathrm{K}_{|S|^2}(1)$ is the set of all closed linear subspaces $V$ of $\mathcal{H}_n$ such that $\langle |S|^2\psi, \; \psi\rangle \geq 1 \cdot \|\psi\|^2$ for all $\psi \in V$. By min-max theorem (see [20], p.84), if we enumerate the eigenvalues (counting also multiplicities) of the positive compact operator $T$ as usual $\lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq 0$ we get

$$\#\{n : \; \lambda_n(|S|^2) \geq 1\} \geq \dim \operatorname{Ker}(I - S) \geq \dim \operatorname{Ker} \mathcal{D}_A. \tag{2.6}$$

Furthermore, we have shown that $K$ is compact and self-adjoint, while

$$|S|^2 = (EK)^*(EK) = K^2.$$

Thus

$$\begin{aligned}\#\{n:\ \lambda_n(|S|^2)\geq 1\} &= \#\{n:\ \lambda_n^+(K)\geq 1\}+\#\{n:\ \lambda_n^-(K)\leq -1\}\\ &= \#\{n:\ \lambda_n^+(K)\geq 1\}+\#\{n:\ \lambda_n^+(-K)\geq 1\}.\end{aligned} \tag{2.7}$$

Lemma 2.2.8 and the min-max theorem give

$$\#\{n:\ \lambda_n^+(\pm K)\geq 1\}\leq \#\{n:\ \lambda_n(L)\geq 1\}, \tag{2.8}$$

since $L$ is positive so $\lambda_n^+(L)=\lambda_n(L)$, where eigenvalues of $L$ are enumerated as usual (thank to [26]). It follows from (2.8) and (2.7) that

$$\#\{n:\ \lambda_n(|S|^2)\geq 1\}\leq 2\cdot\#\{n:\ \lambda_n(L)\geq 1\}. \tag{2.9}$$

By the Birman-Schwinger principle (see [14], [45]), we have

$$\#\{n:\ \lambda_n(L)\geq 1\}=\ N(|p|-|A|), \tag{2.10}$$

where

$$N(|p|-|A|):=\#\{\text{non-positive eigenvalues of the operator } |p|-|A|\}.$$

Indeed we may see the flavour of the Birman-Schwinger principle as follows: Let $\lambda\geq 1$ be an eigenvalue of $L$ with corresponding eigenfunction $\phi$. Then,

$$\begin{aligned}&L\phi=\lambda\phi\\ &\Longleftrightarrow \lambda\sqrt{|p|}\phi=|A|\frac{1}{\sqrt{|p|}}\phi\\ &\Longleftrightarrow \lambda|p|\psi=|A|\psi \text{ for } \psi \text{ such that } \sqrt{|p|}\psi=\phi\\ &\Longleftrightarrow (|p|-\frac{1}{\lambda}|A|)\psi=0.\end{aligned}$$

So, 0 is eigenvalue of the operator $|p|-\dfrac{1}{\lambda}|A|$ with eigenfunction $\psi$. Then, the Birman-Schwinger principle gives us (2.10). Back to our main arguments we obtain from (2.6), (2.9) and (2.10) that

$$\dim\operatorname{Ker}\mathcal{D}_A\leq\#\{n:\ \lambda_n(|S|^2)\geq 1\}\leq 2N(|p|-|A|). \tag{2.11}$$

Now we can apply Daubechies inequality (see [18]) to get

$$\begin{aligned}N(|p|-|A|)&\leq C_1\int_{\mathbb{R}^n}|A|^n\ dx\\ &=C_1\|A\|_{L^n}^n.\end{aligned} \tag{2.12}$$

Then, (2.11) and (2.12) give

$$\dim\operatorname{Ker}\mathcal{D}_A\leq 2C_1\int_{\mathbb{R}^n}|A|^n\ dx,$$

or

$$\dim\operatorname{Ker}\mathcal{D}_A\leq C\int_{\mathbb{R}^n}|A|^n\ dx$$

with $C=2C_1$, which completes the justification for Theorem 2.2.10. ■

Theorem 2.2.10 was given in [12], but in their proof the authours have used the operator inequality

$$K^2 \le L^2 \tag{2.13}$$

We see in Lemma 2.2.8 that

$$-L \le K \le L.$$

*Remark* 2.2.11. We observe that $K^2 \le L^2$ would follow from $-L \le K \le L$ if $K$ and $L$ commute. However, here $K$ and $L$ do not commute. I would like to thank Dr. G Jameson for his advice in this remark.

We will show below that the operator inequality (2.13) is wrong. Indeed, we have the following.

**Proposition 2.2.12.** *Given $r > 1$ there exists a potential $A = (A_1, A_2, A_3)$ with $|A| \in L^3(\mathbb{R}^3)$ and $\phi \in \mathcal{H}_3$ with*

$$\langle K^2\phi,\ \phi\rangle > r\langle L^2\phi,\ \phi\rangle. \tag{2.14}$$

*Proof.* Let $\eta \in \mathbb{R}^3$ be a unit vector, choose $\alpha > 3$ and set

$$A(x) = (1 + |x|^2)^{-\frac{\alpha}{2}}(\cos(2r\langle\eta, x\rangle), \sin(2r\langle\eta, x\rangle), 0).$$

Clearly, $A$ is smooth, while

$$|A(x)| = (1 + |x|^2)^{-\frac{\alpha}{2}}$$

so $|A| \in L^3(\mathbb{R}^3)$ (using the fact that $\alpha > 3$). Furthermore,

$$\begin{aligned} \frac{\sigma \cdot A}{|A|} &= \cos(2r\langle\eta, x\rangle)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(2r\langle\eta, x\rangle)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{-2ir\langle\eta,x\rangle} \\ e^{2ir\langle\eta,x\rangle} & 0 \end{pmatrix}. \end{aligned} \tag{2.15}$$

Now let $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\mathrm{supp}(\chi) \subset B_1 := \{\xi : \ |\xi| < 1\}$ (the open unit ball in $\mathbb{R}^3$) and $\int_{\mathbb{R}^3} |\chi|^2 dx = 1$. Let $f$ be the inverse Fourier transform of $\xi \longmapsto \chi(\xi + (1 + 2r)\eta)$ (a translated version of $\chi$). Finally, set

$$\phi_1 = |p|^{\frac{1}{2}}(|A|^{-1}f) \text{ and } \phi = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}.$$

Now $f \in \mathsf{S}$ (the Schwartz class) and $|A|^{-1} = (1 + |x|^2)^{\frac{\alpha}{2}}$ so $|A|^{-1}f \in \mathsf{S}$. It follows that $\phi_1$ (and hence $\phi$) is smooth and $\phi \in \mathcal{H}_3$. A direct calculation gives

$$L\phi = |p|^{-\frac{1}{2}}|A||p|^{-\frac{1}{2}}\begin{pmatrix} |p|^{\frac{1}{2}}(|A|^{-1}f) \\ 0 \end{pmatrix} = \begin{pmatrix} |p|^{-\frac{1}{2}}f \\ 0 \end{pmatrix}$$

so

$$\langle L^2\phi,\ \phi\rangle = \|L\phi\|^2 = \langle |p|^{-1}f, f\rangle_{L^2} = \int_{\mathbb{R}^3} |\xi|^{-1}|\widehat{f}(\xi)|^2 d\xi.$$

Since $\widehat{f}(\xi) = \chi(\xi + (1+2r)\eta)$ we get

$$\xi \in \operatorname{supp}(\widehat{f}) \Rightarrow \xi + (1+2r)\eta \in B_1 \Rightarrow |\xi| > |(1+2r)\eta| - 1 = 2r$$

and thus

$$\langle L^2\phi,\ \phi\rangle < \frac{1}{2r}\int_{\mathbb{R}^3} |\widehat{f}(\xi)|^2 d\xi = \frac{1}{2r}\int_{\mathbb{R}^3} |\chi|^2 dx = \frac{1}{2r}. \tag{2.16}$$

On the other hand

$$K\phi = |p|^{-\frac{1}{2}}(\sigma\cdot A)|p|^{-\frac{1}{2}}\begin{pmatrix}|p|^{\frac{1}{2}}(|A|^{-1}f)\\ 0\end{pmatrix} = |p|^{-\frac{1}{2}}\frac{\sigma\cdot A}{|A|}\begin{pmatrix}f\\0\end{pmatrix} = \begin{pmatrix}0\\ |p|^{-\frac{1}{2}}(e^{2ir\langle\eta,x\rangle}f)\end{pmatrix},$$

using (2.15). Setting $g = e^{2ir\langle\eta,x\rangle}f$ we thus get

$$\langle K^2\phi,\ \phi\rangle = \|K\phi\|^2 = \langle |p|^{-1}g, g\rangle_{L^2} = \int_{\mathbb{R}^3} |\xi|^{-1}|\widehat{g}(\xi)|^2 d\xi.$$

However $\widehat{g}(\xi) = \widehat{f}(\xi - 2r\eta) = \chi(\xi+\eta)$ so

$$\xi \in \operatorname{supp}(\widehat{g}) \Rightarrow \xi + \eta \in B_1 \Rightarrow (0 <)\quad |\xi| < 2$$

and hence

$$\langle K^2\phi,\ \phi\rangle > \frac{1}{2}\int_{\mathbb{R}^3} |\widehat{g}(\xi)|^2 d\xi = \frac{1}{2}\int_{\mathbb{R}^3} |\chi|^2 dx = \frac{1}{2}. \tag{2.17}$$

Estimate (2.14) clearly follows from (2.16) and (2.17). ■

*Remark* 2.2.13. The potential $A$ is smooth and satisfies $|A| = (1+|x|^2)^{-\frac{\alpha}{2}}$. It follows that the corresponding magnetic field $B = \operatorname{curl} A$ is also smooth, while a straightforward check gives $|B| \le C(1+|x|^2)^{-\frac{\alpha}{2}}$ for some constant $C$. Since $\alpha > 3$ was arbitrary we can ensure that $A$ and $B$ have arbitrary algebraic decay.
Using Fourier transforms it is easy to see that $\phi_1$ and its derivatives of arbitrary order belong to $L^2$; thus $\phi_1 \in H^s$ (the Sobolev space of order $s$) for any $s \in \mathbb{R}$. However $\widehat{\phi}_1(\xi)$ has a $|\xi|^{\frac{1}{2}}$ type singularity at 0, which will prevent $\phi_1$ from having rapid decay. A straightforward scaling argument applied to the inverse Fourier transform of $\phi_1$ shows that we have $|\phi_1(x)| \le c'(1+|x|^2)^{-\frac{7}{4}}$ for some constant $c'$. Using an approximation argument it should be possible to obtain (2.14) with some $\phi \in C_0^\infty$.

## 2.3 An estimate on $n_A(T)$ in three dimensions

For a given $T > 0$, there are at most a finite set of $t$, $0 \le t \le T$ such that $\dim\operatorname{Ker}\mathcal{D}_{tA} \ne 0$ (see [12]). The proof with flavour we met in the previous section is based on Fredholm theory and a much shorter version of a similar result in [10], but for $\dim\operatorname{Ker}\mathcal{P}_{tA}$. However, we will show a stronger result by proving that the estimate works not only for the number of zero modes for each operator $\mathcal{D}_{tA}$, but also for the total of zero modes for all Weyl-Dirac operators $\mathcal{D}_{tA}$, $0 \le t \le T$ (see Theorem 2.3.6). To prepare, we will prove the following lemmas.

**Lemma 2.3.1.** *If* $0 < \beta < 1$ *and* $n \in \mathbb{N}$*, then* $\sum_{k=1}^{n} k^{-\beta} \le \dfrac{1}{1-\beta} n^{1-\beta}$.

*Proof.* Consider the function

$$\begin{aligned} f &: [0, \infty) \longrightarrow \mathbb{R} \\ &: \ x \longmapsto x^{-\beta}. \end{aligned}$$

One addition is that $f(0) = 0$. Now take a partition of the interval $[0, n]$ into $n$ intervals of length 1. Since $f$ is decreasing on $(0, n)$ the area of each rectangle, which is defined by $[k, k+1]$ as the base and $f(k+1)$ as its height, is less than or equal to the area of region which is bounded by the horizontal axis, lines $x = k,\ x = k+1$ and the graph of function $f$. Summing up we get

$$\sum_{k=0}^{n-1} f(k+1) \le \sum_{k=0}^{n-1} \int_k^{k+1} f(x)\ dx,$$

or

$$\sum_{k=1}^{n} k^{-\beta} \le \int_0^n x^{-\beta}\ dx.$$

■

Another fact we will use in the proof of Theorem 2.3.6 is the following.

**Lemma 2.3.2.** *If* $0 < q < 1$*, then*

$$\left(\frac{3}{3-q}\right)^{\frac{3}{q}} < \frac{27}{8}. \tag{2.18}$$

*Proof.* We can obtain (2.18) by proving that

$$f(x) = \frac{1}{(1-x)^{\frac{1}{x}}} \quad \text{is increasing on } (0,1). \tag{2.19}$$

(Hence in particular $f(x) \le f(\frac{1}{3}) = \dfrac{27}{8}$ for $0 < x \le \dfrac{1}{3},\ x = \dfrac{q}{3}$). To prove (2.19) we note that

$$-\log(1-x) \le \frac{x}{1-x} \quad \text{for } 0 < x < 1, \tag{2.20}$$

for instance by comparing the Maclaurin's series for functions on both sides of (2.20). Observe that $\log f(x) = -\dfrac{1}{x}\log(1-x)$ and for $0 < x < 1$

$$\log' f(x) = \frac{f'(x)}{f(x)} = \frac{1}{x^2}\log(1-x) + \frac{1}{x(1-x)} = \frac{1}{x^2}\big(\log(1-x) + \frac{x}{1-x}\big) \ge 0.$$

Now (2.19) follows since $f'(x) \ge 0$ for $0 < x < 1$. ■

*Remark* 2.3.3. I would like to thank Dr G. Jameson for his advice to show this less strange proof for Lemma 2.18 compared to the initial version.

We want to apply the result of Cwikel (see, for example Theorem XI.22, p.47 in [44]) and obtain the compactness of a class of operators such as $\mathcal{T}_1$'s later. We will consider the concept of $L^q$-weakness as follows.

**Definition 2.3.4.** A function $f$ is *weak-*$L^q(\mathbb{R}^n)$ for $1 \le q < \infty$ if

$$\|f\|_{L^q_w} := \sup_t (t^q \mu\{x : \ |f(x)| > t\})^{\frac{1}{q}} < +\infty,$$

where $\mu$ is the usual Lebesgue measure on $\mathbb{R}^n$.

The set of all weak-$L^q(\mathbb{R}^n)$ functions (for each $q$ and $n$) is denoted by $L^q_w(\mathbb{R}^n)$. $\|\cdot\|_{L^q_w}$ is not actually a norm on $L^q_w$ (it does not satisfy the triangle inequality). It is straight-forward to check that $\|x^{-\frac{n}{q}}\|_{L^q_w} = \mathrm{vol}(B_n)^{\frac{1}{q}}$ where $\mathrm{vol}(B_n)$ is the volume of the unit ball in $\mathbb{R}^n$; thus $x^{-\frac{n}{q}} \in L^q_w(\mathbb{R}^n)$ even though $x^{-\frac{n}{q}} \notin L^{q'}(\mathbb{R}^n)$ for any $q'$. Refer to [43] or [36] for more details.

The following result is an easy consequence of the result in [16].

**Theorem 2.3.5.** (See [44], p. 47-49) *Let $2 < q < \infty$ and suppose that $g \in L^q_w(\mathbb{R}^n)$ and $f \in L^q(\mathbb{R}^n)$. Then $f(x)g(-i\nabla)$ is a compact operator with singular values $\mu_j$ satisfying*

$$\mu_j \le C(q,n) j^{-q} \|f\|_{L^q} \|g\|_{L^q_w}, \quad j \ge 1.$$

Now time to state and prove the main result of this section.

**Theorem 2.3.6.** *Let $|A|$ be in $L^3(\mathbb{R}^3)$. For an arbitrary $T > 0$ we have the following estimate*

$$n_A(T) \le CT^3 \cdot \|A\|^3_{L^3},$$

*where $C$ is a constant, not dependent on $T$ or on $A$.*

Here we recall that

$$n_A(T) = \sum_{0 \le t \le T} \dim \mathrm{Ker}\, \mathcal{D}_{tA}.$$

*Proof.* We see that

$$(\sigma \cdot A)^2 = (\sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3)^2 = \begin{pmatrix} A_3 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 \end{pmatrix}^2 = |A|^2, \tag{2.21}$$

where $|A|^2 := |A|^2 \mathbb{I}_2$ with $\mathbb{I}_2$ is the $2 \times 2$ identity matrix.
Similarly, we also have

$$(\sigma \cdot p)^2 = -\Delta = |p|^2. \tag{2.22}$$

It follows from $(\sigma \cdot A)^2 = |A|^2$ and $(\sigma \cdot p)^2 = |p|^2$ that

$$|A||p|^{-2}|A| = |A|(\sigma \cdot p)^{-1}(\sigma \cdot p)^{-1}|A|. \tag{2.23}$$

Let $\mathcal{U} = \dfrac{\sigma \cdot A}{|A|}$ and $\mathcal{T} = (\sigma \cdot p)^{-1}(\sigma \cdot A)$. We remark that $\mathcal{U}$ is unitary. Furthermore, it follows from (2.23) that

$$|A||p|^{-2}|A| = \mathcal{U}^*\mathcal{T}^*\mathcal{T}\mathcal{U}. \tag{2.24}$$

Let $\mathcal{T}_1 = |A||p|^{-1}$. Then, we also have

$$|A||p|^{-2}|A| = \mathcal{T}_1\mathcal{T}_1^*. \tag{2.25}$$

Now we can show the compactness of $\mathcal{T}_1$. Truly, it is because function $f(x) = |A| \in L^3(\mathbb{R}^3)$, and $g(x) = |x|^{-1} \in L^3_w(\mathbb{R}^3)$, we can apply Cwikel's result for $q = 3$ (see Theorem 2.3.5). Apart from the compactness for $\mathcal{T}_1$ Theorem 2.3.5 tells us that

$$\mu_j(\mathcal{T}_1) \leq C_1\|A\|_{L^3} j^{-\frac{1}{3}} = \alpha j^{-\frac{1}{3}}, \tag{2.26}$$

where

$$\alpha := \ C_1\|A\|_{L^3},$$

and $C_1$ independent of $A$. Here we recall that singular values of $\mathcal{T}_1$, including multiplicity, are arranged as $\mu_1(\mathcal{T}_1) \geq \mu_2(\mathcal{T}_1) \geq \cdots \geq 0$.
We notice that $\mathcal{U}$ is unitary, so $\mu_j(\mathcal{T}) = \mu_j(\mathcal{T}_1)$ and hence

$$\mu_j(\mathcal{T}) \leq \alpha j^{-\frac{1}{3}}. \tag{2.27}$$

Now it follows from the result by Lemma 2.3.1 above we have for any $0 < q < 1$ (in fact we can take $0 < q < 3$)

$$\sum_{j=1}^{n} \big(\mu_j(\mathcal{T})\big)^q \leq \alpha^q \sum_{j=1}^{n} j^{-\frac{q}{3}} \leq \alpha^q \frac{3}{3-q} n^{1-\frac{q}{3}}. \tag{2.28}$$

Now let $\mathcal{N}_T := \ \#\{j \ : \ |\lambda_j(\mathcal{T})| \geq \dfrac{1}{T}\}$, so $\dfrac{1}{T} \leq |\lambda_{\mathcal{N}_T}(\mathcal{T})|$. Next we will apply the *localisation of eigenvalues* for a compact operator ((see [47])); that is, if $K$ is is a compact operator, and $\lambda_j$ and $\mu_j$ are the eigenvalues and singular values of $K$ (including multiplicity), then we have the localisation of eigenvalues for $K$ and $0 < q < 1$ as

$$|\lambda_n| \leq \left[\frac{1}{n}\sum_{j=1}^{n} \mu_j^q\right]^{\frac{1}{q}}. \tag{2.29}$$

So it follows from (2.29) that

$$\frac{1}{T} \leq \left(\frac{1}{\mathcal{N}_T}\sum_{j=1}^{\mathcal{N}_T} (\mu_j(\mathcal{T}))^q\right)^{\frac{1}{q}}, \text{ for } 0 < q < 1.$$

Applying the inequality (2.28) for $0 < q < 1$, we get

$$\frac{1}{T} \leq \left(\alpha^q \frac{3}{3-q} \mathcal{N}_T^{-\frac{q}{3}}\right)^{\frac{1}{q}} = \alpha \left(\frac{3}{3-q}\right)^{\frac{1}{q}} \mathcal{N}_T^{-\frac{1}{3}}.$$

That means $\mathcal{N}_T \le \alpha^3 \left(\frac{3}{3-q}\right)^{\frac{3}{q}} T^3 \le \frac{27}{8}\alpha^3 T^3$, as the result of Lemma 2.18. Thus we have

$$\mathcal{N}_T \le \frac{27}{8} C_1 T^3 \|A\|_{L^3}^3. \tag{2.30}$$

Now we see that if $\psi \in \operatorname{Ker} \mathcal{D}_{tA}$, then $\psi$ is an eigenvector of $\mathcal{T}$ with the eigenvalue $\frac{1}{t}$. Indeed, we have

$$\mathcal{D}_{tA}\psi = 0 \Longleftrightarrow \sigma \cdot p\psi = \sigma \cdot tA\psi \Longleftrightarrow \psi = t(\sigma \cdot p)^{-1}(\sigma \cdot A)\psi \Longleftrightarrow \mathcal{T}\psi = \frac{1}{t}\psi.$$

Therefore,

$$\psi \in \operatorname{Ker} \mathcal{D}_{tA} \Longleftrightarrow \psi \text{ is the eigenfunction of } \mathcal{T} \text{ with the eigenvalue } \lambda = \frac{1}{t}.$$

We notice that for $0 \le t \le T$, then

$$\lambda = \frac{1}{t} \ge \frac{1}{T}.$$

So, we have

$$n_A(T) \le \mathcal{N}_T.$$

And then our conclusion follows from (2.30) with a re-selection of the constant $C = \frac{27}{8}C_1$. ∎

## 2.4 An estimate on $n_A(T)$ in two dimensions

In the three dimensional case of the previous section we applied the result of Cwikel and obtained the compactness of operator $\mathcal{T}$ (actually $\mathcal{T}$ is in the Schatten class $\mathbf{S}_q$ for some $q \ge 1$) as well as important estimates for the singular values of $\mathcal{T}$. We used the fact that the function $|x|^{-1} \in L^3_w(\mathbb{R}^3)$ and we can apply Cwikel's result for $q = 3$. However Cwikel's result does not include $q = 2$, despite the fact that $|x|^{-1} \in L^2_w(\mathbb{R}^2)$ as well. It follows that we cannot obtain directly the estimate for $n_A(T)$ in two dimensions by the same method we used in three dimensions above. However, we can change some arguments, so that at last we can again apply Cwikel's result to obtain the following.

**Theorem 2.4.1.** *Let $|A|$ be in $L^2(\mathbb{R}^2) \bigcap L^r(\mathbb{R}^2)$ for some $r > 2$. For an arbitrary $T > 0$ we have the following estimate*

$$n_A(T) \le CT^2 \cdot \|A\|_{L^2}^2,$$

*where the constant $C$ is independent of $T$ and $A$.*

*Proof.* Let $\mathcal{U} = \dfrac{\sigma \cdot A}{|A|}$ and $\mathcal{V} = \dfrac{\sigma \cdot p}{|p|}$. Observe that $\mathcal{U}$ and $\mathcal{V}$ are unitary. Applying the Birman-Schwinger principle we have if 0 is eigenvalue for $\mathcal{D}_{tA}$, then $\dfrac{1}{t}$ is eigenvalue (with the same multiplicity) for $\mathcal{S} = |p|^{-\frac{1}{2}}(\sigma \cdot A)|p|^{-\frac{1}{2}}\mathcal{V}$. Next, we write $\mathcal{S} = |p|^{-\frac{1}{2}}|A|^{\frac{1}{2}}\mathcal{U}|A|^{\frac{1}{2}}|p|^{-\frac{1}{2}}\mathcal{V}$. Let $\mathcal{R}_1 = |A|^{\frac{1}{2}}|p|^{-\frac{1}{2}}$. Then, we have

$$\mathcal{S} = \mathcal{R}_1^* \mathcal{U} \mathcal{R}_1 \mathcal{V}. \tag{2.31}$$

It is time to look back to Cwikel's result for $\mathcal{R}_1$. We write $\mathcal{R}_1 = |A|^{\frac{1}{2}}|p|^{-\frac{2}{4}}$. Therefore, we can apply Cwikel's result for $p = 4,\ n = 2, f(x) = |A|^{\frac{1}{2}} \in L^4(\mathbb{R}^2)$ since $|A| \in L^2(\mathbb{R}^2)$ and $|x|^{-\frac{2}{4}} \in L^4_w(\mathbb{R}^2)$. Then, we obtain the compactness of $\mathcal{R}_1$, then $\mathcal{R}_1^*$. We also notice that since $\mathcal{V}$ is unitary, then

$$\mu_j(\mathcal{S}) = \mu_j(\mathcal{R}_1^* \mathcal{U} \mathcal{R}_1), \tag{2.32}$$

where the nonzero singular values are enumerated as usual. It also follows from Cwikel's result that

$$\mu_j(\mathcal{R}_1) \leq C_1 \left\{ \int_{\mathbb{R}^2} (|A|^{\frac{1}{2}})^4 \right\}^{\frac{1}{4}} j^{-\frac{1}{4}} = C_1 \|A\|_{L^2}^{\frac{1}{2}} j^{-\frac{1}{4}}, \tag{2.33}$$

where $C_1$ is independent of $j$ and $A$ as well. We observe that for any $q,\ 0 < q < 1$

$$\begin{aligned}
&\sum_{j=1}^{N} [\mu_j(\mathcal{R}_1^* \mathcal{U} \mathcal{R}_1)]^q \\
&= \sum_{j=1}^{N} |\lambda_j(\mathcal{R}_1^* \mathcal{U} \mathcal{R}_1)|^q \quad \text{since } \mathcal{R}_1^* \mathcal{U} \mathcal{R}_1 \text{ is self-adjoint} \\
&= \sum_{j=1}^{N} |\lambda_j(\mathcal{R}_1 \mathcal{R}_1^* \mathcal{U})|^q \quad \text{using } \lambda_j(AB) = \lambda_j(BA) \\
&\leq \sum_{j=1}^{N} [\mu_j(\mathcal{R}_1 \mathcal{R}_1^* \mathcal{U})]^q \quad \text{using the weak Weyl inequality (see [47], p.85)} \\
&= \sum_{j=1}^{N} [\mu_j(\mathcal{R}_1 \mathcal{R}_1^*)]^q \quad \text{since } \mathcal{U} \text{ is unitary} \\
&= \sum_{j=1}^{N} \lambda_j^q(\mathcal{R}_1 \mathcal{R}_1^*) \quad \text{since } \mathcal{R}_1 \mathcal{R}_1^* \geq 0 \\
&= \sum_{j=1}^{N} \lambda_j^q(\mathcal{R}_1^* \mathcal{R}_1) \quad \text{since } \lambda_j(AB) = \lambda_j(BA) \\
&= \sum_{j=1}^{N} [\mu_j(\mathcal{R}_1)]^{2q}.
\end{aligned}$$

Now the estimate above along with (2.31), (2.32) and (2.33) allows us to do exactly the same as in three dimensions and obtain the estimate for the case of two dimensions. ■

*Remark.* We also see that the arguments above also work for the case of three dimensions. But we have to write $\mathcal{R}_1 = |A|^{\frac{1}{2}}|p|^{-\frac{1}{2}} = |A|^{\frac{1}{2}}|p|^{-\frac{3}{6}}$. Then, we can apply Cwikel's result with $f(x) = |A|^{\frac{1}{2}} \in L^6(\mathbb{R}^3)$, $g(x) = |x|^{-\frac{3}{6}} \in L^6_w(\mathbb{R}^3)$ and obtained the compactness of $\mathcal{R}_1$ as well as the estimation $\mu_j(\mathcal{S}) \leq C_1\|A\|_{L^3} j^{-\frac{1}{3}}$.

