Math. Proc. Camb. Phil. Soc. (2004), **136**, 383 © 2004 *Cambridge Philosophical Society* 383
DOI: 10.1017/S0305004103007230 *Printed in the United Kingdom*

Asymptotic expansion for closed orbits in homology classes for Anosov flows

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(*Received* 2 *April* 2002)

Abstract

In this paper we give an asymptotic expansion including error terms for the number of closed orbits in the homology classes for homological full Anosov flows. In particular we obtain formulae concerning the coefficients of error terms which depend on the homology classes.

1. *Introduction*

Let M be a C^∞ compact manifold. We call a C^1 flow $\phi_t : M \to M$ an Anosov flow if the tangent bundle TM has a continuous splitting

$$TM = E^0 \oplus E^u \oplus E^s$$

into $D\phi_t$-invariant subbundles such that:

(1) E^0 is the one dimensional bundle tangent to the flow;

(2) there exist $C, \lambda > 0$ such that

$$\|D\phi_t|E^s\| \leqslant Ce^{-\lambda t} \qquad \text{for } t \geqslant 0,$$
$$\|D\phi_{-t}|E^u\| \leqslant Ce^{-\lambda t} \qquad \text{for } t \geqslant 0.$$

We say that ϕ is transitive if there is a dense orbit. We shall restrict attention to transitive flows. We can model an Anosov flow by a suspended flow over a shift of finite type [**2**].

We assume that M is a C^∞ compact manifold with first Betti number $b > 0$. For simplicity, we assume that the first homology group of the manifold M is torsion free. We know that there is a isomorphism between $H_1(M, \mathbb{Z})$ and $\mathbb{Z}^b$. So we can identify the homology group $H_1(M, \mathbb{Z})$ with $\mathbb{Z}^b$. There are $C^1(M)$ functions $F = (F_1, \ldots, F_b)$. For a closed orbit γ of ϕ, $\int_\gamma F = (\int_\gamma F_1, \ldots, \int_\gamma F_b)$ represents the homology class of γ, say, $[\gamma] \in H_1(M, \mathbb{Z})$. Let M_ϕ be the set of ϕ-invariant probability measures on M. For $u \in \mathbb{R}^b$, we define the function $\beta(u)$: $\mathbb{R}^b \to \mathbb{R}$ by

$$\beta(u) = P(\langle u, F\rangle) = \sup_{m \in M_\phi} \left\{ h_m + \langle u, \int F dm\rangle \right\}.$$

$\beta(u)$ is analytic and strictly convex on $\mathbb{R}^b$. We say that an Anosov flow ϕ is homologically full if every homology class contains a closed orbit. If ϕ is homologically full then it is topologically weak mixing. The function β is bounded below and there exists a unique $\xi \in \mathbb{R}^b$ for which the infimum is attained [**11**].

It is easy to see that $\nabla\beta(\xi) = 0$. Let $m_{\langle u,F\rangle}$ be the equilibrium state of $\langle u, F\rangle$ and $h^* = \beta(\xi)$. Since

$$\nabla\beta(u) = \int F dm_{\langle u,F\rangle},$$

we have

$$h^* = \sup\left\{h_m : \int F dm = 0, m \in M_\phi\right\}.$$

It is well known that $\beta(u)$ can be continued analytically in a neighbourhood of ξ in $\mathbb{C}^b$.

Let Γ be the set of closed orbits for Anosov flow ϕ. When $\gamma \in \Gamma$, let $l(\gamma)$ denote the length of γ, i.e, $l(\gamma) = \int_\gamma 1$. If ϕ is a homologically full transitive Anosov flow, for $\alpha \in H_1(M, \mathbb{Z})$ Sharp [**11**] obtained

$$\pi(T, \alpha) := \#\{\gamma \in \Gamma, l(\gamma) \leqslant T, [\gamma] = \alpha\} \sim ce^{\langle \xi,\alpha'\rangle}\frac{e^{Th^*}}{T^{b/2+1}} \quad \text{as } T \longrightarrow \infty,$$

where ξ, h^* are the constants we mentioned above and α' is the torsion-free part of α.

In [**8**], Pollicott and Sharp used the results of Dolgopyat's work ([**3**]) on Anosov flows to obtain a more detailed expansion for $\pi(T, \alpha)$ when ϕ is a homologically full transitive Anosov flow. That is, there exists $\delta > 0$, such that for $N = [2\delta]$, there exist $c_1, c_2, \ldots, c_N$ such that

$$\pi(T, \alpha) = \frac{e^{Th^*}}{T^{b/2+1}}\left(\sum_{n=0}^{N}\frac{c_n}{T^{n/2}} + O(T^{-\delta})\right) \quad \text{as } T \longrightarrow \infty. \qquad (*)$$

In this paper, we shall see that if n is odd, then $c_n = 0$. We also give formulae for c_n to describe how the c_n depend on the homology class α. The main result is the following.

THEOREM. *Let M be a compact manifold with Betti number $b > 0$ and let $\phi_t: M \to M$ be a homologically full transitive Anosov flow. There exist $\xi \in \mathbb{R}^b$, $h^* > 0$ and $\delta > 0$ such that for $\alpha \in H_1(M, \mathbb{Z}) \cong \mathbb{Z}^b$, we have*

$$\pi(T, \alpha) = \frac