*Remark.* In fact the additional assumption compared to the case of three dimensions that $|A| \in L^r(\mathbb{R}^2)$ for some $r > 2$ helps only to guarantee the Weyl-Dirac operators to be expressed in the operator sum as the result of Balinsky and Evans in [12]. We can use $L^2$ estimates on $|A|$ for the remainder of arguments above.

## 2.5 An estimate for the zero modes of massless Dirac operators

We now turn to the massless Dirac operator

$$\mathsf{T} := \alpha \cdot p + Q(x),\ x \in \mathbb{R}^3, \tag{2.34}$$

where $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ is the triple of $4 \times 4$ Dirac matrices

$$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix},\ j = 1, 2, 3,$$

with the $2 \times 2$ zero matrix $0_2$, and $Q(x)$ is a $4 \times 4$ Hermitian matrix-valued function. In mathematical physics we often meet the operators

$$\alpha \cdot (p - A(x)) + V(x)\mathbb{I}_4,$$

where $(V, A)$ is an electromagnetic potential and $\mathbb{I}_4$ is the $4 \times 4$ identity matrix. It is obvious that the family of these operators is a subset of the class of operators (2.34).

Zero modes and their properties for massless Dirac operators are investigated, for instance, in [49] and [50]. There, assuming that $Q(x)$ is Hermitian for each $x \in \mathbb{R}^3$ and each element $q_{jk}(x)$ for $j, k = 1, \ldots, 4$ of $Q(x)$ is measurable and satisfies

$$|q_{jk}(x)| \leq C(1 + |x|^2)^{-\frac{\rho}{2}},\ \text{for some } \rho > 1,$$

Saitō *et al.* show that if $f$ is a zero mode for the massless Dirac operator $\mathsf{T}$, then $f$ is a continuous function on $\mathbb{R}^3$ and satisfies

$$|f(x)| \leq C(1 + |x|^2)^{-1} \text{ for all } x \in \mathbb{R}^3.$$

Moreover, Saitō *et al.* also prove that

$$\lim_{r\to\infty} r^2 f(r\omega) = -\frac{i}{4\pi}\ (\alpha \cdot \omega) \int_{\mathbb{R}^3} Q(y) f(y)\, dy,$$

uniformly with respect to $\omega \in \mathbb{S}^2$.

Motivated by [49], [50] and [13] we want to obtain a similar estimate as we got in Theorem 2.3.6 for zero modes of massless Dirac operators. To do that we assume that $\|Q(\cdot)\|_4 \in L^3(\mathbb{R}^3)$, where $\|\cdot\|_4$ is any matrix norm on $4\times 4$ matrices. With this condition in [13] Balinsky *et al.* show that $Q$ is a small perturbation of $\alpha \cdot p$. Then, it follows from the Kato-Rellich theorem (see Theorem 1.11.2 ) that the operator $\mathsf{T}$ can be defined as the operator sum of $\alpha \cdot p$ and the multiplicative operator by $Q$. Moreover, we also know the domain of the self-adjoint $\mathsf{T}$ is $\left[H^1(\mathbb{R}^3)\right]^4$, the space of 4-component spinors in $\left[L^2(\mathbb{R}^3)\right]^4$ with first derivatives (in the distributional sense) in $\left[L^2(\mathbb{R}^3)\right]^4$. In this case we call a *zero mode* a four-component spinor $f \in \left[H^1(\mathbb{R}^3)\right]^4$ such that $\mathsf{T}f = 0$.

Balinsky *et al.* confirm in that case they obtain a similar result as the case of the Weyl-Dirac operator. That is the massless Dirac operators with scaled potential $\mathsf{T}_t := \alpha \cdot p + tQ,\ t \geq 0$ can have a zero mode for only a countable set values of $t$, while

$$\dim \operatorname{Ker} \mathsf{T} \leq C \int_{\mathbb{R}^3} \|Q(x)\|_4^3 \, dx. \tag{2.35}$$

Please refer to [13] for more details about their assertion.

Hereafter we will show the better estimate than (2.35). Specifically, let

$$n_Q(T) := \sum_{0\leq t\leq T} \dim \operatorname{Ker} \mathsf{T}_t.$$

Then, we have the following.

**Theorem 2.5.1.** *Let $\|Q(\cdot)\|_4$ be in $L^3(\mathbb{R}^3)$. For an arbitrary $T \geq 0$ there are only finite number of $t \in [0,\ T]$ such that $\dim \operatorname{Ker} \mathsf{T}_t \neq 0$. In addition we have the following estimate*

$$n_Q(T) \leq CT^3 \int_{\mathbb{R}^3} \|Q(x)\|_4^3 \, dx,$$

*where $C$ is independent of $T$ and $Q$.*

*Proof.* We use exactly the same argument as we did for Theorem 2.3.6, noticing that $(\alpha \cdot p)^2 = -\Delta \mathbb{I}_4 = |p|^2$ for brevity, where $\mathbb{I}_4$ is the $4\times 4$ identity matrix. ■

*Remark.* We remark that we in fact do not need the self-adjointness of $Q(\cdot)$ when we prove Theorem 2.5.1. However we need this property when showing $\mathsf{T}$ is the operator sum of $\alpha \cdot p$ and $Q$.

## 2.6 An estimate for the eigenfunctions of Dirac operators with positive mass at the threshold energies

In this section we consider the following operator with the vector potential $A$

$$\mathsf{H}_A := \alpha \cdot (p - A) + m\beta, \text{ where } m > 0 \text{ and } \beta = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix},$$

where $\mathbb{I}_2$ is the $2 \times 2$ identity matrix. We follow, for instance, Saitō and Umeda in [51] and call the above operator the *Dirac operator with positive mass.*

In case of $A = 0$ it is well-known that $\mathsf{H}_0$ is essentially self-adjoint on the dense domain $\left[C_0^\infty(\mathbb{R}^3 \setminus \{0\})\right]^4$ and self-adjoint on $\left[H^1(\mathbb{R}^3)\right]^4$. It is also classical that the spectrum of $\mathsf{H}_0$ is purely absolutely continuous and given by

$$\mathrm{Spec}(\mathsf{H}_0) = (-\infty, -m] \cup [m, \ \infty).$$

For instance, see [57], Theorem 1.1 for details.

In [51] while Saitō and Umeda study the Dirac operator with positive mass above, they propose the following condition, which has been used before by Balinsky and Evans in [12]. Here then, we will call it Assumption BE.

**Assumption BE**. Each element $A_j(x)$ is a real-valued measurable function satisfying

$$A_j \in L^3(\mathbb{R}^3).$$

With Assumption BE, Saitō and Umeda show that $\mathsf{H}_A$ is a relatively compact perturbation of the operator $\mathsf{H}_0$. The consequence of this is we can completely define $\mathsf{H}_A$ with the same domain as the one of $\mathsf{H}_0$ from the Kato-Rellich theorem. It also follows that the essential spectrum of $\mathsf{H}_A$ is $(-\infty, \ -m] \cup [m, \ \infty)$ and in the interval $(-m, \ m)$ there exists only discrete spectrum for $\mathsf{H}_A$. So, we can call $\pm m$ *the threshold energies* for $\mathsf{H}_A$.

One of Saitō and Umeda's interests in [51] is the estimate of the dimension for the eigenspaces of the Dirac operators with positive mass at eigenvalues $\pm m$. They consider the class $\mathcal{A}$ of potentials, proposed by Elton in [22]

$$\mathcal{A} = \{A : \ A_j(x) \in C^0(\mathbb{R}^3, \ \mathbb{R}), \ A_j(x) = o(|x|^{-1}) \text{ as } |x| \to +\infty\}.$$

It is not hard to show that $\mathcal{A}$ is a Banach space with the norm $\|A\| = \|(1+|x|)A\|_{L^\infty}$ and $C_0^\infty$ is a dense subspace of $\mathcal{A}$. Then, with such class of potentials $\mathcal{A}$, Saitō and Umeda obtain 'similar' results (to Elton's results in [22]) for $\dim E_{\pm m}(\mathsf{H}_A)$, where $E_{\pm m}(\mathsf{H}_A)$ the eigenspaces of the Dirac operators $\mathsf{H}_A$ at the threshold eigenvalues $\pm m$: (1) The subsets of potentials in $\mathcal{A}$ in which the eigenspaces at eigenvalues $\pm m$

for the corresponding operators $\mathsf{H}_A$ have dimension of $k$ are the same (2) The subset of potentials in $\mathcal{A}$ such that the corresponding $\mathsf{H}_A$ have trivial kernel is open and dense in $\mathcal{A}$ (3) For any non-negative $k$ and any arbitrary open nonempty subset $\Omega$ of $\mathbb{R}^3$ we can find find a smooth potential $A \in \mathcal{A}$ with compact support in $\Omega$ such that the dimensions of eigenspaces at eigenvalues $\pm m$ for $\mathsf{H}_A$ are $k$.

To combine with the results of Balinsky and Evans in [12] Saitō and Umeda also obtain that

- the subset of potentials in $[L^3(\mathbb{R}^3)]^3$ such that the corresponding Dirac operators with positive mass have nontrivial eigenspaces at $\pm m$ is 'sparse'
- there is a constant $C$, which is independent of potentials $A$ such that

$$\dim E_{\pm m}(\mathsf{H}_A) \leq C \int_{\mathbb{R}^3} |A(x)|^3 \, dx.$$

Let

$$n_A(T, \pm m) := \sum_{0 \leq t \leq T} \dim E_{\pm m}(\mathsf{H}_{tA}).$$

Our result here is stronger; that is

**Theorem 2.6.1.** *For arbitrary $T \geq 0$ there are finitely many $t \in [0,\ T]$ such that the eigenspaces $E_{\pm m}(\mathsf{H}_{tA})$ at $\pm m$ for the corresponding Dirac operator with positive mass $\mathsf{H}_{tA}$ is nontrivial. Futhermore,*

$$n_A(T, \pm m) \leq C T^3 \|A\|_{L^3}^3,$$

*and the constant $C$ is independent of $T$ and the potential $A$.*

*Proof.* It follows from [57], Theorem 7.1 that $\mathsf{H}_{tA}\Psi = m\Psi$ if and only if $\Psi = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$ with $\sigma \cdot (p - tA)\psi_1 = 0$. That means $\psi_1$ is a zero mode for the Weyl-Dirac operator $\mathcal{D}_{tA} = \sigma \cdot (p - tA)$. Now we can apply the result of Theorem 2.3.6 and obtain one conclusion above (for $n_A(T, m)$). We repeat the arguments and obtain the remainder. ∎

# Chapter 3

# Dirac operators on $\mathbb{S}^2$

## 3.1 Introduction

In [24] Erdös and Solovej showed a geometric way to study zero modes for $\mathcal{D}_A = \sigma \cdot (-i\nabla - A)$ on $\mathbb{R}^3$ through studying the equivalent problem on the 3-sphere $\mathbb{S}^3$. They gave a family of magnetic fields on $\mathbb{S}^3$ for which they could characterise the spectrum and in some special cases they calculated the dimension of the kernel for the Dirac operator on $\mathbb{S}^3$. Then, based on the conformal equivalence of $\mathbb{R}^3$ to the 3-sphere with a point removed, they gave results about the kernel of $\mathcal{D}_A$ on $\mathbb{R}^3$. However to understand the problem on $\mathbb{S}^3$ we need information about the spectrum of related Dirac operators with magnetic field on $\mathbb{S}^2$. To define a Dirac operator with magnetic field on $\mathbb{S}^2$, or more generally on a manifold, we need Spin$^c$ structures; these are comprised of a Spin$^c$ spinor bundle and a Spin$^c$ connection. These are special cases of vector bundles and connections from differential geometry.

We will introduce Spin$^c$ structures (Spin$^c$ spinor bundles and Spin$^c$ connections) on the unit ball $\mathbb{S}^2$ of $\mathbb{R}^3$, so that we can then construct the Dirac operators with magnetic fields on $\mathbb{S}^2$. We also consider Spin$^c$ structures for $\mathbb{S}^3$ as well as Dirac operators on $\mathbb{S}^3$ with magnetic fields. We will consider a specific class of magnetic fields ($t\mathrm{vol}_{\mathbb{S}^2}$) on $\mathbb{S}^2$ and show explicitly spectrum of the corresponding Dirac operators. We also give a proof of the Aharonov-Casher theorem for $\mathbb{S}^2$. We have already discussed the version of this theorem for $\mathbb{R}^2$ and in fact there are some proofs for the version on $\mathbb{S}^2$; however we will give a proof which reduces the problem to determining the dimension of the kernels of Dirac operators on $\mathbb{S}^2$ which correspond to constant magnetic fields.

## 3.2 Spin$^c$ structures

We will firstly introduce some general concepts from differential geometry which we need later in the thesis. The second part of this section is devoted to the construction of Spin$^c$ structures for $\mathbb{S}^2$.

### 3.2.1 Generalities

We first remind of some concepts and notations considered on a manifold and for convenience we take $M$ to show all. We mean here $M$ is an *$n$-dimensional differentiable manifold.* The main source for this comes from [41] and [46].

We denote by $T_pM$ the tangent vector space at the point $p$ on $M$. We also use $TM$ to denote the tangent bundle over $M$; this is the union of the tangent spaces at each point on $M$. A vector field $X$ is a section of the tangent bundle $TM$; that is, $X$ is a map $M \longrightarrow TM$ which sends a point $p$ on $M$ to a tangent vector in the tangent space $T_pM$ at that point. We denote by $\Gamma(TM)$ the collection of all vector fields defined on $M$. A metric $g$ is an assignment of an inner product $g_p$ to the vector space $T_pM$ for each $p \in M$. The pair $(M, g)$ is called a *Riemannian manifold.*

For any differentiable function $f : M \longrightarrow \mathbb{R}$ and vector field $X \in \Gamma(TM)$ we denote by $Xf$ the derivative of the function $f$ along the vector field $X$; we can think of $Xf(p)$ as the derivative of $f$ at the point $p$ in the direction of $X_p$.

Take $X, Y \in \Gamma(TM)$. We define the *Lie bracket* of $X, Y$ to be the vector field, denoted by $[X, Y]$, such that for any differentiable function $f : M \longrightarrow \mathbb{R}$ we have

$$[X, Y]f = X(Yf) - Y(Xf).$$

If we have a set of local coordinates $(x_1, x_2, \dots, x_n)$ (defined on an open subset $U$ of $M$) we define corresponding *coordinate vector fields* $X_{x_1}, \dots, X_{x_n}$ by

$$X_{x_i} x_j = \delta_{ij}, \quad i, j = 1, \dots, n.$$

*Remark* 3.2.1. Suppose that $f : M \longrightarrow \mathbb{R}$ is a differentiable function. If we consider the restriction of $f$ to the open subset $U$ to be a function of $(x_1, x_2, \dots, x_n)$ we have

$$X_{x_j} f(x_1, x_2, \dots, x_n) = \frac{\partial f}{\partial x_j}(x_1, x_2, \dots, x_n)$$

the $j$-partial derivative of $f$. For this reason the notation $\dfrac{\partial}{\partial x_j}$ is often used for $X_{x_j}$.

To define spinors on an arbitrary manifold $M$ we need to generalise the idea of functions on $M$ taking values in some vector space $\mathcal{V}$ (for spinors on $\mathbb{R}^2$ or $\mathbb{R}^3$ we have $\mathcal{V} = \mathbb{C}^2$). We first introduce a "twisted" version of $M \times \mathcal{V}$ called a *vector bundle.* We may start with a set of charts $U_j$ for $M$. On the union $U_j \times \mathcal{V}$ we consider an equivalence relation $\sim$ between $(p,\ \psi) \in U_j \times \mathcal{V}$ and $(p',\ \psi') \in U_k \times \mathcal{V}$ by $(p,\ \psi) \sim (p',\ \psi')$ if and only if $p = p'$ and $\psi = t_{jk}\psi'$ where given *transitions* $t_{jk} : U_j \cap U_k \longrightarrow GL(\mathcal{V})$ is a smooth *transition map* which satisfies the following conditions;

$$\begin{aligned}
&t_{jj}(p) = I - \quad \text{the identity,}\\
&t_{jk}(p) = t_{kj}^{-1}(p), \quad p \in U_j \cap U_k,\\
&t_{jk}(p)t_{kl}(p) = t_{jl}(p), \quad p \in U_j \cap U_k \cap U_l.
\end{aligned}$$

Then, we will obtain a *vector bundle* $E$ with fibre $\mathcal{V}$; that is $\cup_j U_j \times \mathcal{V}/\sim$. The mapping $\pi : E \longrightarrow M$ given by $\pi(p,\ \psi) = p$ is called the *projection* for this vector bundle. For $p \in M$, then $\pi^{-1}(p)$ is a vector space isomorphic to $\mathcal{V}$; it is called the *fibre* of $E$ at $p$. A *section* $s$ is a smooth map $M \longrightarrow E$ such that $\pi s(p) = p$ for all $p \in M$. Sections in fact generalise the idea of $\mathcal{V}$-valued functions on $M$. The set of all sections on $M$ is often denoted by $\Gamma(M)$.

*Remark* 3.2.2. If $M$ is an $n$-dimensional manifold, then the tangent bundle and cotangent bundle are vector bundles with fibre $\mathbb{R}^n$.

In general there is no generic way of associating vectors in different fibres of a vector bundle; the extra information needed to do this is given by a *connection.* A connection $\nabla$ on a bundle $E$ is a map $\Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$ satisfying

- $\nabla_{fX} s = f \nabla_X s$
- $\nabla_X(fs) = (Xf)s + f\nabla_X s \quad$ for all $X, Y \in \Gamma(TM),\ s \in \Gamma(E)$ and functions $f$
- $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s.$

In case the bundle $E$ is the tangent bundle $TM$ for a Riemannian manifold $(M, g)$ we will consider a special connection. First, we already know that for vector fields $X$ and $Y$, then the Lie bracket $[X,\ Y] = XY - YX$ is a vector field. A connection is called *Torsion-free* if $\nabla_X Y - \nabla_Y X = [X,\ Y]$. The fundamental theorem of Riemannian manifolds guarantees that there is a unique connection for $M$, which is Torison-free and is *compatible* with a given Riemannian metric $g$ in the sense that

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

This connection is called the *Levi-Civita connection* on the Riemannian manifold $(M, g)$.

Since $T_pM$ at each point $p \in M$ is a vector space, there is a dual vector space to $T_pM$. This dual space is called the *cotangent space* at that point; we use $T_p^*M$ to denote this cotangent space. Each element in $T_p^*M$ is called a cotangent vector and the union of all cotangent spaces is called the *contangent bundle*, denoted by $T^*M$. A section of $T^*M$ is called a *one-form.* On a Riemannian manifold $(M, g)$, there is a natural dual connection of the Levi-Civita one on the cotangent bundle $T^*M$, which will also be called the *Levi-Civita connection on one-forms.*

Generalising the construction of the cotangent space, we can consider $\wedge^r T_p^*M$, the space of totally antisymmetric $r$-linear maps on $T_pM$. The union of all these is denoted by $\wedge^r T^*M$ and sections of this bundle are called *$r$-forms.* The notation $\Omega^r(M)$ is used for the space of all $r$-forms on $M$. Note that $\Omega^1(M) = \Gamma(T^*M)$.

Suppose that $M$ and $N$ are manifolds, and $f : M \longrightarrow N$ is a differentiable map. The corresponding *differential map* $f_*$ is a linear map from $T_pM$ to $T_{f(p)}N$ for each

$p \in M$. Moreover, the map $f$ also induces a *pullback* on $r$-forms; this is the map $f^*$ from $\Omega^r(N)$ to $\Omega^r(M)$ given by

$$(f^*\omega)(X_1, X_2, \ldots, X_r) = \omega\big(f_*X_1, f_*X_2, \ldots, f_*X_r\big),$$

where $\omega \in \Omega^r(N)$ and $X_1, X_2, \ldots, X_r \in \Gamma(TM)$.

Suppose that $M$ is a Riemannian $n$-manifold and $U$ is an open subset of $M$. An *orthonormal frame* is a set of vector fields $\{\widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_n\}$ defined on $U$ which provides an orthonormal basis for $T_pM$ at each point $p \in U$. The corresponding dual one-forms are denoted by $\{\widehat{e}^1, \widehat{e}^2, \ldots, \widehat{e}^n\}$; that is

$$\widehat{e}^j(\widehat{e}_k) = \delta_{jk}, \quad j, k = 1, 2, \ldots, n.$$

This set provides an orthonormal basis for $T_p^*M$ for each point $p \in U$.
We can define $r$-forms

$$\widehat{e}^{j_1} \wedge \widehat{e}^{j_2} \wedge \cdots \wedge \widehat{e}^{j_r}, \quad 1 \leq j_1 < j_2 < \cdots < j_r \leq n$$

by

$$(\widehat{e}^{j_1} \wedge \widehat{e}^{j_2} \wedge \cdots \wedge \widehat{e}^{j_r})(\widehat{e}_{k_1}, \widehat{e}_{k_2}, \ldots, \widehat{e}_{k_r}) = \delta_{j_1k_1}\delta_{j_2k_2} \ldots \delta_{j_rk_r}, \text{ for } 1 \leq k_1 < k_2 \cdots < k_r \leq n.$$

Then, the family $\{\widehat{e}^{j_1} \wedge \widehat{e}^{j_2} \wedge \cdots \wedge \widehat{e}^{j_r}, \quad 1 \leq j_1 < j_2 < \cdots < j_r \leq n\}$ provides an orthonormal basis for $r$-forms (defined on $U$).

*Remark* 3.2.3. Actually, $\widehat{e}^{j_1} \wedge \widehat{e}^{j_2} \wedge \cdots \wedge \widehat{e}^{j_r}$ is the *wedge product* or *exterior product* of $\widehat{e}^{j_1}, \widehat{e}^{j_2}, \ldots, \widehat{e}^{j_r}$.

Let $(M, g)$ be an oriented Riemannian manifold. Suppose that $\widehat{e}^1, \widehat{e}^2, \ldots, \widehat{e}^n$ is an oriented orthonormal frame for the cotangent space $T^*M$. Then, we can define the *volume form* $\mathrm{vol}_M := \widehat{e}^1 \wedge \widehat{e}^2 \wedge \cdots \wedge \widehat{e}^n$. If the manifold $M$ is compact we may integrate the volume form over $M$ to obtain the "usual" volume of $M$. Later we will need the *Hodge star operator* $*$ which turns a $k$-form into an $n-k$-form. The Hodge star operator is linear and therefore may be defined on the basis elements $\widehat{e}^{j_1} \wedge \widehat{e}^{j_2} \wedge \cdots \wedge \widehat{e}^{j_k}$, $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ of $\Omega^k(M)$; here we set

$$*(\widehat{e}^{j_1} \wedge \widehat{e}^{j_2} \wedge \cdots \wedge \widehat{e}^{j_k}) = \widehat{e}^{l_1} \wedge \cdots \wedge \widehat{e}^{l_{n-k}},$$

where $\widehat{e}^{j_1} \wedge \cdots \wedge \widehat{e}^{j_k} \wedge \widehat{e}^{l_1} \wedge \cdots \wedge \widehat{e}^{l_{n-k}} = \mathrm{vol}_M$.

For a manifold $M$ there is a natural differential operator $d$, called the *exterior derivative*, which takes $r$-forms to $r+1$-forms. This can be defined as the unique linear operator $\Omega^r(M) \longrightarrow \Omega^{r+1}(M)$, $r = 0, 1, 2, \ldots$ which satisfies the following conditions

- If $f \in \Omega^0(M)$ (that is $f$ is a function), then $df$ is the one-form given by $df(X) = Xf$ for any vector field $X$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{r_1}\alpha \wedge d\beta \quad$ for all $\alpha \in \Omega^{r_1}(M), \beta \in \Omega^{r_2}(M)$
- $d^2\alpha = 0$ for all $\alpha \in \Omega^r(M)$.

### 3.2.2 Spin$^c$ structures

Spinors on a manifold will be defined as sections of a particular type of bundle which is known as *Spin$^c$ spinor bundle*

**Definition 3.2.4.** ([24]) A Spin$^c$ spinor bundle $\Psi$ over a three dimensional Riemannian manifold $M$ is a 2-dimensional complex vector bundle over $M$ with inner product and an isometry $\sigma : \; T^*M \longrightarrow \Psi^{(2)}$, where $\Psi^{(2)} := \{X \in \mathrm{End}(\Psi) : \; X = X^*, \; \mathrm{Tr}(X) = 0\}$.

A Spin$^c$ spinor bundle $\Psi$ over a two dimensional Riemannian manifold is defined almost in the same way as for three dimensional case except that the map $\sigma$ is only required to be a partial injective isometric.

The map $\sigma$ is called the *Clifford multiplication* of the spinor bundle $\Psi$. On $\Psi^{(2)}$ we use $\langle X, \; Y\rangle = \dfrac{1}{2}\,\mathrm{Tr}(XY)$ as the inner product between $X, \; Y \in \Psi^{(2)}$. We may check that $XY + YX = \mathrm{Tr}(XY)\; \mathbb{I}_2 = 2\langle X, Y\rangle \mathbb{I}_2$ for any $X, \; Y \in \Psi^{(2)}$. Therefore, for any $\alpha, \; \beta \in T^*M$ we have

$$\sigma(\alpha)\sigma(\beta) + \sigma(\beta)\sigma(\alpha) = 2\langle \sigma(\alpha), \; \sigma(\beta)\rangle \mathbb{I}_2 = 2\langle \alpha, \; \beta\rangle \mathbb{I}_2, \tag{3.1}$$

where the last equality holds since $\sigma$ is an isometry.

For brevity, in this thesis Spin$^c$ spinor bundles will often be called the *spinor* bundles.

*Remark* 3.2.5. In the simple case where $M$ is $\mathbb{R}^3$, a Spin$^c$ spinor bundle is given by the trivial complex vector bundle $\mathbb{R}^3 \times \mathbb{C}^2$ with Clifford multiplication $\sigma$ defined by $\sigma(\widehat{e}^j) = \sigma_j$, where $\{\widehat{e}^1, \widehat{e}^2, \widehat{e}^3\}$ is the standard orthonormal basis for one-forms on $\mathbb{R}^3$ and $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices.

For Spin$^c$ spinor bundle the connection we need is called a *Spin$^c$ connection.* These are defined as follows.

**Definition 3.2.6.** ([24]) A connection $\nabla$ on a spinor bundle $\Psi$ over $M$ is called a *Spin$^c$ connection* if for all vector fields $X \in \Gamma(TM)$ we have

- $X\langle \xi, \; \eta\rangle = \langle \nabla_X \xi, \; \eta\rangle + \langle \xi, \; \nabla_X \eta\rangle$ for all spinor sections $\xi, \eta$
- $[\nabla_X, \; \sigma(\alpha)] = \sigma(\nabla_X \alpha)$ for all one-forms $\alpha$ on $M$.

We notice that in Definition 3.2.6 above $\nabla_X \alpha$ means the Levi-Civita connection acting on one-forms.

Now we set up the concept of magnetic fields for connections on a spinor bundle $\Psi$ over $M$.

**Definition 3.2.7.** The *curvature tensor* of the Spin$^c$ connection $\nabla$ is defined as

$$\mathcal{R}_\Psi(X,\ Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,\ Y]}\xi,$$

where $\xi$ is a spinor section and $X,\ Y$ are vector fields on $M$. Then, the *magnetic field* $\beta$ is defined to be the two-form given by

$$\beta(X,\ Y) = \frac{i}{2}\operatorname{Tr}[\mathcal{R}_\Psi(X,\ Y)] \quad \text{for all vector fields } X, Y. \tag{3.2}$$

*Remark* 3.2.8. Although magnetic fields are most naturally defined on manifolds as two-forms this does not amount to a change in view point when working on $\mathbb{R}^2$ or $\mathbb{R}^3$ (where magnetic fields we previously consider as scalar functions and vector fields respectively; see (1.10) and (1.1)).
Firstly, consider the case $M = \mathbb{R}^2$. Let $\{\widehat{e}_1, \widehat{e}_2\}$ be the usual orthonormal basis for $T\mathbb{R}^2$ and $\{\widehat{e}^1, \widehat{e}^2\}$ the dual orthonormal basis for $T^*\mathbb{R}^2$. A Spin$^c$ spinor bundle is given by $\Psi = \mathbb{R}^2 \times \mathbb{C}^2$ (the trivial bundle) with Clifford multiplication $\sigma$ given by $\sigma(\widehat{e}^j) = \sigma_j,\ j = 1, 2$-the Pauli matrices; that means spinors are simply $\mathbb{C}^2$-valued functions on $\mathbb{R}^2$. A Spin$^c$ connection can be defined by setting $\nabla_{\widehat{e}_j} = \partial_j - iA_j,\ j = 1, 2$, where $A_1, A_2$ are $\mathbb{R}$-valued functions on $\mathbb{R}^2$; we can put these functions together to give the magnetic potential $A_1\widehat{e}^1 + A_2\widehat{e}^2$ which is now viewed as a one-form. Now, observe that $[\widehat{e}_1, \widehat{e}_2] = 0$ since $[\widehat{e}_1, \widehat{e}_2]f = (\partial_1\partial_2 - \partial_2\partial_1)f = 0$ for any smooth function $f$ on $\mathbb{R}^2$. Then, we have

$$\begin{aligned}\mathcal{R}_\Psi(\widehat{e}_1,\ \widehat{e}_2)\xi &= (\nabla_{\widehat{e}_1}\nabla_{\widehat{e}_2} - \nabla_{\widehat{e}_2}\nabla_{\widehat{e}_1})\xi\\ &= [(\partial_1 - iA_1)(\partial_2 - iA_2) - (\partial_2 - iA_2)(\partial_1 - iA_1)]\xi\\ &= -i(\partial_1 A_2 - \partial_2 A_1)\xi.\end{aligned}$$

Thus, the magnetic field $\beta$ of Definition 3.2.7 is given by $\beta(\widehat{e}_1, \widehat{e}_2) = \frac{i}{2}\operatorname{Tr}\mathcal{R}_\Psi(\widehat{e}_1,\ \widehat{e}_2) = \partial_1 A_2 - \partial_2 A_1$, or

$$\beta = (\partial_1 A_2 - \partial_2 A_1)\widehat{e}_1 \wedge \widehat{e}_2;$$

notice that $\partial_1 A_2 - \partial_2 A_1$ is the scalar function we previously considered as the magnetic field.
There is a similar version for $\mathbb{R}^3$ and we obtain the magnetic field

$$B_{23}\widehat{e}^2 \wedge \widehat{e}^3 + B_{31}\widehat{e}^3 \wedge \widehat{e}^1 + B_{12}\widehat{e}^1 \wedge \widehat{e}^2$$

for a given magnetic potential $A = A_1\widehat{e}^1 + A_2\widehat{e}^2 + A_3\widehat{e}^3$; here $B_{23} = \partial_2 A_3 - \partial_3 A_2$, $B_{31} = \partial_3 A_1 - \partial_1 A_3$ and $B_{12} = \partial_1 A_2 - \partial_2 A_1$ are the components of the magnetic field $(B_{23}, B_{31}, B_{12})$ considered as a vector field.

### 3.2.3 Spin$^c$ structures on $\mathbb{S}^2$

The equation for the unit ball $\mathbb{S}^2$ in $\mathbb{R}^3$ is $x_1^2 + x_2^2 + x_3^2 = 1$. Denote $\mathbb{S}^2 \backslash \{(0, 0, -1)\}$ and $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$ by $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ respectively. Each point $p$ in $\mathbb{S}^2$ with the usual Cartesian coordinates $(x_1, x_2, x_3)$ is characterised by a pair $(\theta, \phi)$ in spherical coordinates, where

$$\phi = \begin{cases} \tan^{-1}\dfrac{x_2}{x_1} & \text{if } x_1 > 0 \text{ and } x_2 \geq 0 \\[2ex] \tan^{-1}\dfrac{x_2}{x_1} + 2\pi & \text{if } x_1 > 0 \text{ and } x_2 < 0 \\[2ex] \tan^{-1}\dfrac{x_2}{x_1} + \pi & \text{if } x_1 < 0 \\[2ex] \dfrac{\pi}{2} & \text{if } x_1 = 0 \text{ and } x_2 > 0 \\[2ex] \dfrac{3\pi}{2} & \text{if } x_1 = 0 \text{ and } x_2 < 0 \\[2ex] 0 & \text{if } x_1 = x_2 = 0, \end{cases}$$

and $\theta = \cos^{-1} x_3$. We notice that $\phi \in [0,\ 2\pi)$ and $\theta \in [0,\ \pi]$. For brevity we will denote $\sin\theta,\ \cos\theta,\ \sin\phi,\ \cos\phi$ and $e^{i\phi}$ by $s,\ c,\ s_\phi,\ c_\phi$ and $\omega$, respectively.

Now we will construct spinor bundles on $\mathbb{S}^2$. For each $n \in \mathbb{Z}$ we define a spinor bundle $\Psi_n$ on $\mathbb{S}^2$ as follows: $\Psi_n = \Psi_n^+ \cup \Psi_n^- / \sim$, where $\Psi_n^+ = \mathbb{S}_+^2 \times \mathbb{C}^2$, $\Psi_n^- = \mathbb{S}_-^2 \times \mathbb{C}^2$, and $\sim$ is an equivalence relation between $(p, \xi^+) \in \Psi_n^+$ and $(p, \xi^-) \in \Psi_n^-$ given by $(p, \xi^+) \sim (p, \xi^-) \iff \xi^- = U_n \xi^+$, in which

$$U_n = \begin{pmatrix} \omega^{-n+1} & 0 \\ 0 & \omega^{-n-1} \end{pmatrix} = \omega^{-n} \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} = \omega^{-n} W,$$

where

$$W = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \in SU(2).$$

Thus, for each $p \in \mathbb{S}_+^2 \cap \mathbb{S}_-^2$ the transition map

$$\begin{aligned} (\mathbb{S}_+^2 \cap \mathbb{S}_-^2) \times \mathbb{C}^2 &\longrightarrow (\mathbb{S}_+^2 \cap \mathbb{S}_-^2) \times \mathbb{C}^2 \\ (p, \xi^+) &\longmapsto (p, \xi^-), \end{aligned}$$

is given by $\xi^- = U_n \xi^+$.

Denote by $\{\widehat{e}_\theta, \widehat{e}_\phi\}$ the orthonormal basis on $T\mathbb{S}^2$ with $\widehat{e}_\theta = X_\theta, \widehat{e}_\phi = \dfrac{1}{s} X_\phi$, where $X_\theta, X_\phi$ are the coordinate vector fields (see Remark 3.2.1). We observe that the Lie bracket of $\widehat{e}_\theta$ and $\widehat{e}_\phi$ is $-\dfrac{c}{s}\widehat{e}_\phi$; indeed

$$[\widehat{e}_\theta, \widehat{e}_\phi] = [X_\theta, \frac{1}{s} X_\phi] = \partial_\theta(\frac{1}{s}\partial_\phi \cdot) - \frac{1}{s}\partial_\phi(\partial_\theta \cdot) = -\frac{c}{s^2}\partial_\phi = -\frac{c}{s}\widehat{e}_\phi. \tag{3.3}$$

The dual is given by $\widehat{e}^\theta = d\theta$, $\widehat{e}^\phi = sd\phi$ and $\{\widehat{e}^\theta,\ \widehat{e}^\phi\}$ is an orthonormal basis for $T^*\mathbb{S}^2$. The volume form for $\mathbb{S}^2$ is $\mathrm{vol}_{\mathbb{S}^2} = \widehat{e}^\theta \wedge \widehat{e}^\phi = sd\theta \wedge d\phi$. Applying the Hodge star operator we obtain, for instance

$$*1 = \widehat{e}^\theta \wedge \widehat{e}^\phi,\ \ * \widehat{e}^\theta = \widehat{e}^\phi,\ \ * \widehat{e}^\phi = -\widehat{e}^\theta,\ \ * (\widehat{e}^\theta \wedge \widehat{e}^\phi) = 1.$$

To define the Clifford multiplication $\sigma$ for this spinor bundle $\Psi_n$ we set

$$\sigma(\widehat{e}^\theta) = \sigma^\theta,\ \sigma(\widehat{e}^\phi) = \sigma^\phi,$$

where

$$\sigma^\theta = \sigma^\theta_+ := \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix},\ \sigma^\phi = \sigma^\phi_+ := \begin{pmatrix} 0 & -i\omega^{-1} \\ i\omega & 0 \end{pmatrix}$$

for $\Psi_n^+$, and

$$\sigma^\theta = \sigma^\theta_- := \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix},\ \sigma^\phi = \sigma^\phi_- := \begin{pmatrix} 0 & -i\omega \\ i\omega^{-1} & 0 \end{pmatrix}$$

for $\Psi_n^-$.

Since the Clifford multiplication has been defined using local trivilisations we still need to check the "compatibility" between the definitions on $\Psi_n^+$ and $\Psi_n^-$. That means at any point in the overlap of $\mathbb{S}_+$ and $\mathbb{S}_-$ the transition map between $\Psi_n^+$ and $\Psi_n^-$ must commute with the Clifford multiplication. To check this firstly note that

$$U_n \sigma^\theta_+ U_n^* = \begin{pmatrix} \omega^{-n+1} & 0 \\ 0 & \omega^{-n-1} \end{pmatrix} \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \omega^{n-1} & 0 \\ 0 & \omega^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix} = \sigma^\theta_-$$

and,

$$U_n \sigma^\phi_+ U_n^* = \begin{pmatrix} \omega^{-n+1} & 0 \\ 0 & \omega^{-n-1} \end{pmatrix} \begin{pmatrix} 0 & -i\omega^{-1} \\ i\omega & 0 \end{pmatrix} \begin{pmatrix} \omega^{n-1} & 0 \\ 0 & \omega^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -i\omega \\ i\omega^{-1} & 0 \end{pmatrix} = \sigma^\phi_-.$$

It follows that if $\xi^- = U_n \xi^+$, then

$$\sigma^\theta_- \xi^- = U_n \sigma^\theta_+ U_n^* \xi^- = U_n \sigma^\theta_+ \xi^+.$$

Similarly, $\sigma^\phi_- \xi^- = U_n \sigma^\phi_+ \xi^+$, so $\sigma^\theta$ and $\sigma^\phi$ are well defined on $\Psi_n$. In the metric on $\Psi_n^{(2)}$ we have $\langle \sigma^\theta,\ \sigma^\theta \rangle = \frac{1}{2}\mathrm{Tr}\ (\sigma^\theta)^2 = \frac{1}{2}\mathrm{Tr}\ \mathbb{I}_2 = 1$ etc. Thus, $\Psi_n$ and $\sigma$ are well defined and satisfy the conditions in Definition 3.2.4. Therefore we obtain a spinor bundle $\Psi_n$ for each $n \in \mathbb{Z}$.
We remark that

$$-i\sigma^\theta \sigma^\phi = -i\sigma^\theta_\pm \sigma^\phi_\pm = \sigma_3. \tag{3.4}$$

Next we will furnish the spinor bundle $\Psi_n$ with a Spin$^c$ connection $\widetilde{\nabla}$. Since $\{\widehat{e}_\theta,\ \widehat{e}_\phi\}$ is a basis for the tangent vector space $T\mathbb{S}^2$ we need only to define $\widetilde{\nabla}$ for this basis.

Hereafter we will use the notation $\widetilde{\nabla}_\theta$ for $\widetilde{\nabla}_{\widehat{e}_\theta}$ and $\widetilde{\nabla}_\phi$ for $\widetilde{\nabla}_{\widehat{e}_\phi}$ (and similarly for other connections to follows).

We begin by defining a particular connection $\widetilde{\nabla}$ on $\Psi_n$ by setting

$$\widetilde{\nabla}_\theta = \widetilde{\nabla}^\pm_\theta = \partial_\theta, \tag{3.5}$$

and

$$\widetilde{\nabla}_\phi = \widetilde{\nabla}^\pm_\phi = \frac{1}{s}\partial_\phi + \frac{is}{2(c\pm 1)}\sigma_3 - \frac{isn}{2(c\pm 1)} = \frac{1}{s}\partial_\phi - \frac{is}{2(c\pm 1)}\left(n - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{3.6}$$

on $\Psi^\pm_n$. Once again we must check compatibility.

Firstly, since $U_n$ is independent of $\theta$, we immediately get $U_n\partial_\theta U_n^* = \partial_\theta$, so $U_n\widetilde{\nabla}^+_\theta U_n^* = \widetilde{\nabla}^-_\theta$. On the other hand

$$\begin{aligned} U_n\partial_\phi U_n^* &= \partial_\phi + \begin{pmatrix} \omega^{-n+1} & 0 \\ 0 & \omega^{-n-1} \end{pmatrix}\begin{pmatrix} i(n-1)\omega^{n-1} & 0 \\ 0 & i(n+1)\omega^{n+1} \end{pmatrix} \\ &= \partial_\phi + i\begin{pmatrix} n-1 & 0 \\ 0 & n+1 \end{pmatrix}. \end{aligned}$$

Then, observing that $U_n\sigma_3U_n^* = \sigma_3$, we get

$$\begin{aligned} U_n\widetilde{\nabla}^+_\phi U_n^* &= \frac{1}{s}U_n\partial_\phi U_n^* - \frac{is}{2(c+1)}U_n\left(n - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)U_n^* \\ &= \frac{1}{s}\partial_\phi + \frac{i}{s}\begin{pmatrix} n-1 & 0 \\ 0 & n+1 \end{pmatrix} - \frac{is}{2(c+1)}\left(n - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \frac{1}{s}\partial_\phi + i\left(\frac{1}{s} - \frac{s}{2(c+1)}\right)\left(n - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \frac{1}{s}\partial_\phi - \frac{is}{2(c-1)}\left(n - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \widetilde{\nabla}^-_\phi. \end{aligned}$$

Thus, for any tangent vector field $X$, we have $U_n\widetilde{\nabla}^+_X U_n^* = \widetilde{\nabla}^-_X$. Hence

$$\widetilde{\nabla}^-_X\xi^- = U_n\widetilde{\nabla}^+_X U_n^*\xi^- = U_n\widetilde{\nabla}^+_X\xi^+,$$

whenever $\xi^- = U_n\xi^+$.
Therefore $\widetilde{\nabla}^\pm_X$ can be used to define a connection on the spinor bundle $\Psi_n$. To become a Spin$^c$ connection on $\Psi_n$, $\widetilde{\nabla}$ must satisfy two conditions in Definition 3.2.6 as well.

Before we check these two conditions for $\widetilde{\nabla}$ we need an expression for the Levi-Civita connection on one-forms on $\mathbb{S}^2$; this standard calculation can be conveniently summarised by the formulae (see (7.14),(7.57) and (7.25) in [41], for example)

$$\nabla_\theta\widehat{e}^\theta = \nabla_\theta\widehat{e}^\phi = 0, \quad \nabla_\phi\widehat{e}^\theta = \frac{c}{s}\widehat{e}^\phi \quad \text{and } \nabla_\phi\widehat{e}^\phi = -\frac{c}{s}\widehat{e}^\theta.$$

Since the two conditions in Definition 3.2.6 are linear in $X$ and $\alpha$ we need only check them for $X = \widehat{e}_\theta,\ \widehat{e}_\phi$ and $\alpha = \widehat{e}^\theta,\ \widehat{e}^\phi$.

The first condition follows easily from the fact that $\widehat{e}_\theta = X_\theta$ and $\widehat{e}_\phi = \dfrac{1}{s} X_\phi$ (recall Remark 3.2.1), while $\dfrac{isn}{2(c \pm 1)}$ and $\dfrac{is}{2(c \pm 1)}\sigma_3$ are anti-hermitian.

Now we will check the second condition for convenience).
Since $\omega = e^{i\phi}$ is independent of $\theta$ we get

$$[\widetilde{\nabla}_\theta,\ \sigma^\theta] = 0 = \sigma(\nabla_\theta \widehat{e}^\theta) \quad \text{and} \quad [\widetilde{\nabla}_\theta,\ \sigma^\phi] = 0 = \sigma(\nabla_\theta \widehat{e}^\phi).$$

On the other hand we have

$$\begin{aligned}
[\widetilde{\nabla}_\phi\ ,\ \sigma^\theta] &= \frac{1}{s}\partial_\phi \begin{pmatrix} 0 & \omega^{\mp 1} \\ \omega^{\pm 1} & 0 \end{pmatrix} + \frac{is}{2(c \pm 1)} \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \begin{pmatrix} 0 & \omega^{\mp 1} \\ \omega^{\pm 1} & 0 \end{pmatrix} \right] \\
&= \frac{1}{s} \begin{pmatrix} 0 & \mp i\omega^{\mp 1} \\ \pm i\omega^{\pm 1} & 0 \end{pmatrix} \pm \frac{is}{2(1 \pm c)} 2 \begin{pmatrix} 0 & \omega^{\mp 1} \\ -\omega^{\pm 1} & 0 \end{pmatrix} \\
&= \mp \Big( -\frac{1}{s} + \frac{s}{1 \pm c} \Big) \begin{pmatrix} 0 & -i\omega^{\mp 1} \\ i\omega^{\pm 1} & 0 \end{pmatrix} \\
&= \frac{c}{s} \begin{pmatrix} 0 & -i\omega^{\mp 1} \\ i\omega^{\pm 1} & 0 \end{pmatrix} \ \text{since}\ \frac{s}{1 \pm c} = \frac{1 \mp c}{s} \\
&= \sigma(\frac{c}{s}\widehat{e}^\phi) \\
&= \sigma(\nabla_\phi \widehat{e}^\theta).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
[\widetilde{\nabla}_\phi\ ,\ \sigma^\phi] &= \frac{1}{s}\partial_\phi \begin{pmatrix} 0 & -i\omega^{\mp 1} \\ i\omega^{\pm 1} & 0 \end{pmatrix} + \frac{is}{2(c \pm 1)} \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \begin{pmatrix} 0 & -i\omega^{\mp 1} \\ i\omega^{\pm 1} & 0 \end{pmatrix} \right] \\
&= \frac{1}{s} \begin{pmatrix} 0 & (-i) \mp i\omega^{\mp 1} \\ (i) \pm i\omega^{\pm 1} & 0 \end{pmatrix} \pm \frac{is}{2(1 \pm c)} 2 \begin{pmatrix} 0 & -i\omega^{\mp 1} \\ -i\omega^{\pm 1} & 0 \end{pmatrix} \\
&= \mp \Big( -\frac{1}{s} + \frac{s}{1 \pm c} \Big) \begin{pmatrix} 0 & (-i)(-i)\omega^{\mp 1} \\ (i)i\omega^{\pm 1} & 0 \end{pmatrix} \\
&= -\frac{c}{s} \begin{pmatrix} 0 & \omega^{\mp 1} \\ \omega^{\pm 1} & 0 \end{pmatrix} \\
&= \sigma(-\frac{c}{s}\widehat{e}^\theta) \\
&= \sigma(\nabla_\phi \widehat{e}^\phi).
\end{aligned}$$

Suppose that $A$ is a one-form on $\mathbb{S}^2$ written as $A_\theta \widehat{e}^\theta + A_\phi \widehat{e}^\phi$. Since the multiplication operators $-iA_\theta$ and $-iA_\phi$ are anti-hermitian, we may similarly as above define a more general connection $\nabla$ on $\Psi_n$ by setting

$$\nabla_\theta = \widetilde{\nabla}_\theta^\pm - iA_\theta \quad \text{and} \quad \nabla_\phi = \widetilde{\nabla}_\phi^\pm - iA_\phi. \tag{3.7}$$

The bundles $\Psi_n$ have been introduced so we can define spinors on $\mathbb{S}^2$ as a generalised version of $\mathbb{C}^2$-valued functions on $\mathbb{S}^2$ (namely sections of $\Psi_n$). For some calculations it will be helpful to have a corresponding generalisation of scalar valued functions on $\mathbb{S}^2$; this requires the introduction of the *line bundles* or bundles with fibre $\mathbb{C}$. For each $n \in \mathbb{Z}$ we define a line bundle $L_n$ over $\mathbb{S}^2$ as follows: $L_n = L_n^+ \cup L_n^- / \sim'$, where $L_n^+ = \mathbb{S}^2_+ \times \mathbb{C}$, $L_n^- = \mathbb{S}^2_- \times \mathbb{C}$, and $\sim'$ is an equivalence relation between $(p, \zeta^+) \in L_n^+$ and $(p, \zeta^-) \in L_n^-$ given by $(p, \zeta^+) \sim' (p, \zeta^-) \iff \zeta^- = \omega^{-n}\zeta^+$.

Thus, for each $p \in L_n^+ \cap L_n^-$, the transition map

$$\begin{aligned}(\mathbb{S}^2_+ \cap \mathbb{S}^2_-) \times \mathbb{C} &\longrightarrow (\mathbb{S}^2_+ \cap \mathbb{S}^2_-) \times \mathbb{C}\\ (p, \zeta^+) &\longmapsto (p, \zeta^-)\end{aligned}$$

is given by

$$\zeta^- = \omega^{-n}\zeta^+. \tag{3.8}$$

Comparing with the transition map for $\Psi_n$ it follows that

$$\Psi_n = L_{n-1} \oplus L_{n+1}. \tag{3.9}$$

On the line bundle $L_n$ we can define a connection $\nabla$ by setting

$$\nabla_\theta = \nabla_\theta^\pm = \partial_\theta \tag{3.10}$$

$$\nabla_\phi = \nabla_\phi^\pm = \frac{1}{s}\partial_\phi - \frac{ins}{2(c \pm 1)} = \frac{1}{s}\big(\partial_\phi + \frac{in}{2}(c \mp 1)\big), \tag{3.11}$$

on $L_n^\pm$. We have to check compatibility, but we can repeat same arguments as we have done for the case $\Psi_n^\pm$ before.

## 3.3 Dirac operators on $\mathbb{S}^2$

Using the Spin$^c$ structures introduced on $\mathbb{S}^2$ in the preceeding section, we can now define Dirac operators on $\mathbb{S}^2$.

Let $A = A_\theta \widehat{e}^\theta + A_\phi \widehat{e}^\phi = A_\theta d\theta + A_\phi s d\phi \in \Omega^1(\mathbb{S}^2)$ and let $\nabla$ denote the Spin$^c$ connection on $\Psi_n$ given by (3.7). Set $D = -i\nabla$; that is

$$D_\theta = D_\theta^\pm = -i\widetilde{\nabla}_\theta^\pm - A_\theta = -i\partial_\theta - A_\theta,$$

and

$$D_\phi = D_\phi^\pm = -i\widetilde{\nabla}_\phi^\pm - A_\phi = -\frac{i}{s}\partial_\phi - \frac{s}{2(c \pm 1)}(n - \sigma_3) - A_\phi.$$

Now we can define the Dirac operator with a magnetic potential $A = A_\theta \widehat{e}^\theta + A_\phi \widehat{e}^\phi$ on the manifold $\mathbb{S}^2$ above as follows.

**Definition 3.3.1.** The *Dirac operator* with magnetic potential $A$ is the operator $\mathcal{D}_A = \sigma^\theta D_\theta + \sigma^\phi D_\phi$, where $D_\theta, D_\phi$ are given as above. This operator acts on spinors, or sections of the Spin$^c$ bundle $\Psi_n$ on $\mathbb{S}^2$. In the case $A = 0$ we denote $\mathcal{D}_A$ by $\mathcal{D}$.

We recall that the volume form of $\mathbb{S}^2$ is $\mathrm{vol}_{\mathbb{S}^2} = \widehat{e}^\theta \wedge \widehat{e}^\theta = s d\theta \wedge d\phi$. The (formal) adjoint of $D_\theta, D_\phi$ acting on sections of $\Psi_n^\pm$ can then be calculated as

$$(D_\theta^\pm)^* = \frac{-i}{s}\partial_\theta(s\cdot) - A_\theta = -i\partial_\theta - \frac{ic}{s} - A_\theta = D_\theta^\pm - \frac{ic}{s},$$

and $(D_\phi^\pm)^* = D_\phi^\pm$.
As is the case for operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ we have the following.

**Proposition 3.3.2.** *The Dirac operator $\mathcal{D}_A$ is (formally) self-adjoint.*

*Remark* 3.3.3. By "formal" we mean $\langle \mathcal{D}_A\xi_1, \xi_2\rangle = \langle \xi_1, \mathcal{D}_A\xi_2\rangle$ for smooth sections $\xi_1, \xi_2$ of $\Psi_n$ (with the $L^2$ inner product). For a discussion of self-adjointness in the sense of unbounded operators see the arguments after Proposition 3.3.6.

*Proof for Proposition 3.3.2.* We have $\mathcal{D}_A^* = (D_\theta)^*(\sigma^\theta\cdot) + (D_\phi)^*(\sigma^\phi\cdot)$. Now

$$\begin{aligned}(D_\theta^\pm)^*(\sigma_\pm^\theta\cdot) &= (-i\partial_\theta - \frac{ic}{s} - A_\theta)(\sigma_\pm^\theta\cdot)\\ &= \sigma_\pm^\theta D_\theta^\pm - i\partial_\theta(\sigma_\pm^\theta) - \frac{ic}{s}\sigma_\pm^\theta\\ &= \sigma_\pm^\theta D_\theta^\pm - \frac{ic}{s}\sigma_\pm^\theta.\end{aligned}$$

Next, we have

$$\begin{aligned}(D_\phi^\pm)^*(\sigma_\pm^\phi\cdot) &= \Big( -\frac{i}{s}\partial_\phi - \frac{s}{2(c\pm 1)}(n-\sigma_3) - A_\phi\Big)(\sigma_\pm^\phi\cdot)\\ &= \sigma_\pm^\phi D_\phi^\pm - \frac{i}{s}\partial_\phi(\sigma_\pm^\phi) + \frac{s}{2(c\pm 1)}[\sigma_3,\ \sigma^\phi].\end{aligned}$$

Then,

$$(D_\theta^\pm)^*(\sigma_\pm^\theta\cdot) + (D_\phi^\pm)^*(\sigma_\pm^\phi\cdot) = \sigma_\pm^\theta D_\theta^\pm + \sigma_\pm^\phi D_\phi^\pm + K,$$

where

$$K = -\frac{ic}{s}\sigma_\pm^\theta - \frac{i}{s}\partial_\phi(\sigma_\pm^\phi) + \frac{s}{2(c\pm 1)}[\sigma_3,\ \sigma_\pm^\phi].$$

We will show that $K = 0$. Truly, we have

$$\begin{aligned}K &= \frac{-ic}{s}\begin{pmatrix} 0 & \omega^{\mp 1}\\ \omega^{\pm 1} & 0\end{pmatrix} - \frac{i}{s}\begin{pmatrix} 0 & (-i)(\mp i)\omega^{\mp 1}\\ i(\pm i)\omega^{\pm 1} & 0\end{pmatrix} + \frac{s}{2(c\pm 1)}2\begin{pmatrix} 0 & -i\omega^{\mp 1}\\ -i\omega^{\pm 1} & 0\end{pmatrix}\\ &= \frac{i}{s}\Big( -c \pm 1 + (c\mp 1)\Big)\begin{pmatrix} 0 & \omega^{\mp 1}\\ \omega^{\pm 1} & 0\end{pmatrix} = 0,.\end{aligned}$$

since $\dfrac{s}{c\pm 1} = -\dfrac{c\mp 1}{s}$. Therefore $\mathcal{D}_A$ is self-adjoint. ∎The Dirac operator $\mathcal{D}_A$ is a first order differential operator on $\mathbb{S}^2$. A collection of results, known as Lichnerowicz-Weitzerböck formulae, relate the second order differential operator $\mathcal{D}_A^2$ to other second order differential operators. The next result is an example of such a formula, which will be useful in subsequent calculations.

**Proposition 3.3.4.** *We have*

$$\mathcal{D}_A^2 = (D_\theta)^* D_\theta + (D_\phi)^* D_\phi + \frac{1}{2} - \big(\frac{n}{2} + B\big)\sigma_3, \tag{3.12}$$

*where $B$ is given by*

$$B = \frac{1}{s}\partial_\theta(sA_\phi) - \frac{1}{s}\partial_\phi A_\theta. \tag{3.13}$$

*Remark.* On the right hand side of (3.12) the operator $(D_\theta)^* D_\theta + (D_\phi)^* D_\phi$ is a Laplacian. The other (zero-order) terms on the right hand side of (3.12) we associated to the connection $\nabla$. The first is $\frac{1}{4}R$ where $R = 2$ is the scalar curvature of $\mathbb{S}^2$. The final term is connected to the curvature of $\nabla$, and hence the magnetic field; see Definition 3.2.7 and the remark after the proof of the current result for some more details.

The $\mathbb{R}^3$ version of (3.12) is (1.6), namely that for the Weyl-Dirac operator $\sigma \cdot (D - A)$ we have

$$\big[\sigma \cdot (D - A)\big]^2 = (D - A)^2 - \sigma \cdot B,$$

where $B = (\partial_2 A_3 - \partial_3 A_2,\ \partial_3 A_1 - \partial_1 A_3,\ \partial_1 A_2 - \partial_2 A_1)$; in this case the scalar curvature is 0.

*Proof of Proposition 3.3.4.* We have

$$\begin{aligned}(\mathcal{D}_A^\pm)^2 &= [(D_\theta^\pm)^*(\sigma_\pm^\theta \cdot) + (D_\phi^\pm)^*(\sigma_\pm^\phi \cdot)][\sigma_\pm^\theta D_\theta^\pm + \sigma_\pm^\phi D_\phi^\pm] \\ &= (D_\theta^\pm)^*(\sigma_\pm^\theta)^2 D_\theta^\pm + (D_\phi^\pm)^*(\sigma_\pm^\phi)^2 D_\phi^\pm + (D_\theta^\pm)^* \sigma_\pm^\theta \sigma_\pm^\phi D_\phi^\pm + (D_\phi^\pm)^* \sigma_\pm^\phi \sigma_\pm^\theta D_\theta^\pm \\ &= (D_\theta^\pm)^* D_\theta^\pm + (D_\phi^\pm)^* D_\phi^\pm + i\sigma_3[(D_\theta^\pm)^* D_\phi^\pm - (D_\phi^\pm)^* D_\theta^\pm],\end{aligned}$$

since

$$(\sigma_\pm^\theta)^2 = (\sigma_\pm^\phi)^2 = \mathbb{I}_2,\ \sigma_\pm^\theta \sigma_\pm^\phi = i\sigma_3,\ \sigma_\pm^\phi \sigma_\pm^\theta = -i\sigma_3,$$

and

$$\Big[\big(D_\theta^\pm\big)^*,\ \sigma_3\Big] = \Big[\big(D_\phi^\pm\big)^*,\ \sigma_3\Big] = 0.$$

Now, we have

$$\begin{aligned}(D_{\theta^\pm})^* D_\phi^\pm - (D_\phi^\pm)^* D_\theta^\pm &= \Big( - i\widetilde{\nabla}_\theta^\pm - A_\theta - \frac{ic}{s}\Big)\Big( - i\widetilde{\nabla}_\phi^\pm - A_\phi\Big) \\ &- \Big( - i\widetilde{\nabla}_\phi^\pm - A_\phi\Big)\Big( - i\widetilde{\nabla}_\theta^\pm - A_\theta\Big) \\ &= -\Big[\Big(\widetilde{\nabla}_\theta^\pm - iA_\theta\Big)\Big(\widetilde{\nabla}_\phi^\pm - iA_\phi\Big) \\ &- \Big(\widetilde{\nabla}_\phi^\pm - iA_\phi\Big)\Big(\widetilde{\nabla}_\theta^\pm - iA_\theta\Big) + \frac{c}{s}\Big(\widetilde{\nabla}_\phi^\pm - iA_\phi\Big)\Big].\end{aligned}$$

However from (3.3) we have

$$[\widehat{e}_\theta\ ,\ \widehat{e}_\phi] = -\frac{c}{s}\widehat{e}_\phi.$$

So,

$$\frac{c}{s}\big(\widetilde{\nabla}^{\pm}_{\phi} - iA_{\phi}\big) = -\big(\widetilde{\nabla}^{\pm} - iA\big)_{[\widehat{e}_{\theta}\ ,\ \widehat{e}_{\phi}]}.$$

Thus, we obtain

$$(D^{\pm}_{\theta})^{*}D^{\pm}_{\phi} - (D^{\pm}_{\phi})^{*}D^{\pm}_{\theta} = -\mathcal{R}_{\Psi^{\pm}_{n}}(\widehat{e}_{\theta},\ \widehat{e}_{\phi}).$$

Now all we need to do is to show that $\mathcal{R}_{\Psi^{\pm}_{n}}(\widehat{e}_{\theta},\ \widehat{e}_{\phi}) = \frac{i}{2}\sigma_3 - i(\frac{n}{2} + B)$. Truly, we have

$$\begin{aligned}
\mathcal{R}_{\Psi^{\pm}_{n}}(\widehat{e}_{\theta},\ \widehat{e}_{\phi}) &= \Big(\widetilde{\nabla}^{\pm}_{\theta} - iA_{\theta}\Big)\Big(\widetilde{\nabla}^{\pm}_{\phi} - iA_{\phi}\Big) \\
&- \Big(\widetilde{\nabla}^{\pm}_{\phi} - iA_{\phi}\Big)\Big(\widetilde{\nabla}^{\pm}_{\theta} - iA_{\theta}\Big) - (\widetilde{\nabla}^{\pm} - iA)_{[\widehat{e}_{\theta}\ ,\ \widehat{e}_{\phi}]} \\
&= (\partial_{\theta} - iA_{\theta})\Big(\frac{1}{s}\partial_{\phi} - \frac{isn}{2(c\pm 1)} + \frac{is}{2(c\pm 1)}\sigma_3 - iA_{\phi}\Big) \\
&- \Big(\frac{1}{s}\partial_{\phi} - \frac{isn}{2(c\pm 1)} + \frac{is}{2(c\pm 1)}\sigma_3 - iA_{\phi}\Big)(\partial_{\theta} - iA_{\theta}) + \frac{c}{s}\big(\widetilde{\nabla}^{\pm}_{\phi} - iA_{\phi}\big) \\
&= (\partial_{\theta} - iA_{\theta} + \frac{c}{s})\Big(\frac{1}{s}\partial_{\phi} - \frac{isn}{2(c\pm 1)} + \frac{is}{2(c\pm 1)}\sigma_3 - iA_{\phi}\Big) \\
&- \Big(\frac{1}{s}\partial_{\phi} - \frac{isn}{2(c\pm 1)} + \frac{is}{2(c\pm 1)}\sigma_3 - iA_{\phi}\Big)(\partial_{\theta} - iA_{\theta}).
\end{aligned}$$

Then, expanding and cancelling terms we get

$$\begin{aligned}
\mathcal{R}_{\Psi^{\pm}_{n}}(\widehat{e}_{\theta},\ \widehat{e}_{\phi}) &= \frac{i}{2}(\mp 1 - c)\frac{1}{c\pm 1}(n - \sigma_3) - i\big(\partial_{\theta}A_{\phi} + \frac{c}{s}A_{\phi} - \partial_{\phi}A_{\theta}\big) \\
&= -\frac{i}{2}(n - \sigma_3) - i\Big(\frac{1}{s}\partial_{\theta}(sA_{\phi}) - \partial_{\phi}A_{\theta}\Big) \\
&= \frac{i}{2}\sigma_3 - i(\frac{n}{2} + B).
\end{aligned}$$

This concludes the proof for this proposition. ∎

*Remark* 3.3.5. In the proof of Proposition 3.3.4 we have obtained

$$\mathcal{R}_{\Psi_n}(\widehat{e}_{\theta},\ \widehat{e}_{\phi}) = \frac{i}{2}\sigma_3 - i(\frac{n}{2} + B).$$

It follows from Definition 3.2.7 that the magnetic field on $\mathbb{S}^2$ corresponding to the Spin$^c$ connection $\nabla$ on $\Psi_n$ is

$$\frac{i}{2}\operatorname{Tr}\mathcal{R}_{\Psi_n} = \Big(\frac{n}{2} + B\Big)\widehat{e}_{\theta}\wedge\widehat{e}_{\phi}; \tag{3.14}$$

This magnetic field consists of two parts; there is a constant part $\frac{n}{2}$ (essentially coming from the "twist" in the bundle $\Psi_n$) and a variable part given by $B$ (which comes from the magnetic potential $A$). Notice that the constant part remains even when we take $A = 0$.

Next we observe that

**Proposition 3.3.6.** *We have*

$$\mathcal{D}_A(\sigma_3 \cdot) = -\sigma_3 \mathcal{D}_A. \tag{3.15}$$

*Proof.* Since $\sigma_3 \sigma^\theta_\pm \sigma_3 = -\sigma^\theta_\pm$ and $\sigma_3 \sigma^\phi_\pm \sigma_3 = -\sigma^\phi_\pm$ we get

$$\sigma_3 \big(\sigma^\theta_\pm D^\pm_\theta + \sigma^\phi_\pm D^\pm_\phi\big) \sigma_3 = -\big(\sigma^\theta_\pm D^\pm_\theta + \sigma^\phi_\pm D^\pm_\phi\big).$$

Thus $\sigma_3 \mathcal{D}_A = -\sigma_3 \mathcal{D}_A$. ■

The operator $\mathcal{D}_A$ is an example of a self-adjoint *elliptic differential operator* on the compact manifold $\mathbb{S}^2$. There is a well-developed general theory for such operators (see [32], Section 17.5, for instance); we will briefly outline some aspects of this theory, with a summary of the results that we need appearing in Proposition 3.3.7.

Initially we have defined $\mathcal{D}_A$ to be acting on $\Gamma(\Psi_n)$, or smooth sections of $\Psi_n$. Using the formal adjoint of $\mathcal{D}_A$ (which is just $\mathcal{D}_A$ by Proposition 3.3.2) we can extend the definition of $\mathcal{D}_A$ to act on distributional sections of $\Psi_n$. Define $L^2(\Psi_n)$ to be the subspace of those distributions obtained by completing $\Gamma(\Psi_n)$ in the norm given by

$$\|\xi\|^2 = \int_{\mathbb{S}^2} |\xi|^2 \ \mathrm{vol}_{\mathbb{S}^2}$$

(where $|\cdot|$ represents the fibre norm in $\Psi_n$). We can then consider $\mathcal{D}_A$ to be an unbounded operator on $L^2(\Psi_n)$ with domain given by those $\xi \in L^2(\Psi_n)$ with $\mathcal{D}_A \xi \in L^2(\Psi_n)$; this gives a self-adjoint operator.

The *principal symbol* of the differential operator $\mathcal{D}_A$ is the function

$$\rho_{\mathcal{D}_A} : \quad T^*\mathbb{S}^2 \longrightarrow \mathrm{End}(\Psi_n)$$

obtained by replacing $D_\theta$ and $D_\phi$ in the definition of $\mathcal{D}_A$ with the corresponding components of the cotangent vector; more precisely,

$$\rho_{\mathcal{D}_A}(\alpha) = \sigma^\theta \alpha_\theta + \sigma^\phi \alpha_\phi = \sigma(\alpha)$$

for $\alpha = \alpha_\theta \widehat{e}^\theta + \alpha_\phi \widehat{e}^\phi$. Since $\sigma(\alpha)^2 = \|\alpha\|^2 \, \mathbb{I}_2$ (recall (3.1)), it is clear that $\rho_{\mathcal{D}_A}(\alpha)$ is invertible in $\mathrm{End}(\Psi_n)$ whenever $\alpha$ is non-zero; this is precisely the condition that $\mathcal{D}_A$ is an *elliptic operator*. General theory for self-adjoint elliptic differential operators on compact manifolds now shows that $\mathcal{D}_A$ has purely discrete spectrum (in other words, $\mathrm{Spec}(\mathcal{D}_A)$ consists only of eigenvalues with finite multiplicity).

Now suppose that $\mathcal{D}_A \xi = \lambda \xi$ for some $\xi \in L^2(\Psi_n)$ and $\lambda \in \mathbb{R}$. Elliptic regularity implies $\xi \in \Gamma(\Psi_n)$ (that is $\xi$ must be smooth), while Proposition 3.3.6 gives

$$\mathcal{D}_A(\sigma_3 \xi) = -\lambda(\sigma_3 \xi).$$

Since $\sigma_3$ is invertible (recall that $\sigma_3^2 = \mathbb{I}_2$) we immediately see that the spectrum of $\mathcal{D}_A$ is symmetric about 0.

We summarise the above observations in the following result.

**Proposition 3.3.7.** *The operator $\mathcal{D}_A$ is an unbounded sefl-adjoint opeartor on $L^2(\Psi_n)$ which has purely discrete spectrum and its spectrum is symmetric about 0.*

We will now show that an arbitrary magnetic field (two-form) with integer flux $n$ on $\mathbb{S}^2$ determines a Spin$^c$ connection and hence a Dirac operator on $\Psi_n$. Furthermore, all such Dirac operators have the same spectrum, so the magnetic field determines the spectrum.

**Proposition 3.3.8.** *Suppose that $f$ is a smooth function on $\mathbb{S}^2$ and*

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} f \ \mathrm{vol}_{\mathbb{S}^2} = n \in \mathbb{Z}.$$

*Then, there is a Spin$^c$ connection $\nabla_A$ on the Spin$^c$ bundle $\Psi_n$ with corresponding magnetic field $f$ $\mathrm{vol}_{\mathbb{S}^2}$. Furthermore $\nabla_A$ is unique up to gauge equivalence. It follows that the spectrum (including multiplicities) of the corresponding Dirac operator $\mathcal{D}_A$ is determined by $f$.*

*Proof.* Recall that $\Omega^r(\mathbb{S}^2)$ is the set of $r$-forms on $\mathbb{S}^2$ and $d$ denotes the usual exterior derivative acting from $\Omega^r(\mathbb{S}^2)$ to $\Omega^{r+1}(\mathbb{S}^2)$. We need the *adjoint exterior derivative operator* for $d$, denoted by $\delta : \ \Omega^r(\mathbb{S}^2) \longrightarrow \Omega^{r-1}(\mathbb{S}^2)$, in which $\delta = *d*$, where $*$ is the Hodge $*$. The *Laplacian* acting on $\Omega^r(\mathbb{S}^2)$ is defined as $-\Delta_r := \ \delta d + d\delta$. We will call an $r$-form $\omega$ on $\mathbb{S}^2$ *harmonic* if $-\Delta_r\omega = 0$. Denote by $\mathrm{Harm}^r(\mathbb{S}^2)$ the set of all harmonic $r$-forms of $\mathbb{S}^2$.

Back to the proof of Proposition 3.3.8 we have $f$ $\mathrm{vol}_{\mathbb{S}^2} \in \Omega^2(\mathbb{S}^2)$. Now the *Hodge decomposition theorem* for $\mathbb{S}^2$ (see, for instance Theorem 7.52 in [41]) gives

$$\Omega^2(\mathbb{S}^2) = d\Omega^1(\mathbb{S}^2) \oplus \delta\Omega^3(\mathbb{S}^2) \oplus \mathrm{Harm}^2(\mathbb{S}^2) = d\Omega^1(\mathbb{S}^2) \oplus \mathrm{Harm}^2(\mathbb{S}^2)$$

since $\Omega^3(\mathbb{S}^2) = \{0\}$. We observe that $\mathrm{Harm}^2(\mathbb{S}^2) = \{\beta \in \Omega^2(\mathbb{S}^2) : \ -\Delta_2\beta = 0\} = \{b \ \mathrm{vol}_{\mathbb{S}^2} : b \in \mathbb{R}\}$; indeed, if $b$ is a function on $\mathbb{S}^2$ and $\beta = b \ \mathrm{vol}_{\mathbb{S}^2} \in \Omega^2(\mathbb{S}^2)$, then since $d\beta = 0$ we get

$$\begin{aligned}
&-\Delta_2\beta = 0\\
\Longleftrightarrow\ & d\delta\beta = 0\\
\Longleftrightarrow\ & d * d * (b \ \mathrm{vol}_{\mathbb{S}^2}) = 0\\
\Longleftrightarrow\ & d * db = 0\\
\Longleftrightarrow\ & b \in \mathrm{Harm}^0(\mathbb{S}^2)\\
\Longleftrightarrow\ & b \text{ is a constant.}
\end{aligned}$$

Therefore, we may find $A \in \Omega^1(\mathbb{S}^2)$ and a constant $b$ such that

$$f \ \mathrm{vol}_{\mathbb{S}^2} = dA + b \ \mathrm{vol}_{\mathbb{S}^2}.$$

It follows from Stokes' theorem and the given assumption that

$$n = \frac{1}{2\pi} \int_{\mathbb{S}^2} f \ \mathrm{vol}_{\mathbb{S}^2} = \frac{1}{2\pi} \int_{\mathbb{S}^2} dA + \frac{1}{2\pi} \int_{\mathbb{S}^2} b \ \mathrm{vol}_{\mathbb{S}^2} = 0 + 2b = 2b.$$

Thus

$$b = \frac{n}{2}.$$

Take $\nabla_A$ on $\Psi_n$ and use (3.14) to get corresponding magnetic field

$$\beta = dA + \frac{n}{2} \ \mathrm{vol}_{\mathbb{S}^2} = f \ \mathrm{vol}_{\mathbb{S}^2}.$$

To show uniqueness we suppose that $A, A' \in \Omega^1(\mathbb{S}^2)$ give the same magnetic field; that is

$$dA + \frac{n}{2} \ \mathrm{vol}_{\mathbb{S}^2} = dA' + \frac{n}{2} \ \mathrm{vol}_{\mathbb{S}^2}.$$

Then we have $d(A' - A) = 0$. It follows that $A' - A = dg$ for some smooth function $g$ (since $H^1(\mathbb{S}^2) = 0$). Now we may check that

$$\nabla_{A'} = e^{ig} \nabla_A e^{-ig},$$

showing that $\nabla'_A$ and $\nabla_A$ are gauge equivalent. It follows that we also have

$$\mathcal{D}_{A'} = e^{ig} \mathcal{D}_A e^{-ig}$$

for the corresponding Dirac operators; in particular $\mathcal{D}_A$ and $\mathcal{D}_{A'}$ are unitarily equivalent so they then have the same spectrum. ∎

*Remark* 3.3.9. The integer number $n$ is called the *Chern number* for $\Psi_n$.

## 3.4 The Laplacian on the line bundle $L_n$ and its spectrum

Now we return to line bundle $L_n$ and the connection $\nabla$ over $L_n$ defined at the end of Section 3.2.3. In order to calculate the spectrum of the Dirac operator $\mathcal{D}$ on $\Psi_n$ we will firstly consider the spectrum of an auxiliary operator which is defined using this connection.

First, it follows from (3.10) and (3.11) that

$$(\nabla^{\pm}_{\theta})^* = -\frac{1}{s} \partial_\theta (s \cdot) \tag{3.16}$$

and

$$(\nabla^{\pm}_{\phi})^* = -\frac{1}{s} \big( \partial_\phi + \frac{in}{2} (c \mp 1) \big). \tag{3.17}$$

Then we can define the *Laplacian* $-\Delta_n$ acting on the line bundle $L_n$ as follows.

**Definition 3.4.1.** We call the *Laplacian* and denote by $-\Delta_n$ the operator

$$\nabla_\theta^*\nabla_\theta + \nabla_\phi^*\nabla_\phi = -\frac{1}{s}\partial_\theta(s\partial_\theta\cdot) - \frac{1}{s^2}\left(\partial_\phi + \frac{in(c\mp 1)}{2}\right)^2$$

acting on $\Gamma(L_n)$, sections of the line bundle $L_n$.

*Remark* 3.4.2. The line bundle $L_0$ is the trivial bundle $\mathbb{S}^2\times\mathbb{C}$, so sections of $L_0$ are simply functions on $\mathbb{S}^2$. The operator $-\Delta_0$ is then the Laplace-Beltrami operator on $\mathbb{S}^2$ with its usual metric.

Let $L^2(L_n)$ be the completion of the linear space of sections $\xi\in\Gamma(L_n)$ in the norm given by

$$\|\xi\|^2 = \int_{\mathbb{S}^2} |\xi|^2 \ \mathrm{vol}_{\mathbb{S}^2}.$$

We can repeat arguments similar to the ones at the end of the previous section for $\mathcal{D}_A$ to see that $-\Delta_n$ can be defined in $L^2(L_n)$ to be an elliptic differential and unbounded self-adjoint operator which has purely discrete spectrum. This spectrum can be determined precisely as follows.

**Theorem 3.4.3.** *For each $n\in\mathbb{Z}$ the spectrum of $-\Delta_n$ is purely discrete and its eigenvalues are of the form*

$$\left(j+\frac{|n|+1}{2}\right)^2 - \frac{n^2+1}{4} = j(j+|n|+1) + \frac{|n|}{2},$$

*where $j$ is in any non-negative integer. Moreover, the multiplicity of this eigenvalue is $2j+|n|+1$.*

*Remark.* Specialising to the case of $n=0$, we see that the spectrum of $-\Delta_0$ consists of eigenvalues of the form $j(j+1)$ with multiplicity $2j+1$, where $j$ is a non-negative interger; this is a well-known result for the Laplace-Beltrami operator (see [19], p.49, for example).

*Proof of Theorem 3.4.3.* To investigate the spectrum of $-\Delta_n$ we need only find all $\lambda\in\mathbb{R}$ and $\xi\in\Gamma(L_n)$ such that

$$-\Delta_n\xi = \lambda\xi. \tag{3.18}$$

More precisely we must find $\lambda\in\mathbb{R}$ and $\xi^\pm\in\Gamma(L_n^\pm)$ such that

$$-\Delta_n\xi^\pm = \lambda\xi^\pm. \tag{3.19}$$

and

$$\xi^- = \omega^{-n}\xi^+. \tag{3.20}$$

Define an operator $L_\phi$ on $\Gamma(L_n)$ by setting

$$L_\phi = L_\phi^\pm = -i\partial_\phi \mp \frac{n}{2} \tag{3.21}$$

on $\Gamma(L_n^\pm)$. We can check compatibility and we then see that $L_\phi$ is self-adjoint and commutes with $-\Delta_n$. Thus, we can choose eigenfunctions for $-\Delta_n$ which are also eigenfunctions for $L_\phi$. Actually, $L_\phi$ will play the role of $L_3$ in the same process of looking for orbital angular momentum as in [31], p. 155-117.

Now suppose that $L_\phi \xi = m\xi$, for some $\xi \in \Gamma(L_n)$ and $m \in \mathbb{R}$. As before, this means we need $\xi^\pm \in \Gamma(L_n^\pm)$ with

$$\xi^- = \omega^{-n}\xi^+ \iff \xi^-(\theta,\phi) = e^{-in\phi}\xi^+(\theta,\phi). \tag{3.22}$$

Then, we need

$$\begin{aligned} & L_\phi^\pm \xi^\pm = m\xi^\pm \\ \iff & \big(-i\partial_\phi \mp \frac{n}{2}\big)\xi^\pm = m\xi^\pm \\ \iff & \xi^\pm(\theta,\ \phi) = u^\pm(\theta) e^{im_\pm \phi} \end{aligned}$$

for some functions $u^\pm(\theta)$, where we set $m_\pm = m \pm \dfrac{n}{2}$. We observe that

$$m_\pm = m_\mp \pm n. \tag{3.23}$$

On the other hand we want $\xi^\pm(\theta,\ \phi + 2\pi) = \xi^\pm(\theta,\ \phi)$, so $m_\pm$ must be integral; thus we must have $m \in \mathbb{Z} + \dfrac{n}{2}$. We also need the compatibility between $\xi^+ \in \Gamma(L_n^+)$ and $\xi^- \in \Gamma(L_n^-)$ as given by condition (3.22). It means

$$u^-(\theta)e^{im_-\phi} = \omega^{-n}u^+(\theta)e^{im_+\phi},$$

which follows provided

$$u^- = u^+ \quad \text{since } m_- = -n + m_+ \quad \text{and } \omega = e^{i\phi}.$$

By using the spectral decomposition of $L_\phi$ we can write $L^2(L_n)$ as $\oplus_{m\in\mathbb{Z}+\frac{n}{2}}\mathcal{H}_m$, in which restricted to each $\mathcal{H}_m$ the operator $L_\phi$ is just $mI$ (where $I$ is the identity). Furthermore, an element $\xi \in \mathcal{H}_m$ has the form $u(\theta)e^{im_\pm\phi}$ on $L_n^\pm$, for some function $u$.

Since $-\Delta_n$ commutes with $L_\phi$ we can now solve (3.18) on each subspace $\mathcal{H}_m$ separately: Thus, for each $m \in \mathbb{Z} + \frac{n}{2}$, we need to find all $\lambda \in \mathbb{R}$ and functions $u$ such that

$$-\Delta_n\big(u(\theta)e^{im_\pm\phi}\big) = \lambda u(\theta)e^{im_\pm\phi} \tag{3.24}$$

on $L_n^\pm$; the condition that our eigenfunctions should belong to $L^2(L_n)$ becomes

$$\int_0^\pi |u(\theta)|^2\ s d\theta < +\infty. \tag{3.25}$$

We notice that $-\dfrac{1}{s}\partial_\theta(s\partial_\theta u) = -\partial_\theta(\dfrac{1}{s}\partial_\theta(su)) - \dfrac{1}{s^2}u$, so equation (3.24) become

$$-\partial_\theta\big(\frac{1}{s}\partial_\theta(su)\big) + \frac{1}{s^2}[(m_\pm + \frac{n}{2}(c \mp 1))^2 - 1]u - \lambda u = 0. \tag{3.26}$$

Now we put $x := c$, then $s = (1-x^2)^{\frac{1}{2}}$, $\partial_\theta = -s\partial_x$ or $\partial_x = -\dfrac{1}{s}\partial_\theta$; and $w := su$, then (3.26) becomes

$$\partial_x^2 w - \frac{1}{(1-x^2)^2}\left[(m_\pm + \frac{n}{2}(x \mp 1))^2 - 1\right] w + \frac{\lambda}{1-x^2} w = 0.$$

With the help of (3.23) we can write this as

$$\partial_x^2 w + \left[\frac{1-m_\mp^2}{4(x\pm 1)^2} + \frac{1-m_\pm^2}{4(x\mp 1)^2} + \frac{2\lambda + 1 - m_\pm m_\mp}{2(1-x^2)}\right] w = 0. \tag{3.27}$$

We remark that (3.27) is in the type of $g_2(x)\partial_x^2 y + g_1(x)\partial_x y + g_0(x) y = 0$, where $g_2(x) = 1$, $g_1(x) = 0$, and

$$g_0(x) = \frac{1}{4}\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\frac{1-\beta^2}{(1+x)^2} + \frac{2n(n+\alpha+\beta+1) + (\alpha+1)(\beta+1)}{2(1-x^2)}.$$

Solutions of (3.27) are expressed in the type of $(1-x)^{\frac{\alpha+1}{2}}(1-x)^{\frac{\beta+1}{2}} p_k^{(\alpha,\beta)}(x)$ with $p_k^{(\alpha,\beta)}(x)$, the Jacobi polynomials, $k = 0,\ 1,\ 2, \dots$ (see [1], p.781). We remark that coefficients of solutions above must obey those following conditions

$$\begin{cases} \beta^2 = (m_+ - n)^2 = m_-^2 \\ \alpha^2 = m_+^2 \\ 2k(k+\alpha+\beta+1) + (\alpha+1)(\beta+1) = 2\lambda + 1 - m_+ m_-. \end{cases} \tag{3.28}$$

It follows that

$$\begin{cases} \beta^2 = m_-^2 \\ \alpha^2 = m_+^2 \\ 2k(k+\alpha+\beta+1) + (\alpha+1)(\beta+1) = 2\lambda + 1 - m_+ m_- \end{cases} \tag{3.29}$$

We notice that $p_k^{(\alpha,\beta)}(1) = \begin{pmatrix} k+\alpha \\ k \end{pmatrix}$ and $p_k^{(\alpha,\beta)}(-1) = (-1)^k \begin{pmatrix} k+\beta \\ k \end{pmatrix}$ (see [1], p.777). Then $w(x) \sim (1-x)^{\frac{\alpha+1}{2}}$ as $x \uparrow 1$ and $w(x) \sim (1+x)^{\frac{\beta+1}{2}}$ as $x \downarrow -1$. Now, we observe that a function $r(\theta)e^{im\phi}$, where $r(\theta) \sim \theta^\nu$ as $\theta \to 0$ is smooth at 0 only if $\nu \geq |m|$. Therefore, it follows from the necessary condition for the smoothness of $u$ coming from the elliptic regularity, where $u = (1-x^2)^{-\frac{1}{2}} w$ that

$$\frac{\alpha}{2} \geq \frac{|m_+|}{2} \text{ and } \frac{\beta}{2} \geq \frac{|m_-|}{2}.$$

It follows that

$$\alpha \geq |m_+| \text{ and } \beta \geq |m_-|.$$

Then, it follows from (3.29) that $\alpha = |m_+|$, and $\beta = |m_-|$, where $m_+ = m_- + n$. We also get, in particular, that

$$\alpha \geq 0;\ \beta \geq 0. \tag{3.30}$$

Now we get from (3.29) that

$$2\lambda + 1 - m_+m_- = 2k(k + |m_+| + |m_-| + 1) + (|m_+| + 1)(|m_-| + 1).$$

Then, we have

$$\begin{aligned}
\lambda &= k^2 + (|m_+| + |m_-| + 1)k + \frac{1}{2}(|m_+m_-| + m_+m_- + |m_+| + |m_-|) \\
&= k^2 + (|m_+| + |m_-| + 1)k + \frac{1}{4}[m_+^2 + m_-^2 + 2|m_+m_-| + 2|m_+| + 2|m_-| + \\
&+ 1 - (m_+^2 + m_-^2 - 2m_+m_- + 1)] \\
&= \left(k + \frac{|m_+| + |m_-| + 1}{2}\right)^2 - \frac{n^2+1}{4} \\
&= \left(k + \frac{|m_+| + |m_+ - n| + 1}{2}\right)^2 - \frac{n^2+1}{4}.
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}
&- \Delta_n^+ \phi_{k,m_+} = \mu_{k,m_+}\phi_{k,m_+}, \text{ where} \\
&\phi_{k,m_+} = (1-x)^{\frac{|m_+|}{2}}(1+x)^{\frac{|m_-|}{2}} p_k^{(|m_+|,|m_-|)}(x)\omega^{m_+} \\
&\mu_{k,m_+} = \left(k + \frac{|m_+| + |m_+ - n| + 1}{2}\right)^2 - \frac{n^2+1}{4} \text{ with } m_- = m_+ - n.
\end{aligned}$$

By investigating the expression $k + \dfrac{|m_+| + |m_+ - n| + 1}{2}$, we see that there is a $j \in \mathbb{Z},\ j \geq 0$ such that

$$k + \frac{|m_+| + |m_+ - n| + 1}{2} = \frac{|n| + 1}{2} + j. \tag{3.31}$$

We also get that

$$\begin{cases} -j \leq m_+ \leq n + j & \text{if } n \geq 0 \\ n - j \leq m_+ \leq j & \text{if } n < 0. \end{cases}$$

Therefore if we denote by $E_{n,j}$ the eigenspace of eigenvalue $\left(j + \dfrac{|n|+1}{2}\right)^2 - \dfrac{n^2+1}{4}$ (in which $n$, $j$ satisfy the relation (3.31)) for $-\Delta_n$, then $E_{n,j}$ is generated by $2j+1+n$ in case of $n \geq 0$ or $2j+1-n$ in case of $n < 0$ independent functions $\phi_{j,m}$ (in place of $\phi_{k,m_+}$). Then, we obtain the conclusion of the theorem. ■

*Remark* 3.4.4. We may determine an orthonormal basis of eigenfuctions for the Laplacian above. We already have

$$-\Delta_n^+ \phi_{k,m_+} = \mu_{k,m_+}\phi_{k,m_+}, \text{ where}$$

$$\phi_{k,m_+}(x) = (1-x)^{\frac{|m_+|}{2}}(1+x)^{\frac{|m_-|}{2}} p_k^{(|m_+|,|m_-|)}(x)\omega_+^m$$

$$\mu_{k,m_+} = \left(\frac{|n|+1}{2} + j\right)^2 - \frac{n^2+1}{4} \text{ with } m_- = m_+ - n$$

and

$$k + \frac{|m_+| + |m_+ - n| + 1}{2} = \frac{|n|+1}{2} + j,\ j \in \mathbb{Z},\ j \geq 0.$$

Hereafter we will denote $\phi_{k,m_+}(x)$ by $\phi_{j,m}(x)$. Next, we determine the norm of $\phi_{j,m}(x)$. It comes to calculate the following integral in the polar coordinates $(\theta, \phi)$

$$\begin{aligned} \int_0^\pi sd\theta \int_0^{2\pi} d\phi\ |\phi_{j,m}(\theta,\phi)|^2 &= 2\pi \int_0^\pi (1-c)^{|m|}(1+c)^{|m-n|}|p_k^{(|m|,|m-n|)}(c)|^2 sd\theta \\ &= 2\pi \int_{-1}^1 (1-x)^{|m|}(1+x)^{|m-n|}|p_k^{(|m|,|m-n|)}(x)|^2\ dx. \end{aligned}$$

To apply formulae 2.2 and 2.1 in [1], p. 774 we get the value of the above integral as follows

$$2\pi \frac{2^{|m|+|m-n|+1}}{2k+|m|+|m-n|+1} \frac{\Gamma(k+|m|+1)\Gamma(k+|m-n|+1)}{k!\Gamma(k+|m|+|m-n|+1)}.$$

At first we will consider $n \geq 0$. Then, we notice that $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}$ and get

$$\|\phi_{j,m}\|^2 = \frac{2\pi \cdot 2^{|m|+|m-n|+1}}{n+1+2j} \frac{(k+|m|)!(k+|m-n|)!}{k!(k+|m|+|m-n|)!}.$$

Now we split in three different cases of $m$ and we get

$$\begin{aligned} \|\phi_{j,m}\|^2 &= \frac{4\pi}{n+1+2j} \begin{cases} 2^{n-2m} \cdot \dfrac{(k-m)!(k-m+n)!}{k!(k-2m+n)!} & : m < 0 \\ 2^n \cdot \dfrac{(k+m)!(k-m+n)!}{k!(k+n)!} & : 0 \leq m \leq n \\ 2^{2m-n} \cdot \dfrac{(k+m)!(k+m-n)!}{k!(k+2m-n)!} & : m > 0 \end{cases} \\ &= \frac{4\pi}{n+1+2j} \begin{cases} 2^{n-2m} \cdot \dfrac{j!(j+n)!}{(j+m)!(j+n-m)!} & : m < 0 \\ 2^n \cdot \dfrac{(j+m)!(j+n-m)!}{j!(j+n)!} & : 0 \leq m \leq n \\ 2^{2m-n} \cdot \dfrac{j!(j+n)!}{(j+n-m)!(j+m)!} & : m > n. \end{cases} \end{aligned}$$

It follows that we have to multiply each $\phi_{j,m}(x)$ by the following constant so that we can obtain a family of eigenfunctions with unit norm for the Laplacian:

$$\frac{\sqrt{n+1+2j}}{2\sqrt{\pi}} \begin{cases} 2^{m-\frac{n}{2}} \cdot \sqrt{\dfrac{(j+m)!(j+n-m)!}{j!(j+n)!}} & : m < 0 \\ 2^{-\frac{n}{2}} \cdot \sqrt{\dfrac{j!(j+n)!}{(j+m)!(j+n-m)!}} & : 0 \leq m \leq n \\ 2^{-m+\frac{n}{2}} \cdot \sqrt{\dfrac{(j+m)!(j+n-m)!}{j!(j+n)!}} & : m > 0. \end{cases}$$

The case $n < 0$ can be obtained from the case of $n \geq 0$ by replacing $n$ with $-n$ and $m$ with $-m$. We can see this from the following

$$|m| = |-m|; \ |m-n| = |n-m| = |(-m)-(-n)|.$$

## 3.5 Constant magnetic fields on $\mathbb{S}^2$

In this section we will determine the spectrum of the Dirac operator $\mathcal{D}$ in the case $A = 0$; that is when the magnetic field is only $\frac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$. Firstly we need a preliminary result.

Looking back to relation (3.9) we know that $\Psi_n = L_{n-1} \oplus L_{n+1}$. Thus, a spinor in $\Psi_n$ can be described as a pair of spinors on $L_{n-1}$ and $L_{n+1}$ and we can think of the Dirac operator $\mathcal{D}$ as a $2 \times 2$ matrix of operators acting on such pairings.

**Proposition 3.5.1.** *We have*

$$\mathcal{D}^2 = \begin{pmatrix} -\Delta_{n-1} - \dfrac{n-1}{2} & 0 \\ 0 & -\Delta_{n+1} + \dfrac{n+1}{2} \end{pmatrix}.$$

*Proof.* It follows from Proposition 3.3.4 for $A = 0$ that

$$\mathcal{D}^2 = (-i\partial_\theta - \frac{ic}{s})(-i\partial_\theta) + \left(\frac{-i}{s}\partial_\phi + \frac{c \mp 1}{2s}(n - \sigma_3)\right)^2 + \frac{1}{2}(1 - n\sigma_3).$$

Comparing this with the definition of the Laplacian (see Definition 3.4.1) now gives the result. ■

Now we will determine the spectrum of the Dirac operator $\mathcal{D}$ in case of having no variable part in its magnetic field; that is the Dirac operator with magnetic field $\frac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$.

**Theorem 3.5.2.** *For each $n \in \mathbb{Z}$, the Dirac operator $\mathcal{D}$ on $\Psi_n$ has purely discrete spectrum with eigenvalues $\pm\sqrt{j(j+|n|)}$ for $j \in \mathbb{N}_0$. Moreover, the eigenvalue $\pm\sqrt{j(j+|n|)}$ has a multiplicity of $2j + |n|$.*

*Proof.* It follows from Proposition 3.5.1 that

$$\mathcal{D}^2 = \begin{pmatrix} -\Delta_{n-1} - \frac{1}{2}(n-1) & 0 \\ 0 & -\Delta_{n+1} + \frac{1}{2}(n+1) \end{pmatrix}.$$

Then the set of eigenvalues of $\mathcal{D}^2$ is the union of the eigenvalues of $-\Delta_{n-1}-\dfrac{1}{2}(n-1)$ and $-\Delta_{n+1}+\dfrac{1}{2}(n+1)$. For convenience we will first consider $n>0$. By Theorem 3.4.3 we know that the eigenvalues of $-\Delta_n$ are given by

$$\begin{aligned}\left(j+\frac{n+1}{2}\right)^2-\frac{n^2+1}{4}&=\left(j+\frac{n+1}{2}\right)^2-(\frac{n+1}{2})^2+\frac{n}{2}\\&=j(j+n+1)+\frac{n}{2}\end{aligned}$$

for $j\in\mathbb{N}_0$. Then the eigenvalues of $-\Delta_{n-1}-\dfrac{1}{2}(n-1)$ are $j(j+n)$ for $j\in\mathbb{N}_0$. Similarly, we can write

$$\begin{aligned}\left(j+\frac{n+1}{2}\right)^2-\frac{n^2+1}{4}&=\left(j+\frac{n+1}{2}\right)^2-(\frac{n-1}{2})^2-\frac{n}{2}\\&=(j+1)(j+n)-\frac{n}{2}.\end{aligned}$$

So the eigenvalues of $-\Delta_{n+1}+\dfrac{1}{2}(n+1)$ are $(j+1)(j+1+n)$ for $j\in\mathbb{N}_0$ or $j(j+n)$ for $j\in\mathbb{N}$.

For $n\le 0$ we see that eigenvalues of $-\Delta_{n-1}-\dfrac{1}{2}(n-1)$ are $(j+1)(j+1-n)$ for $j\in\mathbb{N}_0$, or $j(j-n)$ for $j\in\mathbb{N}$, while the eigenvalues of $-\Delta_{n+1}+\dfrac{1}{2}(n+1)$ are $j(j-n)$ for $j\in\mathbb{N}_0$.

Therefore, for any interger $n$ the eigenvalues of $\mathcal{D}^2$ (defined on $\Psi_n$) are $j(j+|n|)$ for $j\in\mathbb{N}_0$. Looking back to Proposition 3.3.7 we can conclude that for any interger $n$ the spectrum of $\mathcal{D}$ (defined on $\Psi_n$) are the eigenvalues $\pm\sqrt{j(j+|n|)}$ for $j\in\mathbb{N}_0$.

Let $F_{n,j}$ be the eigenspace of $\mathcal{D}^2$ for the eigenvalue $j(j+|n|)$. We firstly consider $j\in\mathbb{N}$. Let $n>0$. Recall that $E_{n,j}$ the eigenspace of $-\Delta_n$ for the eigenvalue $\left(j+\dfrac{|n|+1}{2}\right)^2-\dfrac{n^2+1}{4}$. That means $E_{n-1,j}$ is the eigenspace of $-\Delta_{n-1}-\frac{1}{2}(n-1)$ for the eigenvalue $j(j+n)$, while $E_{n+1,j-1}$ is the eigenspace of $-\Delta_{n+1}+\frac{1}{2}(n+1)$ for the eigenvalue $j(j+n)$. Therefore, it follows from Proposition 3.5.1 that

$$F_{n,j}=\left\{\begin{pmatrix}\xi\\0\end{pmatrix}:\quad \xi\in E_{n-1,j}\right\}\oplus\left\{\begin{pmatrix}0\\\xi\end{pmatrix}:\quad \xi\in E_{n+1,j-1}\right\}.$$

Thus,

$$\dim F_{n,j}=\dim E_{n-1,j}+\dim E_{n+1,j-1}.$$

Then, Theorem 3.4.3 gives

$$\dim F_{n,j}=2j+(n-1)+1+2(j-1)+(n+1)+1=2(2j+n)=2(2j+|n|).$$

The case $n \leq 0$ is treated similarly and we still get $\dim F_{n,j} = 2j + (-n-1) + 1 + 2(j-1) + (-n+1) + 1 = 2(2j-n) = 2(2j+|n|)$.
Now consider $j = 0$. Obviously we cannot get the second eigenspace (since $-\Delta_{n+1} + \frac{1}{2}(n+1) > 0$ in case of $n \geq 0$; and $-\Delta_{n-1} - \frac{1}{2}(n-1) > 0$ in case of $n < 0$). It follows that

$$\dim F_{n,0} = \begin{cases} 2j + (n-1) + 1 = n = |n| & \text{if } n \geq 0 \\ 2(j-1) - n + 2 = -n = |n| & \text{if } n < 0. \end{cases}$$

Combining all we have $\dim F_{n,j} = 2(2j + |n|)$ if $n \in \mathbb{N}$ and $\dim F_{n,0} = |n|$. Now, recall Propostion 3.3.7 and it tells us that $\sigma_3 \mathcal{D} \sigma_3 = -\mathcal{D}$ and the spectrum of $\mathcal{D}$ is symmetric about 0. Therefore, the multiplicity of the eigenvalue $\lambda = \pm\sqrt{j(j+|n|)}$ of the Dirac operator $\mathcal{D}$ is $2j + |n|$. (The case $\lambda = 0$ or $j = 0$ is obviously included in this formula as well). ■

## 3.6 The Aharonov-Casher theorem for $\mathbb{S}^2$

The results from the previous section allows us to obtain the Aharonov-Casher theorem for $\mathbb{S}^2$; indeed, for the spinor bundle $\Psi_n$ along with the Spin$^c$ connection $\widetilde{\nabla}$ given by (3.5) and (3.6), it follows from (3.14) that the corresponding magnetic field for the Dirac operator is $\beta = \frac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$. Therefore, the total flux for this magnetic field is

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} \beta = n.$$

On the other hand it obviously follows from Theorem 3.5.2 above that $\dim \operatorname{Ker} \mathcal{D} = |n|$. Thus we have

$$\dim \operatorname{Ker} \mathcal{D} = |n| = \frac{1}{2\pi}\left| \int_{\mathbb{S}^2} \frac{n}{2}\ \mathrm{vol}_{\mathbb{S}^2} \right|, \tag{3.32}$$

which is the Aharonov-Casher theorem in this case. Now we will show how the general version of this result (for an arbitrary Dirac operator on $\mathbb{S}^2$) can be obtained from this special result.

**Theorem 3.6.1.** (Aharonov-Casher theorem on $\mathbb{S}^2$) *Let $\nabla$ be the Spin$^c$ connection on $\Psi_n$ given by (3.7) with corresponding magnetic field $\beta = \frac{n}{2}\mathrm{vol}_{\mathbb{S}^2} + dA$. Then, for the Dirac operator $\mathcal{D}_A$ on $\mathbb{S}^2$, we have*

$$\dim \operatorname{Ker} \mathcal{D}_A = |n| = \frac{1}{2\pi}\left| \int_{\mathbb{S}^2} \beta \right|.$$

We will prove this result using some ideas from differential geometry as outlined in Section 3.2.1. Note that in particular both the total flux and the dimension of $\operatorname{Ker} \mathcal{D}_A$ depend only on the Spin$^c$ bundle $\Psi_n$ (and not on the choice of $A$).

First, we prove the following result

**Lemma 3.6.2.** *If $A = A_\theta \widehat{e}^\theta + A_\phi \widehat{e}^\phi \in \Omega^1(\mathbb{S}^2)$, then there exist smooth functions $f$, $g$ on $\mathbb{S}^2$ such that*

$$A_\theta = \partial_\theta g - \frac{1}{s}\partial_\phi f \quad \text{and} \quad A_\phi = \frac{1}{s}\partial_\phi g + \partial_\theta f. \tag{3.33}$$

*Proof.* By *Hodge's Theorem* (see, for instance Theorem 7.55 in [41]) we know that $\mathrm{Harm}^1(\mathbb{S}^2) \cong H^1(\mathbb{S}^2)$, in which $H^1(\mathbb{S}^2)$ is the 1st de Rham cohomology group of $\mathbb{S}^2$ (see [41], p.195). On the other hand we also have $H^1(\mathbb{S}^2) = 0$ (or, equivalently any closed two-form on $\mathbb{S}^2$ is exact; see, for instance [41], p.202). Then, applying the Hodge decomposition theorem for $\mathbb{S}^2$ (Theorem 7.52 in [41]) we have

$$\Omega^1(\mathbb{S}^2) = d\Omega^0(\mathbb{S}^2) \oplus \delta\Omega^2(\mathbb{S}^2) \oplus \mathrm{Harm}^1(\mathbb{S}^2) = d\Omega^0(\mathbb{S}^2) \oplus \delta\Omega^2(\mathbb{S}^2).$$

Therefore we can write $A = dg + \delta F$, for some $g \in \Omega^0(\mathbb{S}^2)$ (so $g$ is a smooth function on $\mathbb{S}^2$), and $F \in \Omega^2(\mathbb{S}^2)$. Now, $\widehat{e}^\theta \wedge \widehat{e}^\phi$ is a basis for $\Omega(\mathbb{S}^2)$, so we can write $F = f\widehat{e}^\theta \wedge \widehat{e}^\phi$ for some smooth function $f$ on $\mathbb{S}^2$. Observe that $F = *f$ and $**f = f$. Then, we obtain

$$\begin{aligned}
A &= dg + *df \\
&= (\partial_\theta g)d\theta + (\partial_\phi g)d\phi + (\partial_\theta f)(*d\theta) + (\partial_\phi f)(*d\phi) \\
&= (\partial_\theta g)d\theta + \left(\frac{1}{s}\partial_\phi g\right)(sd\phi) + (\partial_\theta f)(*d\theta) + \left(\frac{1}{s}\partial_\phi f\right)\Big(*sd\phi\Big) \\
&= (\partial_\theta g)\widehat{e}^\theta + \left(\frac{1}{s}\partial_\phi g\right)\widehat{e}^\phi + (\partial_\theta f)(*\widehat{e}^\theta) + \left(\frac{1}{s}\partial_\phi f\right)(*\widehat{e}^\phi) \\
&= \left(\partial_\theta g - \frac{1}{s}\partial_\phi f\right)\widehat{e}^\theta + \left(\partial_\theta f + \frac{1}{s}\partial_\phi g\right)\widehat{e}^\phi,
\end{aligned}$$

completing the result. ∎

Using the smooth functions $f$ and $g$ from Lemma 3.6.2 we can define multiplicative transformations $e^{\pm ig}$ and $e^{f\sigma_3} = \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix}$ acting on spinor sections of the spinor bundle $\Psi_n$ over $\mathbb{S}^2$. We remark that $e^{\pm ig}$ and $e^{f\sigma_3}$ are invertible; $\left(e^{\pm ig}\right)^{-1} = e^{\mp ig}$ and $(e^{f\sigma_3})^{-1} = e^{-f\sigma_3}$. Thus $e^{\pm ig}$ is unitary, although $e^{f\sigma_3}$ is not in general. We can use these multiplicative transformations to relate the general Dirac operator $\mathcal{D}_A$ to that with $A = 0$.

**Lemma 3.6.3.** *We have*

$$\mathcal{D}_A = e^{ig}e^{f\sigma_3}\mathcal{D}e^{-f\sigma_3}e^{-ig}.$$

*Remark.* The unitary map $e^{ig}$ induces a usual gauge transformation in $A$ while the map $e^{f\sigma_3}$ induces a "real gauge transformation"; the latter is specific to two dimensions.

*Proof of Lemma 3.6.3.* We observe that scalar functions and $\sigma_3$ commute with $e^{f\sigma_3}$. Furthermore we also have

$$\begin{aligned}
\partial_\theta\big(e^{f\sigma_3}.\big) &= \begin{pmatrix} \partial_\theta\big(e^{f}.\big) & 0 \\ 0 & \partial_\theta\big(e^{-f}.\big) \end{pmatrix} \\
&= \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix} \begin{pmatrix} \partial_\theta + (\partial_\theta f) & 0 \\ 0 & \partial_\theta - (\partial_\theta f) \end{pmatrix} \\
&= e^{f\sigma_3}\big(\partial_\theta + (\partial_\theta f)\sigma_3\big).
\end{aligned}$$

Similarly, we also have

$$\partial_\phi\big(e^{f\sigma_3}.\big) = e^{f\sigma_3}\big(\partial_\phi + (\partial_\phi f)\sigma_3\big).$$

On the other hand we have

$$\begin{aligned}
\sigma_\pm^\theta e^{f\sigma_3} &= \begin{pmatrix} 0 & \omega^{\mp 1} \\ \omega^{\pm 1} & 0 \end{pmatrix} \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix} = \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix} \begin{pmatrix} 0 & \omega^{\mp 1} \\ \omega^{\pm 1} & 0 \end{pmatrix} \\
&= e^{-f\sigma_3}\sigma_\pm^\theta.
\end{aligned}$$

Similarly, $\sigma_\pm^\phi e^{f\sigma_3} = e^{-f\sigma_3}\sigma_\pm^\phi$. Thus,

$$\begin{aligned}
e^{f\sigma_3}\mathcal{D}e^{-f\sigma_3} &= e^{f\sigma_3}\big(\sigma_\pm^\theta(-i\widetilde{\nabla}_\theta^\pm) + \sigma_\pm^\phi(-i\widetilde{\nabla}_\phi^\pm)\big) \\
&= e^{f\sigma_3}\left(\sigma_\pm^\theta e^{f\sigma_3}\big(-i\widetilde{\nabla}_\theta^\pm - i(\partial_\theta f)\sigma_3\big) + \sigma_\pm^\phi e^{f\sigma_3}\big(-i\widetilde{\nabla}_\phi^\pm - \frac{i}{s}(\partial_\phi f)\sigma_3\right) \\
&= e^{f\sigma_3}e^{-f\sigma_3}\left(\sigma_\pm^\theta(-i\widetilde{\nabla}_\theta^\pm) - i\partial_\theta f\sigma_\pm^\theta\sigma_3 + \sigma_\pm^\phi(-i\widetilde{\nabla}_\phi^\pm) - \frac{i}{s}\partial_\phi f\sigma_\pm^\phi\sigma_3\right) \\
&= \sigma_\pm^\theta(-i\widetilde{\nabla}_\theta^\pm + \frac{1}{s}\partial_\phi f) + \sigma_\pm^\phi(-i\widetilde{\nabla}_\phi^\pm - \partial_\theta f),
\end{aligned}$$

since $\sigma_\pm^\theta\sigma_3 = -i\sigma_\pm^\phi$ and $\sigma_\pm^\phi\sigma_3 = -i\sigma_\pm^\theta$. We also notice that $e^{\pm ig}$ commutes with all the $\sigma$'s, while $\partial_\theta(e^{-ig}.) = e^{-ig}(\partial_\theta - (i\partial_\theta g))$ and $\partial_\phi(e^{-ig}.) = e^{-ig}(\partial_\phi - (i\partial_\phi g))$. Thus

$$\begin{aligned}
e^{ig}e^{f\sigma_3}\mathcal{D}e^{-f\sigma_3}e^{-ig} &= \sigma_\pm^\theta(-i\widetilde{\nabla}_\theta^\pm - (\partial_\theta g - \frac{1}{s}\partial_\phi f)) + \sigma_\pm^\phi(-i\widetilde{\nabla}_\phi^\pm - (\frac{1}{s}\partial_\phi g + \partial_\theta f)) \\
&= \sigma_\pm^\theta(-i\widetilde{\nabla}_\theta^\pm - A_\theta) + \sigma_\pm^\phi(-i\widetilde{\nabla}_\phi^\pm - A_\phi) \\
&= \mathcal{D}_A,
\end{aligned}$$

with the help of (3.33). ■

*Proof of Theorem 3.6.1.* It follows from the result of Lemma 3.6.3 that $\xi \in \operatorname{Ker}\ \mathcal{D}_A$ if and only if $\xi = e^{-f\sigma_3}e^{ig}\eta$ for some $\eta \in \operatorname{Ker}\ \mathcal{D}$. Therefore, we have

$$\dim\operatorname{Ker}\ \mathcal{D} = \dim\operatorname{Ker}\mathcal{D}_A.$$

By Stokes' theorem,

$$\int_{\mathbb{S}^2} dA = 0,$$

so (3.32) gives us

$$\dim \operatorname{Ker}\ \mathcal{D}_A = |n| = \frac{1}{2\pi}\left| \int_{\mathbb{S}^2} \frac{n}{2}\ \mathrm{vol}_{\mathbb{S}^2} \right| = \frac{1}{2\pi}\left| \int_{\mathbb{S}^2} \left(\frac{n}{2}\mathrm{vol}_{\mathbb{S}^2} + dA\right) \right| = \frac{1}{2\pi}\left| \int_{\mathbb{S}^2} \beta \right|,$$

completing the result. ■

*Remark* 3.6.4. Our proof of the Ahanorov-Casher theorem on $\mathbb{S}^2$ is based fully on results we have obtained in this Chapter. In [24] Erdös and Solovej prove the same result using the relationship between Dirac operators on $\mathbb{S}^2$ and $\mathbb{R}^2 \cong \mathbb{C}$; they turn the problem into the similar problem in $\mathbb{R}^2$, where the techniques used in the proof of the Aharonov-Casher theorem in $\mathbb{R}^2$ can be applied (as was done in the proof of Theorem 1.4.1 in Chapter 1).

# Chapter 4

# Zero modes for Weyl-Dirac operators on $\mathbb{R}^3$

## 4.1 Introduction

In this chapter we will return to discuss Weyl-Dirac operators on $\mathbb{R}^3$, the main theme of this thesis. First, we will introduce the geometric construction used by Erdös and Solovej in [24] to construct a certain class of Weyl-Dirac operators on $\mathbb{R}^3$ with zero modes. They considered magnetic fields, or two-forms on $\mathbb{R}^3$ which are obtained by pulling back arbitrary two-forms on $\mathbb{S}^2$, firstly to $\mathbb{S}^3$ using the Hopf map, and then to $\mathbb{R}^3$ using inverse strereographic projection.

For the remainder of the chapter we return our attention to the quantity $n_A(T)$, the total number of zero modes for the Weyl-Dirac operator $\mathcal{D}_{tA}$ with scaled potential $tA$, as we vary $t$ from 0 to $T$. Using the fact that the original Loss-Yau example of a zero mode is the simplest of the Erdös-Solovej class of examples, together with the explicit spectral calculation in Section 3.5, we determine an explicit formula for $n_{A_{LY}}(T)$ (a formula for $A_{LY}$ is given in (1.16)).

In the next part of this chapter we obtain a relationship between the Dirac operator on $\mathbb{S}^2$ and the Dirac operator on $\mathbb{R}^2$. This one is a preparation step to show a lower bound for the number of zero modes of the Weyl-Dirac operators on $\mathbb{R}^3$. Actually, we need to obtain an estimate on the number of the "small" eigenvalues for Dirac operators on $\mathbb{S}^2$ through a similar one for Pauli operators on a disc.

In the final part of the chapter we consider $n_A(T)$ for general magnetic potentials $A$ with corresponding magnetic fields in the class considered by Erdös and Solovej in [24]. In this case an explicit $O(T^2)$ lower bound is obtained (see Theorem 4.5.1). The construction of Erdös and Solovej reduces this problem to the study of "small" eigenvalues of Dirac operators on $\mathbb{S}^2$ as studied in Section 4.4.

## 4.2 The construction of Erdös and Solovej

In this section we will introduce the construction of Erdös and Solovej from [24]. The purpose is to construct a large class of magnetic fields on $\mathbb{R}^3$ such that the corresponding Weyl-Dirac operators have zero modes with a prescribed multiplicity. The key idea is to consider a class of magnetic fields on $\mathbb{S}^3$, which are in fact the pullbacks of two-forms on $\mathbb{S}^2$, and then use the conformal invariance of the dimension of kernels of Dirac operators.

### 4.2.1 The class of magnetic fields

We will use the *Hopf map*, here denoted by $\kappa$. It acts from $\mathbb{S}^3$ to $\mathbb{S}^2$ and we will explain a little bit more about this mapping: Here $\mathbb{S}^3$ and $\mathbb{S}^2$ are unit balls in $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Given $(x_1, x_2, x_3, x_4) \in \mathbb{S}^3$ we have $\kappa(x_1, x_2, x_3, x_4) = (\xi_1, \xi_2, \xi_3)$, where $(\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$ is given by

$$\begin{aligned}
\xi_1 &= 2(x_1x_3 + x_2x_4) \\
\xi_2 &= 2(x_2x_3 - x_1x_4) \\
\xi_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2.
\end{aligned}$$

*Remark.* We can check that $\kappa$ is surjective. Furthermore, we can check that $\kappa_*$ actually is a surjective partial isometry between $\mathbb{S}^3$ with usual metric and $\mathbb{S}^2$ with $\frac{1}{4}g_{\mathbb{S}^2}$ as its metric (see [24], Lemma 7.1). Then, $\kappa$ is a *Riemannian submersion.* In fact this is one of the important properties of the Hopf map which is used in [24]; a Riemannian submersion between Riemannian manifolds $M$ and $N$ allows us to pull back Spin$^c$ structures including spinor bundles (with Clifford multiplication), Spin$^c$ connections and finally lift Dirac operator from $N$ to $M$.

Let $\tau^{-1}$ be the stereographic projection from $\mathbb{S}^3 \setminus \{(0,0,0,-1)\}$ to $\mathbb{R}^3$; that is

$$\tau^{-1}(x_1, x_2, x_3, x_4) = \Big(\frac{x_1}{1+x_4},\ \frac{x_2}{1+x_4},\ \frac{x_3}{1+x_4}\Big) \in \mathbb{R}^3$$

for all $(x_1, x_2, x_3, x_4) \in \mathbb{S}^3$. The inverse of this map gives us a smooth map $\tau : \mathbb{R}^3 \longrightarrow \mathbb{S}^3$. Now denote by $\iota :\ \mathbb{R}^3 \longrightarrow \mathbb{S}^2$, where $\iota = \kappa \circ \tau$.

Recall that two-forms on $\mathbb{S}^2$ are $f\mathrm{vol}_{\mathbb{S}^2}$, where $f$ is a smooth function on $\mathbb{S}^2$, and any two-form on $\mathbb{S}^2$ is closed. Then, the pullback of any two-form on $\mathbb{S}^2$ by $\iota^*$ is also a closed two-form on $\mathbb{R}^3$. Thus $\iota^*(f\mathrm{vol}_{\mathbb{S}^2})$ is a magnetic field on $\mathbb{R}^3$ for any smooth function $f$ on $\mathbb{S}^2$ (see [41]). Therefore, we can define the following class of magnetic fields on $\mathbb{R}^3$.

**Definition 4.2.1.** Let $\mathcal{B}_{ES}$ be the class of magnetic fields on $\mathbb{R}^3$ given by

$$\mathcal{B}_{ES} = \big\{\iota^*(f\mathrm{vol}_{\mathbb{S}^2}) : \quad f \in C^\infty(\mathbb{S}^2)\big\}.$$

*Remark.* We may see from [22] that

$$B_{LY} = \iota^*\left(\frac{3}{4}\mathrm{vol}_{\mathbb{S}^2}\right), \tag{4.1}$$

where $B_{LY}$ is given in (1.17). We then have

$$\iota^*(\mathrm{vol}_{\mathbb{S}^2}) = \frac{16}{(1+|x|^2)^3}\begin{pmatrix} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}^T. \tag{4.2}$$

Using $\iota^*(f\mathrm{vol}_{\mathbb{S}^2}) = (f \circ \iota)\iota^*(\mathrm{vol}_{\mathbb{S}^2})$ we observe that $B \in \mathcal{B}_{ES}$ iff (as a vector field)

$$B(x) = f(\iota(x))\frac{16}{(1+|x|^2)^3}\begin{pmatrix} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}^T \tag{4.3}$$

for some $f \in C^\infty(\mathbb{S}^2)$. In particular it follows that the decay rate of a general element of $\mathcal{B}_{ES}$ is $O(|x|^{-4})$ as $|x| \to \infty$.

### 4.2.2 The construction

We have given the general definition of a Spin$^c$ spinor bundle and Spin$^c$ connection for an arbitrary two or three dimensional manifold (see Definition 3.2.4 and Definition 3.2.6, respectively). We have also given a more detailed description of possible Spin$^c$ bundles and connections on $\mathbb{S}^2$ (see Subsection 3.2.3) and $\mathbb{R}^2$ (see Remark 3.2.8). The situation on $\mathbb{S}^3$ is actually simpler; it is known (see [24] for example) that, up to isomorphism, there is a unique Spin$^c$ bundle on $\mathbb{S}^3$ (which we can take to be the trivial bundle $\mathbb{S}^3 \times \mathbb{C}^2$), while for an arbitrary magnetic field on $\mathbb{S}^3$ (that is, an arbitrary closed two form) there is a Spin$^c$ connection and corresponding Dirac operator on $\mathbb{S}^3$ with this magnetic field. Furthermore, this connection and hence Dirac operator, are unique up to gauge equivalence; in particular, from a spectral point of view, the Dirac operator is determined by the magnetic field. The situation on $\mathbb{R}^3$ is similar (see the discussion on $\mathcal{D}_A$ in Section 1.1 of Chapter 1 for more details).

Now we will consider the construction of Erdös and Solovej from [24]. Let $B \in \mathcal{B}_{ES}$ in (4.3), so $B = \tau^*\, b, \quad b = \kappa^*\beta$, where $\beta = f\mathrm{vol}_{\mathbb{S}^2}$ for some $f \in C^\infty(\mathbb{S}^2)$. Denote by $\Phi$ the *total flux* of $\beta$; that is

$$\Phi = \frac{1}{2\pi}\int_{\mathbb{S}^2} \beta.$$

For each integer $k$, we define $\beta_k := \left[f - \frac{1}{2}(\Phi - k)\right]\mathrm{vol}_{\mathbb{S}^2}$. We see that

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} \beta_k = \frac{1}{2\pi}\int_{\mathbb{S}^2} f\ \mathrm{vol}_{\mathbb{S}^2} - \frac{(\Phi - k)}{4\pi}\int_{\mathbb{S}^2} \mathrm{vol}_{\mathbb{S}^2} = k,$$

since $\int_{\mathbb{S}^2} \mathrm{vol}_{\mathbb{S}^2} = 4\pi$. We can then use the result of Propostion 3.3.8 and get the Dirac operator on $\Psi_k$, which we denote by $\mathcal{D}_{\beta_k,\mathbb{S}^2}$.

Denote by $S_k$ the set

$$S_k = \begin{cases} \{-k+\Phi-\frac{1}{2}\} & \text{if } k>0 \\ \emptyset & \text{if } k=0 \\ \{k-\Phi-\frac{1}{2}\} & \text{if } k<0. \end{cases}$$

Denote by $\mathcal{D}_{b,\mathbb{S}^3}$ the Dirac operator on $\mathbb{S}^3$ with magnetic field $b$. Then, Erdös and Solovej showed how the spectrum of $\mathcal{D}_{b,\mathbb{S}^3}$ can be expressed through the positive spectrum of the Dirac operators $\mathcal{D}_{\beta_k,\mathbb{S}^2}$ on $\mathbb{S}^2$ with magnetic fields $\beta_k$ as follows.

**Theorem 4.2.2.** (Theorem 8.1 in [24]) *Suppose that $\nabla_{\mathbb{S}^3}$ is a Spin$^c$ connection on $\Psi_{\mathbb{S}^3}$ with magnetic field $b$. Suppose that* $\mathrm{Spec}^+(\mathcal{D}_{\beta_k,\mathbb{S}^2})$ *is the positive spectrum of that Dirac operator $\mathcal{D}_{\beta_k,\mathbb{S}^2}$ on $\mathbb{S}^2$ with magnetic $\beta_k$. Then, we have*

- *the spectrum of $\mathcal{D}_{b,\mathbb{S}^3}$ is given by*

$$\mathrm{Spec}(\mathcal{D}_{b,\mathbb{S}^3}) = \bigcup_{k\in\mathbb{Z}} \left( S_k \ \cup \left\{ \pm\sqrt{\lambda^2+(k-\Phi)^2} - \frac{1}{2} : \ \lambda \in \mathrm{Spec}^+(\mathcal{D}_{\beta_k,\mathbb{S}^2}) \right\} \right) \tag{4.4}$$

- *the multiplicity of an eigenvalue in* $\mathrm{Spec}(\mathcal{D}_{b,\mathbb{S}^3})$ *is equal to the number of ways it can be written as $\pm\sqrt{\lambda^2+(k-\Phi)^2}-\frac{1}{2}$ with $k\in\mathbb{Z}$ and $\lambda \in$* $\mathrm{Spec}^+(\mathcal{D}_{\beta_k,\mathbb{S}^2})$ *counted with multiplicity, or as an element in $S_k$ counted with multiplicity $|k|$.*

By the invariance property of the dimension of the kernel for the Dirac operators up to conformal transformations Erdös and Solovej then showed that

**Theorem 4.2.3.** (Theorem 8.7 in [24]) *Let $\mathcal{D}_{B,\mathbb{R}^3}$ be the Dirac operator on $\mathbb{R}^3$ with magnetic two-form $B$, and $\mathcal{D}_{b,\mathbb{S}^3}$ the Dirac operator on $\mathbb{S}^3$ with magnetic two-form $b$. Then,*

$$\dim \mathrm{Ker}\, \mathcal{D}_{B,\mathbb{R}^3} = \dim \mathrm{Ker}\, \mathcal{D}_{b,\mathbb{S}^3}.$$

For any magnetic field $B \in \mathcal{B}_{ES}$ Theorem 4.2.2 and Theorem 4.2.3 shows us where zero modes of the Weyl-Dirac operator $\mathcal{D}_{B,\mathbb{R}^3}$ come from. More precisely, there are only two types of zero modes of $\mathcal{D}_{B,\mathbb{R}^3}$; that is

- Type I zero modes of $\mathcal{D}_{B,\mathbb{R}^3}$. These are zero modes coming from the set $S_k$. In particular, $\mathcal{D}_{B,\mathbb{R}^3}$ has exactly $|k|$ zero modes if

$$\mp(k-\Phi) - \frac{1}{2} = 0 \Longleftrightarrow \Phi = k \pm \frac{1}{2} \quad \text{for } k\in\mathbb{Z}, \ \pm k>0. \tag{4.5}$$

- Type II zero modes of $\mathcal{D}_{B,\mathbb{R}^3}$. These are zero modes coming from strictly positive eigenvalues of $\mathcal{D}_{\beta_k,\mathbb{S}^2}$; this happens when

$$\sqrt{\lambda^2 + (k-\Phi)^2} - \frac{1}{2} = 0 \Longleftrightarrow \lambda^2 + (k-\Phi)^2 = \frac{1}{4} \tag{4.6}$$

for some $\lambda \in \mathrm{Spec}^+(\mathcal{D}_{\beta_k,\mathbb{S}^2})$. In particular we need $\lambda$ to satisfy

$$0 < \lambda < \sqrt{\frac{1}{4} - (k-\Phi)^2} \leq \frac{1}{2}.$$

We observe that we cannot have Type I and Type II zero modes simultaneously, since (4.6) cannot be satisfied by $\lambda > 0$ when $\Phi = k \pm \frac{1}{2}$.

Considering Type I zero modes we thus arrive at the following corollary of Theorem 4.2.2 and Theorem 4.2.3

**Corollary 4.2.4.** *Suppose $B = i^*(f\mathrm{vol}_{\mathbb{S}^2}) \in \mathcal{B}_{ES}$, where $f \in C^\infty(\mathbb{S}^2)$ satisfies*

$$\Phi = \frac{1}{2\pi}\int_{\mathbb{S}^2} f\mathrm{vol}_{\mathbb{S}^2} = \pm(k + \frac{1}{2})$$

*for some $k \in \mathbb{N}$. Then,*

$$\dim \mathrm{Ker}\, \mathcal{D}_{B,\mathbb{R}^3} = k.$$

Ultimately this is the result used by Erdös and Solovej to construct zero mode producing magnetic fields on $\mathbb{R}^3$. The simplest (non-trivial) example is given by the constant function $f = \frac{3}{4}$, for which $\Phi = \frac{3}{2}$ and $k = 1$; the corresponding magnetic field is the original example given by Loss and Yau (see (1.17)). We notice that in [24] Erdös and Solovej do not investigate the Type II zero modes.

## 4.3 The Loss-Yau example revisited

Looking back to Theorem 3.5.2 we see that we know all eigenvalues and their multiplicities for the Dirac operator with magnetic field $\frac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$ on $\mathbb{S}^2$. The Chern number for the spinor bundle here is

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{n}{2}\,\mathrm{vol}_{\mathbb{S}^2} = n,$$

while the positive spectrum of the Dirac operator is the set $\{\sqrt{j(j+|n|)},\ j \in \mathbb{N}\}$, where the eigenvalue $\sqrt{j(j+|n|)}$ has multiplicity $2j + |n|$.

We will now consider the magnetic field $b_0 = \kappa^*(\frac{1}{2}\mathrm{vol}_{\mathbb{S}^2})$ on $\mathbb{S}^3$ where $\kappa$ is the Hopf map as above. Therefore, in the notation of the previous section we are taking $f = \frac{1}{2}$

as a constant function on $\mathbb{S}^2$. Additionally, $tb_0 = \kappa^*(\frac{t}{2}\mathrm{vol}_{\mathbb{S}^2})$, for $t \geq 0$ is a scaling of the initial magnetic field $b_0$. We pullback the magnetic field $tb_0$ by $\tau^*$ as above. Then we can get a magnetic field $tB_0 \in \mathcal{B}_{ES}$ on $\mathbb{R}^3$ and the corresponding Weyl-Dirac operator $\mathcal{D}_{tB_0,\mathbb{R}^3}$ defined on $\mathbb{R}^3$ as discussed in Theorem 4.2.3; in particular, $tB_0 = \iota^*(\frac{t}{2}\mathrm{vol}_{\mathbb{S}^2})$. Observe from (4.1) that

$$B_{LY} = \frac{3}{2}B_0. \tag{4.7}$$

Denote by $\mathcal{D}_{tb_0,\mathbb{S}^3}$ the Dirac operators on $\mathbb{S}^3$ with magnetic fields $tb_0$. We have

$$\Phi_{tb_0} = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{t}{2}\mathrm{vol}_{\mathbb{S}^2} = t.$$

It then follows from Theorem 3.5.2 and Theorem 4.2.2 that

$$\mathrm{Spec}(\mathcal{D}_{tb_0,\mathbb{S}^3}) = \bigcup_{k\in\mathbb{Z},\ j\in\mathbb{N}} \big(S_k \cup \{\pm\sqrt{j(j+|k|)+(k-t)^2}-\frac{1}{2}\}\big),$$

where

$$S_k = \begin{cases} \{-k+t-\frac{1}{2}\} & \text{if } k>0 \\ \emptyset & \text{if } k=0 \\ \{k-t-\frac{1}{2}\} & \text{if } k<0. \end{cases}$$

Using Maple we can illustrate the spectrum of the Dirac operators $\mathcal{D}_{tb_0,\mathbb{S}^3}$ corresponding to the scaled magnetic field for $tb_0$ as Figure 4.1. As discussed above the multiplicity for all eigenvalues of $\mathcal{D}_{tb_0,\mathbb{S}^3}$ may be determined by the result of Theorem 4.2.2. In paticular we can determined exactly the multiplicity of 0 as an eigenvalue. Therefore, we can determine exactly the total number of zero modes for all of the Weyl-Dirac operators $\mathcal{D}_{tB_0,\mathbb{R}^3},\ t \geq 0$. In turn, this allows us to obtain an explicit formula for the quantity $n_{B_0}(T)$.

**Theorem 4.3.1.** *Let $\mathcal{D}_{tB_0,\mathbb{R}^3}$ be the Weyl-Dirac operators on $\mathbb{R}^3$ with magnetic fields $tB_0$. Then, for any $T>0$ we have*

$$n_{B_0}(T) = \frac{[T-1](1+[T-1])}{2},$$

*where $n_{B_0}(T)$ denotes the number of zero modes for the Weyl-Dirac operators $\mathcal{D}_{tB_0}$ for $0 \leq t \leq T$. Here $[x]$ denotes the nearest integer to $x$, rounded up if $x \in \mathbb{Z}+\frac{1}{2}$.*

*Proof.* Looking back to the discussion after Theorem 4.2.3 we have Type I zero modes for $\mathcal{D}_{tB_0,\mathbb{R}^3}$ when

$$t = \pm(n+\frac{1}{2}) \quad \text{for } n \in \mathbb{N} \quad \text{(with multiplicity } n),$$

or since we are only considering $t \geq 0$,

$$t = n+\frac{1}{2} \quad \text{for } n \in \mathbb{N}.$$

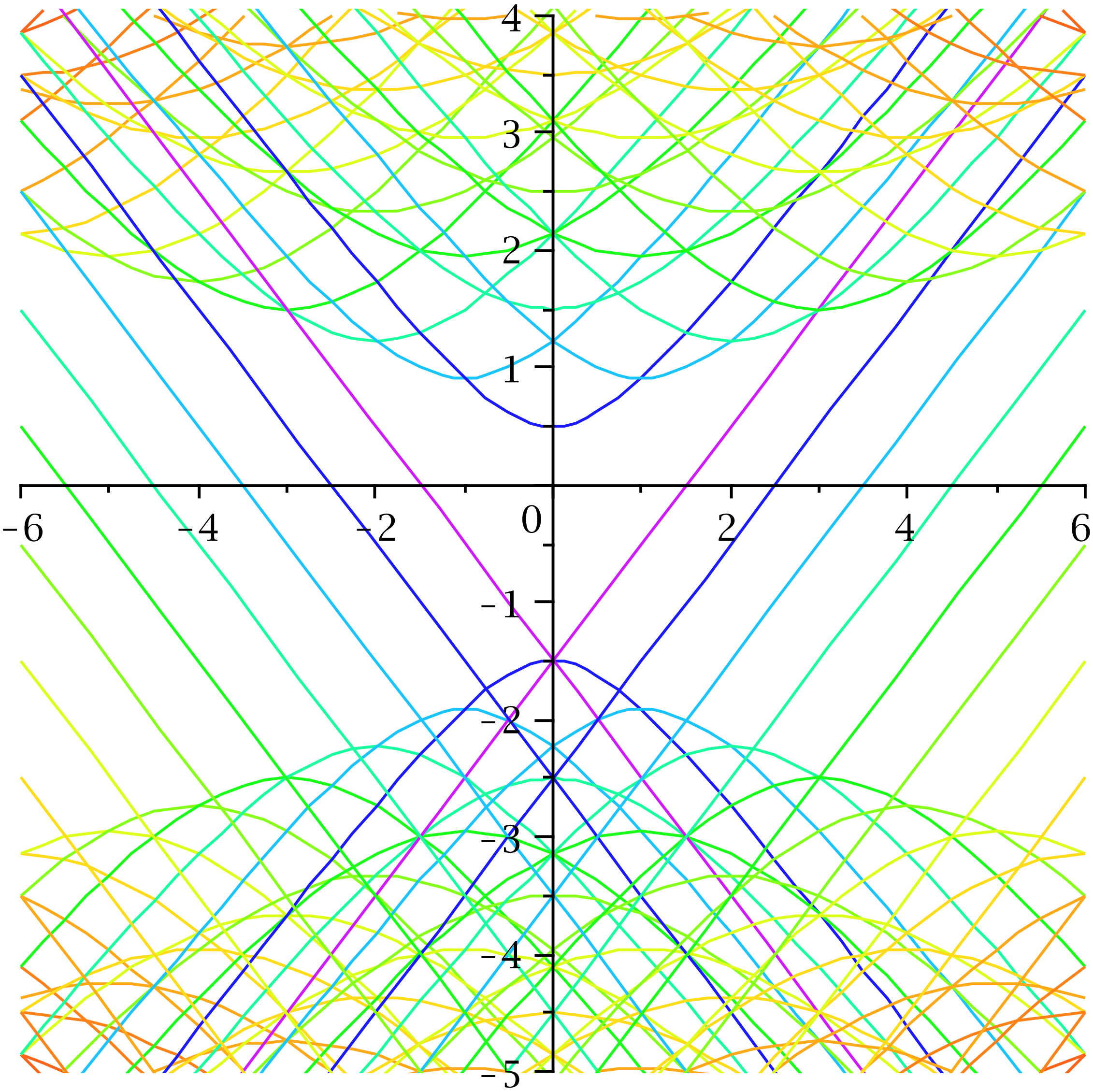


Figure 4.1: Spectrum of the Dirac operators $\mathcal{D}_{tb_0,\mathbb{S}^3}$ when scaling a 'constant' magnetic field $b_0$. *Note*: Horizontal axis: $t$- axis; Vertical axis: "spectrum" axis; Colour of curves: multiplicity for eigenvalues

However there are no Type II zero modes of $\mathcal{D}_{tB_0,\mathbb{R}^3}$ since we cannot have $\lambda^2 + (k - \Phi)^2 = \dfrac{1}{4}$ for $\lambda = \sqrt{j(j+|k|)} \geq 1$ when $j \in \mathbb{N}$. In conclusion zero modes of $\mathcal{D}_{tB_0,\mathbb{R}^3}$ only occur when $t = n + \dfrac{1}{2}$ for $n \in \mathbb{N}$ and each time we have exactly $n$ zero modes. Observe that $t = n + \dfrac{1}{2} \leq T \Longleftrightarrow n \leq T - \dfrac{1}{2} \Longleftrightarrow n \leq [T-1]$. Thus we have

$$n_{B_0}(T) = \sum_{n=1}^{[T-1]} n = \frac{[T-1](1+[T-1])}{2}$$

as required. ■

*Remark* 4.3.2. We can see partly the result of Theorem 4.3.1 in Figure 4.1. We can see the spectrum for $\mathcal{D}_{tB_0,\mathbb{R}^3}$ there: they are all intersections between curves and vertical line passing through $t$ on the horizontal axis. Therefore we do not see any Type II zero modes: intersections with curves. Type I zero modes occur only when $t = 3/2,\ 5/2, \dots$, or only for real number $t$ such that $2t := 3, 5, \dots$. The colour of the line shows us the different multiplicity of zero modes corresponding to different $t$. The intersections between curves show us the different expressions in the type of $\pm\sqrt{\lambda^2 + (k-t)^2} - \frac{1}{2},\ k \in \mathbb{Z}$. Finally, we observe that it follows from (4.7) that

$$n_{B_{LY}}(T) = n_{B_0}\left(\frac{3}{2}T\right) \quad \text{for } T > 0.$$

## 4.4 Small eigenvalues of Dirac operators on $\mathbb{S}^2$

We remark that in the explicit example of the previous section we are able to count $n_{B_0}(T)$ since there were no Type II zero modes; that is zeros of

$$\sqrt{\lambda^2 + (k-t)^2} - \frac{1}{2} \quad \text{for } \lambda \in \operatorname{Spec}^+(\mathcal{D}_{\beta_k,\mathbb{S}^2}).$$

All zero modes of $\mathcal{D}_{tb_0,\mathbb{S}^3}$ were Type I; that is they came from sets $S_k$. However, for a general smooth function $f$ on $\mathbb{S}^2$ we may not have as complete information about $\lambda$ as for the case for $f = \dfrac{1}{2}$; in particular we may obtain some Type II zero modes. For instance, when the Dirac operator on $\mathbb{S}^2$ has an eigenvalue $\lambda$ such that $0 < \lambda < \frac{1}{2}$ it would induce more zero modes for $\mathcal{D}_{tB,\mathbb{R}^3}$ which correspond to eigenvalue 0 of $\mathcal{D}_{tb,\mathbb{S}^3}$ coming from the equation $\sqrt{\lambda^2 + (k-t)^2} - \frac{1}{2} = 0$. In our opinion it is hard to determine such "small" eigenvalues of the Dirac operator on $\mathbb{S}^2$, but we can estimate their number and obtain a lower bound for $n_B(T)$ for a general function $f$. This section will deal with that estimate.

We will first use the idea of conformal transformation between $\mathbb{S}^2$ and $\mathbb{R}^2$ to interchange the concept of the Dirac operators on $\mathbb{S}^2$ and Dirac operators on $\mathbb{R}^2$ as we did for $\mathbb{S}^3$ and $\mathbb{R}^3$ in Theorem 4.2.3. It is because the usual metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$

considered as $\mathbb{C} \cup \{\infty\}$ is conformally equivalent to the standard metric $g_{\mathbb{R}^2}$ on $\mathbb{R}^2$. The main idea is to use once again the stereographic projection from $\mathbb{S}^2 \setminus \{p\}$ to $\mathbb{R}^2$ to pullback the Spin$^c$ structure on $\mathbb{S}^2$ to $\mathbb{R}^2$. Then, we can study some properties of small eigenvalues of the Dirac operator on $\mathbb{S}^2$ through the property of eigenvalues of the Dirac operator on $\mathbb{R}^2$.

We re-use here the notation of Chapter 3. We considered the following Spin$^c$ connection on $\Psi_n$ :

$$\nabla_\theta^\pm = \partial_\theta - iA_\theta \text{ and}$$

$$\nabla_\phi^\pm = \frac{1}{s}\partial_\phi + \frac{is}{2(c \pm 1)}\sigma_3 - \frac{isn}{2(c \pm 1)} - iA_\phi.$$

Then, we can define

$$D_\theta^\pm = -i\nabla_\theta^\pm - A_\theta = -i\partial_\theta - A_\theta,$$

and

$$D_\phi^\pm = -i\nabla_\phi^\pm - A_\phi := \frac{-i}{s}\partial_\phi + \frac{s}{2(c \pm 1)}\sigma_3 - \frac{sn}{2(c \pm 1)} - A_\phi.$$

It follows that we obtained the Dirac operator $\mathcal{D}_{A,\mathbb{S}^2} := \sigma^\theta \mathcal{D}_\theta + \sigma^\phi \mathcal{D}_\phi$ defined on spinors on $\mathbb{S}^2$; namely sections $\Gamma(\Psi_n)$ of $\Psi_n$. Our purpose here is to establish a formula to show the relationship between $\mathcal{D}_{nA,\mathbb{S}^2}$ and the usual Dirac operator on $\mathbb{R}^2$ with some magnetic potential $A'$.

Let $z_+ : \mathbb{S}^2_+ \longrightarrow \mathbb{R}^2$ denote the strereographic projection from the sphere with the south pole removed to the plane. More precisely, for $p = (\sin\theta e^{i\phi}, \ \cos\theta) \in \mathbb{S}^2_+$ (where we are viewing $\mathbb{S}^2$ as the unit sphere in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$), we have $z_+(p) = 2\tan\frac{\theta}{2}\, e^{i\phi} = x + iy$, where

$$x = \frac{2s}{1+c}c_\phi \quad \text{and} \quad y = \frac{2s}{1+c}s_\phi. \tag{4.8}$$

Now set $\mu := \dfrac{s}{1+c}$, so $x = 2\mu c_\phi$ and $y = 2\mu s_\phi$. Also let $\Omega := \dfrac{s}{2\mu} = \dfrac{2}{c+1}$. Set $A' = (A_x + \frac{n}{2}\alpha_x)dx + (A_y + \frac{n}{2}\alpha_y)dy$, where

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \frac{s}{4\mu^2}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix},$$

and

$$\begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \frac{s}{4\mu^2}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} 0 \\ \mu \end{pmatrix}.$$

Denote by $\mathcal{D}_{A',\mathbb{R}^2}$ the usual Dirac operator on $\mathbb{R}^2$ with magnetic potential $A'$; that is the operator

$$\mathcal{D}_{A',\mathbb{R}^2} := \sigma \cdot (-i\nabla - A') = (\sigma_1 \ \sigma_2)\left[\begin{pmatrix} -i\partial_x \\ -i\partial_y \end{pmatrix} - \begin{pmatrix} A_x \\ A_y \end{pmatrix} - \frac{n}{2}\begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}\right].$$

The next result gives the link between the Dirac operators $\mathcal{D}_{A,\mathbb{S}^2}$ and $\mathcal{D}_{A',\mathbb{R}^2}$.

**Theorem 4.4.1.** *We have* $\Omega^{\frac{3}{2}}\mathcal{D}_{A,\mathbb{S}^2}\Omega^{-\frac{1}{2}} = \mathcal{D}_{A',\mathbb{R}^2}$ *in the sense that for all functions* $\eta:\ \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ *we have*

$$(\Omega^{\frac{3}{2}}\mathcal{D}_{A,\mathbb{S}^2}\Omega^{-\frac{1}{2}})(\eta \circ z_+) = (\mathcal{D}_{A',\mathbb{R}^2}\eta) \circ z_+.$$

First, we notice that $\eta \circ z_+$ is a map $\mathbb{S}^2_+ \longrightarrow \mathbb{C}^2$, which we view as a section of $\Psi^+_n$. Second, in Theorem 4.4.1 we are considering $\mathbb{S}^2_+$. A similar result can be obtained for $\mathbb{S}^2_-$. We need to use the stereographic projection $z_-:\ \mathbb{S}^2_- \longrightarrow \mathbb{R}^2$ from the sphere with the north pole removed to the plane; for $p = (-\sin\theta e^{i\phi},\ -\cos\theta) \in \mathbb{S}^2_-$ (where we are viewing $\mathbb{S}^2$ as the unit sphere in $\mathbb{C}\times\mathbb{R} \cong \mathbb{R}^3$) we have $z_-(p) = 2\cot\frac{\theta}{2}\, e^{i\phi} = x+iy$. Then, we will have $A'_-$ and $\Omega_-$ for $\mathbb{S}^2_-$ as well. Notice that $\Omega_- = \dfrac{2}{1-c}$. When necessary we will use notations $\Omega_\pm$ and $A'_\pm$ to correspond to the ones with respect to $\mathbb{S}^2_\pm$. The corresponding statement for Theorem 4.4.1 is that

$$(\Omega_\pm^{\frac{3}{2}}\mathcal{D}_{A,\mathbb{S}^2}\Omega_\pm^{-\frac{1}{2}})(\eta \circ z_\pm) = (\mathcal{D}_{A'_\pm,\mathbb{R}^2}\eta) \circ z_\pm \tag{4.9}$$

for all $\eta:\ \mathbb{R}^2 \longrightarrow \mathbb{C}^2$.
Finally, denote by $\mathbb{H}_\pm$ the northern and southern hemisphere of $\mathbb{S}^2$, respectively. Then, we remark that $z_\pm(\mathbb{H}_\pm) = \mathbb{D}$, where $\mathbb{D}$ is the disc

$$\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq 2\}.$$

*Proof of Theorem 4.4.1* Truly, it follows from our setting

$$x = \frac{2s}{1+c}c_\phi :=\ 2\mu c_\phi \text{ and } y = \frac{2s}{1+c}s_\phi := 2\mu s_\phi,$$

for $\mu := \dfrac{s}{1+c}$ and $\Omega = \dfrac{s}{2\mu}$ that

$$\begin{pmatrix}\partial_\theta\\ \partial_\phi\end{pmatrix} = \begin{pmatrix}\partial_\theta x & \partial_\theta y\\ \partial_\phi x & \partial_\phi y\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} = \begin{pmatrix}\dfrac{2c_\phi}{1+c} & \dfrac{2s_\phi}{1+c}\\ \dfrac{-2ss_\phi}{1+c} & \dfrac{2sc_\phi}{1+c}\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} =$$
$$\begin{pmatrix}\dfrac{x}{s} & \dfrac{y}{s}\\ -y & x\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}.$$

Then, we have

$$\begin{pmatrix}\widehat{e}_\theta\\ \widehat{e}_\phi\end{pmatrix} = \begin{pmatrix}\partial_\theta\\ \dfrac{1}{s}\partial_\phi\end{pmatrix} = \frac{1}{s}\begin{pmatrix}x & y\\ -y & x\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}.$$

Similarly, we have

$$\begin{pmatrix}dx\\ dy\end{pmatrix} = \begin{pmatrix}\partial_\theta x & \partial_\phi x\\ \partial_\theta y & \partial_\phi y\end{pmatrix}\begin{pmatrix}d\theta\\ d\phi\end{pmatrix} = \begin{pmatrix}\dfrac{x}{s} & -y\\ \dfrac{y}{s} & x\end{pmatrix}\begin{pmatrix}d\theta\\ d\phi\end{pmatrix}.$$

Therefore, we obtain

$$\begin{pmatrix} \widehat{e}^{\theta} \\ \widehat{e}^{\phi} \end{pmatrix} = \begin{pmatrix} d\theta \\ sd\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \frac{x}{s} & -y \\ \frac{y}{s} & x \end{pmatrix}^{-1} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

We see that $x^2 + y^2 = 4\mu^2$, then

$$\begin{pmatrix} \widehat{e}^{\theta} \\ \widehat{e}^{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \frac{s}{x^2+y^2} \begin{pmatrix} x & y \\ \frac{-y}{s} & \frac{x}{s} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{s}{4\mu^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Thus, the volume form becomes

$$\widehat{e}^{\theta} \wedge \widehat{e}^{\phi} = (\frac{s}{4\mu^2})^2 (xdx \wedge ydy) \wedge (-ydx + xdy) = \frac{s^2}{4\mu^2} dx \wedge dy.$$

We observe that

$$\sigma^{\theta} = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} = \frac{1}{2\mu} \begin{pmatrix} 0 & x - iy \\ x + iy & 0 \end{pmatrix} = \frac{1}{2\mu}(x\sigma_1 + y\sigma_2),$$

and similarly

$$\sigma^{\phi} = \frac{1}{2\mu}(-y\sigma_1 + x\sigma_2).$$

We simply put both in the following expression

$$\begin{pmatrix} \sigma^{\theta} \\ \sigma^{\phi} \end{pmatrix} = \frac{1}{2\mu} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}.$$

This can help us to change the magnetic potential $A = \begin{pmatrix} A_\theta & A_\phi \end{pmatrix} \begin{pmatrix} \sigma^{\theta} \\ \sigma^{\phi} \end{pmatrix}$ into the magnetic potential on $\mathbb{R}^2$ as $A_x dx + A_y dy = (A_x \ A_y) \begin{pmatrix} dx \\ dy \end{pmatrix}$. Truly, it follows from the expression above we have

$$A = (A_\theta \ A_\phi) \frac{s}{4\mu^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Then,

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \frac{s}{4\mu^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^T \begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} = \frac{s}{4\mu^2} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix}. \tag{4.10}$$

Note that

$$\begin{aligned} \Omega^{\frac{3}{2}} \mathcal{D}_{A,\mathbb{S}^2} \Omega^{-\frac{1}{2}} &= \Omega \mathcal{D}_{A,\mathbb{S}^2} + \Omega^{\frac{3}{2}} \sigma^{\theta} \big( (-i\partial_\theta) \Omega^{-\frac{1}{2}} \big) = \Omega \mathcal{D}_{A,\mathbb{S}^2} + \Omega^{\frac{3}{2}} \sigma^{\theta} \frac{i}{2} \Omega^{-\frac{3}{2}} (\partial_\theta \Omega) \\ &= \Omega \mathcal{D}_{A,\mathbb{S}^2} - \frac{is}{4} \sigma^{\theta} \end{aligned}.$$

Therefore, we have

$$
\begin{aligned}
\Omega^{\frac{3}{2}}\mathcal{D}_{A,\mathbb{S}^2}\Omega^{-\frac{1}{2}} &= \Omega(\sigma^\theta\ \sigma^\phi)\begin{pmatrix}-i\nabla_\theta\\ -i\nabla_\phi\end{pmatrix} - \frac{is}{4}\sigma^\theta\\
&= \Omega(\sigma^\theta\ \sigma^\phi)\Bigg(-i\begin{pmatrix}\partial_\theta\\ \frac{1}{s}\partial_\phi\end{pmatrix} + \begin{pmatrix}0\\ \frac{\mu}{2}\sigma_3\end{pmatrix} - \begin{pmatrix}A_\theta\\ A_\phi + \frac{n\mu}{2}\end{pmatrix}\Bigg) - \frac{is}{4}\sigma^\theta\\
&= \frac{s}{4\mu^2}\begin{pmatrix}x & y\\ -y & x\end{pmatrix}^T\Bigg[-i\frac{1}{s}\begin{pmatrix}x & y\\ -y & x\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} - \begin{pmatrix}A_\theta\\ A_\phi + \frac{n\mu}{2}\end{pmatrix}\Bigg] + \frac{s}{4}\sigma^\phi\sigma_3 - \frac{is}{4}\sigma^\theta\\
&= (\sigma_1\ \sigma_2)\Bigg[\begin{pmatrix}-i\partial_x\\ -i\partial_y\end{pmatrix} - \frac{s}{4\mu^2}\begin{pmatrix}x & -y\\ y & x\end{pmatrix}\begin{pmatrix}A_\theta\\ A_\phi + \frac{n\mu}{2}\end{pmatrix}\Bigg] \text{ since } \sigma^\phi\sigma_3 = i\sigma^\theta\\
&= (\sigma_1\ \sigma_2)\Bigg[\begin{pmatrix}-i\partial_x\\ -i\partial_y\end{pmatrix} - \begin{pmatrix}A_x\\ A_y\end{pmatrix} - \frac{n}{2}\begin{pmatrix}\alpha_x\\ \alpha_y\end{pmatrix}\Bigg],
\end{aligned}
$$

where $\begin{pmatrix}\alpha_x\\ \alpha_y\end{pmatrix} = \frac{s}{4\mu^2}\begin{pmatrix}x & -y\\ y & x\end{pmatrix}\begin{pmatrix}0\\ \mu\end{pmatrix}$. The last expression is clearly the Dirac operator $\mathcal{D}_{A',\mathbb{R}^2}$ on $\mathbb{R}^2$ as expected.

The case on $\mathbb{S}^2_-$ we do all similar steps above and then conclude the justification for Theorem 4.4.1 here. ■

*Remark* 4.4.2. We notice that the magnetic field $\partial_x A_y - \partial_y A_x$ on $\mathbb{R}^2$ will correspond to the magnetic field $\frac{1}{s}\partial_\theta(sA_\phi) - \frac{1}{s}\partial_\phi(A_\theta)$; and $\frac{n}{2}\begin{pmatrix}\alpha_x\\ \alpha_y\end{pmatrix}$ is responsible for the "constant" magnetic field $\frac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$ on $\mathbb{S}^2$. In fact we have implemented a variable substitution for $(\theta,\phi)$ on $\mathbb{S}^2$ to $(x,y)$ on $\mathbb{R}^2$.

Now we will use the result of Theorem 4.4.1 above to show a lower bound for the number of "small" eigenvalues for the Weyl-Dirac operator on $\mathbb{S}^2$ as follows

**Theorem 4.4.3.** *Let $f \in C^\infty(\mathbb{S}^2)$ be a smooth function on $\mathbb{S}^2$ such that*

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} f\ \mathrm{vol}_{\mathbb{S}^2} = 1.$$

*Consider the Weyl-Dirac operator $\mathcal{D}_{nf}$ on $\mathbb{S}^2$ with magnetic field $nf\mathrm{vol}_{\mathbb{S}^2}$ defined on the spinor bundle $\Psi_n$. Then, for each $\varepsilon > 0$*

$$\#\{\text{eigenvalues } \lambda \text{ of } \mathcal{D}_{nf} \text{ such that } |\lambda| < \varepsilon\} \geq \frac{n}{2\pi}\int_{\mathbb{S}^2}|f|\ \mathrm{vol}_{\mathbb{S}^2} + o(n)\ \text{as } n\to\infty.$$

By $o(n)$ we mean that

$$\liminf_{n\to\infty}\frac{1}{n}\Bigg(\#\{\text{eigenvalues } \lambda \text{ of } \mathcal{D}_{nf} \text{ such that } |\lambda| < \varepsilon\} - \frac{n}{2\pi}\int_{\mathbb{S}^2}|f|\ \mathrm{vol}_{\mathbb{S}^2}\Bigg) \geq 0.$$

To prove this we need to apply the similar result for the Pauli operator on a disc initiated by Elton in [23]. First, we will meet the *Pauli operator* $\mathcal{P}_A$ on a disc $\mathbb{D}$ of $\mathbb{R}^2$. That is the self-adjoint operator with Dirichlet condition which is associated with the following quadratic form

$$\langle \mathcal{P}_A \psi,\ \psi \rangle = \int_{\mathbb{D}} |\sigma \cdot (p - A)\psi|^2, \tag{4.11}$$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in [C_0^\infty(\mathbb{D})]^2$, where the magnetic potential $A_x dx + A_y dy$ is good enough (for instance, $A_x, A_y$ are $\mathbb{R}$-valued smooth functions). Since $\langle \mathcal{P}_A \psi,\ \psi \rangle \geq 0$ for $\psi \in [C_0^\infty(\mathbb{D})]^2$ the Friedrichs extension of this quadratic form results in a unique self-adjoint operator associated with the self-adjoint extension for the above quadratic form with core as $[C_0^\infty(\mathbb{D})]^2$; that is the Pauli operator defined on $\mathbb{D}$ (see [43], p.177 for details of discussion). We still keep the notation $\mathcal{P}_A$ for the self-adjoint operator associated with that self-adjoint Friedrichs extension. To prepare for our result later we here remind of Elton's result in [23] which gives an estimate for the number of small eigenvalues of the Pauli operator defined on a disc in $\mathbb{R}^2$.

**Theorem 4.4.4.** (See [23]) *Let $A_x, A_y$ be $\mathbb{R}$-valued and smooth functions defined on the unit disc $\mathbb{D}$. Suppose that $B = \partial_x A_y - \partial_y A_x$. Consider the Pauli operator $\mathcal{P}_{tA}$, defined on $\mathbb{D}$. Then, for a given $\varepsilon > 0$ we have*

$$\#\big\{\text{eigenvalues } \lambda \text{ of } \mathcal{P}_{tA} \text{ such that } |\lambda| < \varepsilon\big\} \geq \frac{t}{2\pi} \int_{\mathbb{D}} |B| dx dy + o(t) \ \textit{as } t \to \infty.$$

Here $o(t)$ means

$$\liminf_{t \to \infty} \frac{1}{t} \bigg( \#\big\{\text{eigenvalues } \lambda \text{ of } \mathcal{P}_{tA} \text{ such that } |\lambda| < \varepsilon\big\} - \frac{t}{2\pi} \int_{\mathbb{D}} |B| dx dy \bigg) \geq 0.$$

Here we will show how to move the result of Theorem 4.4.4 to obtain the result of Theorem 4.4.3. The key idea is to apply the variational method (see [19], for instance) and the relation between the Dirac operators on $\mathbb{S}^2$ and on $\mathbb{R}^2$ (Theorem 4.4.1).

*Proof of Theorem 4.4.3* We observe that for the magnetic field $nf\mathrm{vol}_{\mathbb{S}^2}$ the Chern number for the spinor bundle is

$$\frac{1}{2\pi} \int_{\mathbb{S}^2} nf \ \mathrm{vol}_{\mathbb{S}^2} = n.$$

Therefore, we will deal with $\Psi_n$ , the Spin$^c$ spinor bundle of $\mathbb{S}^2$ with Chern number $n$. On the other hand we may write

$$nf = \frac{n}{2} + n\tilde{f},$$

where the smooth function $\tilde{f}$ satisfies

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} \tilde{f}\ \mathrm{vol}_{\mathbb{S}^2} = 0.$$

It follows from the proof of Proposition 3.3.8 that there is a Spin$^c$ connection on $\Psi_n$ which gives the magnetic field $nf\mathrm{vol}_{\mathbb{S}^2}$ with

$$\nabla^{\pm}_{\theta} = \partial_\theta - inA_\theta \text{ and}$$

$$\nabla^{\pm}_{\phi} = \frac{1}{s}\partial_\phi + \frac{is}{2(c\pm 1)}\sigma_3 - \frac{isn}{2(c\pm 1)} - inA_\phi.$$

For convenience we denote by $\mathcal{D}_{nA,\mathbb{S}^2}$ the operator $\mathcal{D}_{nf}$, the corresponding Dirac operator with magnetic field $nf\mathrm{vol}_{\mathbb{S}^2}$ on $\mathbb{S}^2$. Now, consider the Dirac operators on $\mathbb{R}^2$ resulting in Theorem 4.4.1 with magnetic potential $nA'_{\pm}$ and corresponding magnetic field $nB_{\pm}$. Denoted this operator on $\mathbb{R}^2$ by $\mathcal{D}_{nA'_{\pm},\mathbb{R}^2}$. In fact $nB_{\pm}$ is the combination of two parts: one corresponds to the integral part of magnetic field $\dfrac{n}{2}\mathrm{vol}_{\mathbb{S}^2}$ and the other corresponds to $n\tilde{f}\mathrm{vol}_{\mathbb{S}^2}$.

Re-introduce $\Omega_{\pm}$ and $A'_{\pm}$ from Theorem 4.4.1. Note that $\Omega_{\pm}$ is bounded and bounded away from 0 on $\mathbb{H}_{\pm}$; in particular

$$c_1 := \max_{p\in\mathbb{H}_{\pm}} \Omega_{\pm}^{-1} > 0. \tag{4.12}$$

The result of Theorem 4.4.1 is pointwise, so we have

$$(\mathcal{D}_{nA'_{\pm},\mathbb{R}^2}\eta)\circ z_{\pm} = \left(\Omega_{\pm}^{\frac{3}{2}}\mathcal{D}_{nA,\mathbb{S}^2}\Omega_{\pm}^{-\frac{1}{2}}\right)(\eta\circ z_{\pm}), \tag{4.13}$$

for all functions $\eta :\ \mathbb{D} \longrightarrow \mathbb{C}^2$, where $\eta\circ z_{\pm}$ will be defined on hemispheres $\mathbb{H}_{\pm}$.

Let $V_{\pm}$ be the spectral subspace of $\mathcal{P}_{nA'_{\pm}}$, the Pauli operator operator with Dirichlet boundary condition on $\mathbb{D}$ corresponding to the spectral interval $[0, \frac{\varepsilon^2}{c_1^2}) \subset \mathbb{R}$; that is $V_{\pm}$ is the span of the eigenfunctions of $\mathcal{P}_{nA'_{\pm}}$ with corresponding eigenvalues $\lambda$ satisfying $0 \le \lambda < \dfrac{\varepsilon^2}{c_1^2}$.

Let

$$W_{\pm} = \left\{\xi \in \Gamma(\Psi_n) :\ \xi = \begin{cases}\Omega_{\pm}^{-\frac{1}{2}}\eta_{\pm}\circ z_{\pm} & \text{on } \mathbb{H}_{\pm}\\ 0 & \text{on } \mathbb{H}_{\mp}\end{cases}\right\}, \quad \text{where } \eta_{\pm}\in V_{\pm}.$$

Let $W = W_+ \oplus W_-$. We observe that $\eta$ satisfies the Dirichlet boundary condition on $\mathbb{D}$ so the spinors in $W_{\pm}$ will be continuous on $\mathbb{S}^2$ giving $W \subseteq \mathrm{Dom}(\mathcal{D}_{nA,\mathbb{S}^2})$. We also observe that spinors in $W_+$ and $W_-$ are supported on $\mathbb{H}_+$ and $\mathbb{H}_-$, respectively.

Thus, 0 is the only spinor section in the intersection of $W_+$ and $W_-$. Also $\dim W_\pm = \dim V_\pm$. Therefore, we have

$$\dim\ W = \dim\ W_+ + \dim\ W_- = \dim\ V_+ + \dim\ V_-.$$

Let $\xi = \xi_+ + \xi_- \in W$. Then, there exits $\eta_\pm \in V_\pm$ with

$$\xi_\pm = \Omega_\pm^{-\frac{1}{2}} \eta_\pm \circ z_\pm, \tag{4.14}$$

on $\mathbb{H}_\pm$. Then,

$$\|\xi\|^2 = \|\xi_+\|^2 + \|\xi_-\|^2,$$

where

$$\|\xi_\pm\|^2 = \int_{\mathbb{H}_\pm} |\xi_\pm|^2 \ \mathrm{vol}_{\mathbb{S}^2} = \int_{\mathbb{H}_\pm} \Omega_\pm^{-1} |\eta_\pm \circ z_\pm|^2 \ \mathrm{vol}_{\mathbb{S}^2}.$$

We here observe from the proof of Theorem 4.4.1 that if we subsitute variables from $(\theta, \phi)$ on $\mathbb{S}^2$ to $(x, y)$ on $\mathbb{R}^2$, then the Jacobian of this substitution is $\Omega_\pm^{-2}$. By (4.12), $\frac{1}{c_1} \leq \min_{p \in \mathbb{H}_\pm} \Omega_\pm$. It follows that

$$\|\xi_\pm\|^2 = \int_{\mathbb{D}} |\eta_\pm|^2 (\Omega_\pm \circ z_\pm) \ dxdy \geq \frac{1}{c_1} \int_{\mathbb{D}} |\eta_\pm|^2 \ dxdy = \frac{1}{c_1} \|\eta_\pm\|^2.$$

Then,

$$\|\eta_+\|^2 + \|\eta_-\|^2 \leq c_1 \|\xi\|^2 \tag{4.15}$$

Also,

$$\|\mathcal{D}_{nA,\mathbb{S}^2} \xi\|^2 = \|\mathcal{D}_{nA,\mathbb{S}^2} \xi_+\|^2 + \|\mathcal{D}_{nA,\mathbb{S}^2} \xi_-\|^2,$$

where using (4.13) and (4.12),

$$\begin{aligned}
\|\mathcal{D}_{nA,\mathbb{S}^2} \xi_\pm\|^2 &= \int_{\mathbb{H}_\pm} |\mathcal{D}_{nA,\mathbb{S}^2} \xi_\pm|^2 \ \mathrm{vol}_{\mathbb{S}^2} \\
&= \int_{\mathbb{H}_\pm} \Omega_\pm^{-3} |(\mathcal{D}_{nA'_\pm,\mathbb{R}^2} \eta_\pm) \circ z_\pm|^2 \ \mathrm{vol}_{\mathbb{S}^2} \\
&= \int_{\mathbb{D}} |\mathcal{D}_{nA'_\pm,\mathbb{R}^2} \eta_\pm|^2 (\Omega_\pm^{-1} \circ z_\pm^{-1}) \ dxdy \\
&\leq c_1 \int_{\mathbb{D}} |\mathcal{D}_{nA'_\pm,\mathbb{R}^2} \eta_\pm|^2 \ dxdy \\
&= c_1 \langle \mathcal{P}_{nA'_\pm} \eta_\pm, \ \eta_\pm \rangle,
\end{aligned}$$

where the last line follows from (4.11) (essentially the definition of $\mathcal{P}_{nA'_\pm}$ via its quadratic form). However, from the definition of $V_\pm$,

$$0 \leq \langle \mathcal{P}_{nA'_\pm} \eta_\pm, \ \eta_\pm \rangle < \frac{\varepsilon^2}{c_1^2} \|\eta_\pm\|^2.$$

Putting these calculations together we have

$$\|\mathcal{D}_{nA,\mathbb{S}^2} \xi\|^2 \leq c_1 \frac{\varepsilon^2}{c_1^2} (\|\eta_+\|^2 + \|\eta_-\|^2) \leq c_1^2 \frac{\varepsilon^2}{c_1^2} \|\xi\|^2 = \varepsilon^2 \|\xi\|^2.$$

Then, the variational principle tells us that

$$\#\big\{\text{eigenvalues } \lambda \text{ of } \mathcal{D}_{nA,\mathbb{S}^2} \text{ such that } |\lambda| < \varepsilon\big\} \geq \dim\ W.$$

We will estimate the quantity dim $W$. First we notice that dim $W$ = dim $V_+$ + dim $V_-$. Next, we will apply the result of Theorem 4.4.4. We observe that

$$\dim\ V_\pm = \#\big\{\text{eigenvalues } \lambda \text{ of } \mathcal{P}_{nA'\pm} \text{ such that } |\lambda| < \frac{\varepsilon^2}{c_1^2}\big\}.$$

So we get

$$\dim\ V_\pm \geq \frac{n}{2\pi}\int_{\mathbb{D}} |B_\pm| dxdy + o(n) = \frac{n}{2\pi}\int_{\mathbb{H}_\pm} |f|\ \mathrm{vol}_{\mathbb{S}^2} + o(n),$$

where the last line is since $f\mathrm{vol}_{\mathbb{S}^2}$ is the pullback of $nB_\pm$ by $z_\pm^*$ on $\mathbb{H}_\pm$ (see the proof of Theorem 4.4.1). Therefore, for $\varepsilon > 0$ we have

$$\begin{aligned}
\#\big\{\text{eigenvalues } \lambda \text{ of } \mathcal{D}_{nA,\mathbb{S}^2} & \text{ such that } |\lambda| < \varepsilon\big\} \\
&\geq \dim W = \dim V_+ + \dim V_- \\
&\geq \frac{n}{2\pi}\int_{\mathbb{H}_+} |f|\ \mathrm{vol}_{\mathbb{S}^2} + o(n) + \frac{n}{2\pi}\int_{\mathbb{H}_-} |f|\ \mathrm{vol}_{\mathbb{S}^2} + o(n) \\
&\geq \frac{n}{2\pi}\int_{\mathbb{S}^2} |f|\mathrm{vol}_{\mathbb{S}^2} + o(n).
\end{aligned}$$

Now recall that operator $\mathcal{D}_{nA,\mathbb{S}^2}$ is $\mathcal{D}_{nf}$, then which completes the proof. ■

## 4.5 A lower bound for the number of zero modes corresponding to scaled Erdös and Solovej type magnetic fields

In this section we will apply the result of the previous section and consider the class $\mathcal{B}_{ES}$ of magnetic fields on $\mathbb{R}^3$ as we discussed in Section 4.2 to obtain a lower bound for the number of zero modes for the Weyl-Dirac operators on $\mathbb{R}^3$.
Suppose $f$ is a smooth function on $\mathbb{S}^2$ such that $\frac{1}{2\pi}\int_{\mathbb{S}^2} f\ \mathrm{vol}_{\mathbb{S}^2} = 1$. Consider the closed two-form $\beta = f\mathrm{vol}_{\mathbb{S}^2}$ on $\mathbb{S}^2$. For each non-negative number $t$ we set $\beta_t = tf\mathrm{vol}_{\mathbb{S}^2}$ as the scaled magnetic field on $\mathbb{S}^2$. Using $\kappa^*$, where $\kappa$ is the Hopf map, we pull back these magnetic fields $\beta_t$ and obtain $tb = \kappa^*(\beta_t)$ as magnetic fields on $\mathbb{S}^3$. Finally, we use $\tau^*$ with $\tau$ is the inverse of the stereographic projection considered in Section 4.2 to pull back again to get magnetic fields $tB \in \mathcal{B}_{ES}$ (magnetic field on $\mathbb{R}^3$); that is $tB = \iota^*(\beta_t) \in \mathcal{B}_{ES}$, where $\iota = \kappa \circ \tau$. Denote by $\mathcal{D}_{tB,\mathbb{R}^3}$ the Weyl-Dirac operators on $\mathbb{R}^3$ with magnetic fields $tB$. We will prove the following lower bound for the number of zero modes of $\mathcal{D}_{tB,\mathbb{R}^3}$, $0 < t \leq T$.

**Theorem 4.5.1.** *Consider the Weyl-Dirac operator $\mathcal{D}_{tB,\mathbb{R}^3}$ on $\mathbb{R}^3$ with magnetic field $tB \in \mathcal{B}_{ES}$ with $tB = \iota^*(\beta_t)$, where $\int_{\mathbb{S}^2} \beta = 1$ and $\beta_t = t\beta$. For $T > 0$ denote by $n_B(T)$ the total number of zero modes for $\mathcal{D}_{tB,\mathbb{R}^3}$ with $0 < t \leq T$. We have*

$$n_B(T) \geq \frac{T^2}{2} \frac{1}{2\pi} \int_{\mathbb{S}^2} |f| \text{ vol}_{\mathbb{S}^2} + o(T^2),$$

*as $T \to \infty$.*

Here we remind that $o(T^2)$ means

$$\liminf_{T\to\infty} \frac{1}{T^2}\left( n_B(T) - \frac{T^2}{2} \frac{1}{2\pi} \int_{\mathbb{S}^2} |f| \text{ vol}_{\mathbb{S}^2} \right) \geq 0.$$

*Proof.* It follows from the assumption on $f$ that the total flux $\Phi_t$ of the magnetic field $tB$ is

$$\Phi_t = \frac{1}{2\pi} \int_{\mathbb{S}^2} tf \text{ vol}_{\mathbb{S}^2} = t.$$

To apply the results of Erdös and Solovej (namely, Theorem 4.2.2) we need to consider

$$\beta_{t,k} = \beta_t - \frac{1}{2}(\Phi_t - k)\text{vol}_{\mathbb{S}^2}, \text{ or } \beta_{t,k} = \left[tf - \frac{1}{2}(t-k)\text{vol}_{\mathbb{S}^2}\right]$$

for $k \in \mathbb{Z}$. It follows from the result of Theorem 4.2.2 that

$$\text{Spec}(\mathcal{D}_{tb,\mathbb{S}^3}) = \bigcup_{k\in\mathbb{Z}} \left( S_{t,k} \ \cup \ \left\{ \pm \sqrt{\left(\lambda_j^+(\beta_{t,k})\right)^2 + (k-t)^2} - \frac{1}{2} \right\} \right),$$

where $\lambda_j^+(\beta_{t,k})$ are the positive eigenvalues of $\mathcal{D}_{\beta_{t,k},\mathbb{S}^2}$ including multiplicity, listed so that $0 < \lambda_1^+(\beta_{t,k}) \leq \lambda_2^+(\beta_{t,k}) \leq \cdots$, and

$$S_{t,k} = \begin{cases} \{-k+t-\frac{1}{2}\} & \text{if } k > 0 \\ \emptyset \quad \text{if } k = 0 \\ \{k-t-\frac{1}{2}\} & \text{if } k < 0. \end{cases}$$

Theorem 4.2.3 tells us that if we want to know $\dim \operatorname{Ker} \mathcal{D}_{tB,\mathbb{R}^3}$ and then $n_B(T)$ we need know $\dim \operatorname{Ker} \mathcal{D}_{tb,\mathbb{S}^3}$ for $0 < t \leq T$. Define

$$N_k := \sum_{k-\frac{1}{2}<t\leq k+\frac{1}{2}} \dim \operatorname{Ker} \mathcal{D}_{tB,\mathbb{R}^3}.$$

Since $k + \dfrac{1}{2} \leq T \iff k \leq [T-1]$ we get

$$n_B(T) = \sum_{0\leq t\leq T} \dim \operatorname{Ker} \mathcal{D}_{tB,\mathbb{R}^3} \geq \sum_{k=1}^{[T-1]} N_k. \tag{4.16}$$

We now estimate $N_k$ by considering two different contributions.

Case 1: $t = k + \dfrac{1}{2}$ for $k \in \mathbb{N}$ (see Theorem 4.3.1). In this case we have exactly $k$ Type I zero modes and no Type II zero modes to $N_k$.

Case 2: $k - \dfrac{1}{2} < t < k + \dfrac{1}{2}$. There are no Type I zero modes in this case. We need only to look for the Type II zero modes for $N_k$. Observe that $\mathcal{D}_{\beta_{t,k},\mathbb{S}^2}$ is an analytic Type I family of operators in $t$. Thus, eigenvalues of $\mathcal{D}_{\beta_{t,k},\mathbb{S}^2}$ can be parametrised as continuous functions of $t$ (see [35]). Furthermore, $\dim \operatorname{Ker} \mathcal{D}_{\beta_{t,k},\mathbb{S}^2}$ is fixed (at $k$) for all $t$, so all except the $k$ curves corresponding to $\operatorname{Ker} \mathcal{D}_{\beta_{t,k},\mathbb{S}^2}$ the eigenvalue curves are either everywhere positive or everywhere negative. Thus $t \longmapsto \lambda_j^+(\beta_{t,k})$ is continuous for all $j \in \mathbb{N}$. Let

$$\Lambda(t) := \sqrt{\left(\lambda_j^+(\beta_{t,k})\right)^2 + (k-t)^2} - \frac{1}{2}.$$

We have $\Lambda(t)$ is continuous in $t$. Moreover, we observe that

$$\Lambda(k - \frac{1}{2}) = \sqrt{\left(\lambda_j^+(\beta_{t,k})\right)^2 + \frac{1}{4}} - \frac{1}{2} > 0,$$

and

$$\Lambda(k + \frac{1}{2}) = \sqrt{\left(\lambda_j^+(\beta_{t,k})\right)^2 + \frac{1}{4}} - \frac{1}{2} > 0.$$

On the other hand,

$$\Lambda(k) = \sqrt{\left(\lambda_j^+(\beta_{t,k})\right)^2} - \frac{1}{2}.$$

Therefore, $\Lambda(k) < 0$ if $\lambda_j^+(\beta_{t,k}) < \dfrac{1}{2}$. By Immediate Value Theorem we get

$$\begin{aligned}
\sum_{k-\frac{1}{2}<t<k+\frac{1}{2}} \dim \operatorname{Ker} \mathcal{D}_{tB,\mathbb{R}^3} &\geq 2\#\left\{j : \quad \lambda_j^+(\beta_{k,k}) < \frac{1}{2}\right\} \\
&= \#\left\{\lambda \in \operatorname{Spec}(\mathcal{D}_{k\beta,\mathbb{S}^2}) : \quad 0 < |\lambda| < \frac{1}{2}\right\} \\
&= \#\left\{\lambda \in \operatorname{Spec}(\mathcal{D}_{k\beta,\mathbb{S}^2}) : \quad |\lambda| < \frac{1}{2}\right\} - k,
\end{aligned}$$

since $\dim \operatorname{Ker} \mathcal{D}_{k\beta,\mathbb{S}^2} = k$ and the spectrum of $\mathcal{D}_{k\beta,\mathbb{S}^2}$ is symmetric about 0 (see Proposition 3.3.7).

It is time to apply the estimate from Theorem 4.4.3 for the Dirac operator $\mathcal{D}_{k\beta,\mathbb{S}^2}$; this gives

$$\#\{\lambda \in \operatorname{Spec}(\mathcal{D}_{k\beta,\mathbb{S}^2}) : \quad |\lambda| < \frac{1}{2}\} \geq \frac{k}{2\pi} \int_{\mathbb{S}^2} |f| \operatorname{vol}_{\mathbb{S}^2} + o(k)$$

for each $k = 1, 2, \ldots, [T-1]$. Combining contributions from Case 1 and Case 2 it follows that

$$N_k \geq \frac{k}{2\pi} \int_{\mathbb{S}^2} |f| \operatorname{vol}_{\mathbb{S}^2} + o(k). \tag{4.17}$$

Let

$$x_k = N_k - \frac{k}{2\pi} \int_{\mathbb{S}^2} |f| \operatorname{vol}_{\mathbb{S}^2}$$

for $k = 1, 2, \ldots$. Then, it follows from the definition of $o(k)$ that

$$\liminf_{k\to\infty} \frac{1}{k} x_k \geq 0. \tag{4.18}$$

We will prove the following result.

**Lemma 4.5.2.** *For $\{x_k\}$ satisfying (4.18), then*

$$\sum_{k=1}^{N} x_k \geq o(N^2) \text{ as } N \to \infty,$$

*meaning that*

$$\liminf_{N\to\infty} \frac{1}{N^2} \sum_{k=1}^{N} x_k \geq 0.$$

First, we will apply the result of Lemma 4.5.2 to complete the proof of Theorem 4.5.1. Truly, by Lemma 4.5.2 we have

$$\sum_{k=1}^{[T-1]} x_k = o([T-1]^2), \text{ so } \sum_{k=1}^{[T-1]} N_k - \sum_{k=1}^{[T-1]} k \frac{1}{2\pi} \int_{\mathbb{S}^2} |f| \operatorname{vol}_{\mathbb{S}^2} \geq o([T-1]^2).$$

Since

$$\sum_{j=1}^{[T-1]} j = \frac{[T-1](1+[T-1])}{2} = \frac{T^2}{2} + O(T),$$

we have

$$\begin{aligned} n_B(T) &\geq \frac{T^2}{2} \frac{1}{2\pi} \int_{\mathbb{S}^2} |f| \operatorname{vol}_{\mathbb{S}^2} + O(T) + o([T-1]^2) \\ &= \frac{T^2}{2} \frac{1}{2\pi} \int_{\mathbb{S}^2} |f| \operatorname{vol}_{\mathbb{S}^2} + o(T^2) \end{aligned} \tag{4.19}$$

as $T \to \infty$. That is our conclusion in Theorem 4.5.1. ■

Finally, we will prove the result of Lemma 4.5.2. Truly, for any $\varepsilon > 0$ it follows from (4.18) that there exists a natural number $n$ such that

$$\frac{1}{m} x_m \geq -\varepsilon, \ \forall m \geq n. \tag{4.20}$$

Now we write

$$\frac{1}{N^2}\sum_{k=1}^{N} x_k = \frac{1}{N^2}\sum_{m=1}^{n-1} x_m + \frac{1}{N^2}\sum_{m=n}^{N} x_m. \tag{4.21}$$

It follows from (4.20) that

$$\frac{1}{N^2}\sum_{m=n}^{N} x_m \geq -\varepsilon\frac{1}{N^2}\sum_{m=n}^{N} m \geq -\varepsilon\frac{1}{N^2}\sum_{m=1}^{N} m \geq -\frac{\varepsilon}{2}\frac{N(N+1)}{N^2} \geq -\varepsilon. \tag{4.22}$$

On the other hand we may choose $m_0$ such that $m_0^2 \geq \dfrac{|S_1|}{\varepsilon}$, where

$$S_1 = \frac{1}{N^2}\sum_{m=1}^{n-1} x_m.$$

Then, for all $N \geq \max\{m_0, n\}$ we have

$$\frac{1}{N^2}\sum_{m=1}^{n-1} x_m = \frac{S_1}{N^2} \geq -\frac{|S_1|}{N^2} \geq -\frac{\varepsilon m_0^2}{N^2} \geq -\varepsilon. \tag{4.23}$$

Now looking at (4.21) and using (4.22) and (4.23) we have

$$\frac{1}{N^2}\sum_{k=1}^{N} x_k \geq -2\varepsilon, \text{ for all } \varepsilon > 0. \tag{4.24}$$

Clearly, (4.24) justifies the conclusion in Lemma 4.5.2. ■

# Bibliography


[1] M. Abramowitz and I.A. Stegun (eds.), *Handbook of Mathematical Physics,* Dover Publications, New York(1972).

[2] C. Adam, B. Muratori, and C. Nash, Zero modes of the Dirac operator in three dimensions, *Phys. Rev. D,* **60**(1999), 125001 1-8.

[3] C. Adam, B. Muratori, and C. Nash, Zero modes in finite range magnetic fields, *Modern Physics Letters A,* Vol. 15, **25**(2000), 1577-1581.

[4] C. Adam, B. Muratori, and C. Nash, Degeneracy of zero modes of the Dirac operator in three dimensions, *Physics Letters B,* **485**(2000), 314-318.

[5] C. Adam, B. Muratori, and C. Nash, Multiple zero modes of the Dirac operator in three dimensions, *Phys. Rev. D,* **62**(2000), 085026 1-9.

[6] C. Adam, B. Muratori, and C. Nash, Chern-Simons action for zero-mode supporting gauge fields in three dimensions, *Phys. Rev. D,* **67**(2003), 087703 1-3.

[7] Y. Aharonov and A. Casher, Ground state of a spin$-1/2$ charged particle in a two-dimensional magnetic field, *Phys. Rev. A,* **19**(1979), 2461-2462.

[8] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic field, I. General interactions *Duke Math. Journal,* **45**(1978), 847-883.

[9] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic field: III. Atoms in homogeneous magnetic field, *Commun. Math. Phys.,* **79**(1981), 529-572.

[10] A. A. Balinsky and W. D. Evans, On the Zero Modes of Pauli Operators, *Journal of Functional Analysis,* **179**(2001), 120-135.

[11] A. A. Balinsky, W. D. Evans and R. T. Lewis, Sobolev, Hardy and CLR inequalities associated with Pauli Operators in $\mathbb{R}^3$, *J. Phys., A: Math. Gen.,* **34**(2001), L19-L23.

[12] A. A. Balinsky and W. D. Evans, On the zero modes of Weyl-Dirac operators and their multiplicity, *Bull. London Math. Soc.,* **34**(2002), 236-242.

[13] A. A. Balinsky, W. D. Evans and Y. Saitō, Dirac-Sobolev inequalities and estimates for the zero modes of massless Dirac operators, *Preprint:* arXiv:0712.2051v1 [math.SP].

[14] M. S. Birman, On the spectrum of singular boundary value problems, *Mat. Sbornik,* **55**(1961), 125-174 (in Russian); Engl. transl. in Amer. Math. Soc. Transl., 53(1966), 23-80.

[15] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space,* D. Reidel Publishing Company(1987).

[16] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schrödinger operators, *Annals of Math.,* **106**(1977), 93-100.

[17] H.L. Cycon, R.G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators, With Applications to Quantum Mechanics and Global Geometry*, Berlin, Springer-Verlag(1987).

[18] I. Daubechies, An uncertainty principle for fermions with generalized kinetic energy, *Comm. Math. Phys.,* **90**(1983), 511-520.

[19] B. Davies and Y. Safarov (edited), *Spectral theory and Geometry,* Cambridge University Press, London Mathematical Society Lecture Note Series **273**(1999).

[20] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators,* Oxford University Press, New York(1987).

[21] D. M. Elton, New Examples of Zero Modes, *J. Phys. A,* **33**(2000), 7297-7303.

[22] D. M. Elton, The Local Structure of Zero Mode Producing Magnetic Potentials, *Commun. Math. Phys.,* **229**(2002), 121-139.

[23] D. M. Elton, *In preparation.*

[24] L. Erdös and J. P. Solovej, The kernel of Dirac operators on $\mathbb{S}^3$ and $\mathbb{R}^3$, *Rev. in Math. Phys.,* Vol. 13, 10(2001), 1247-1280.

[25] L. Erdös, V. Vougalter, Pauli operator and Aharonov-Casher theorem for measure valued magnetic fields, *Commun. Math. Phys.,* **225**(2002), 399-421.

[26] W. D. Evans, *Private communication.*

[27] R. L. Frank, E. H. Lieb and R. Seiringer, Number of Bound States of Schrödinger operators with Matrix-Valued Potentials, *Letters in Math. Phys.,* **82**(2007), 107-116.

[28] R. L. Frank, S. Morozov and S. Vugalter, Weakly coupled bound states of Pauli operators, *Preprint:* arXiv:0903.5333v1 [math.SP].

[29] J. Fröhlich, E. Lieb and M. Loss, Stability of Coulomb systems with magnetic fields I. The one-electron atom, *Commun. Math. Phys.*, **104**(1986), 251-270.

[30] M. P. Fry, Paramagnetism, zero modes, and mass singularities in QED in $1+1$, $2+1$ and $3+1$ dimensions, *Physical Review D,* **55**(1997), 968-972.

[31] K. Hannabuss, *An Introduction to Quantum Theory*, Clarendon Press, Oxford University Press(1997).

[32] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, 2nd printing., Springer-Verlag(1994).

[33] K. Jänich, *Vector Analysis*, Springer(2001).

[34] T. Kato, Schrödinger operators with singular potentials, *Israel J. Math.* **13**(1972), 135-148.

[35] T. Kato, *Pertubation Theory for Linear Operators*, 2nd ed., Springer, Berlin(1976).

[36] E. Lieb and M. Loss, *Analysis*, Amer. Math. Soc., Providence, RI(1997).

[37] E. H. Lieb, Lieb-Thirring inequalities, Available at arXiv:math-ph/0003039v1

[38] M. S. Longair, *Theoretical Concepts in Physics: An Alternative View of Theoretical Reasoning in Physics*, 2nd ed., Cambridge University Press(2003).

[39] J. M. Loss and H. T. Yau, Stability of Coulomb systems with magnetic fields III. Zero energy bound states of the Pauli operators, *Commun. Math. Phys.*, **104**(1986), 283-290.

[40] K. Miller, Bound states of quantum mechanical particles in magnetic fields, *PhD thesis*, Princeton University(1982).

[41] M. Nakahara, *Geometry, Topology and Physics,* Graduate student series in Physics, Adam Hilger(1990).

[42] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis,* Academic Press, New York(1972).

[43] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-adjointness,* Academic Press, New York(1975).

[44] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory,* Academic Press, New York(1979).

[45] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators,* Academic Press, New York(1978).

[46] S. Rosenberg, *The Laplacian on a Riemannian Manifold,* London Mathematical Society, Student Texts **31**, Cambridge University Press(1997).

[47] J. R. Retherford, *Hilbert Space: Compact Operators and the Trace Theorem,* Cambridge University Press(1993).

[48] G. Rozenblum and N. Shirokov, Infiniteness of Zero Modes for the Pauli Operator with singular magnetic field, *Journal of Functional Analysis,* **233**(2006), 135-172.

[49] Y. Saitō and T. Umeda, The zero modes and zero resonances of massless Dirac operators, To appear in *Hokkaido. Math. J.*.

[50] Y. Saitō and T. Umeda, The asymptotic limits of zero modes of massless Dirac operators, To appear in *Letters in Math. Phys.*.

[51] Y. Saitō and T. Umeda, Eigenfunctions at the threshold energies of the Dirac operators with positive mass, Preprint.

[52] J. Schwinger, On the bounds states of a given potential, *Proceedings of the National Academy of Sciences of the United States of America,* **47**(1961), 122-129.

[53] B. Simon, *Quantum mechanics for Halmitonians defined as quadratic forms,* Princeton University Press, Princeton, N.J.(1971).

[54] B. Simon, On the number of bound states of two body Schrödinger operators-A review, In *Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann,* edited by E.H. Lieb, B.Simon and A.S. Wightman, Princeton University Press, Princeton, New Jersey(1976).

[55] B. Simon, *Trace ideals and their applications,* American Mathematical Society, Second edition(2005).

[56] B. Simon, *Functional Integration and Quantum Physics,* American Mathematical Society, Providence, Second edition(2005).

[57] B. Thaller, *The Dirac equation,* Springer Verlag(1992).

[58] T. Weidl, Remarks on virtual bound states for semi-bounded operators, *Comm. Partial Diffrential Equations,* **24**(1999), 25-60.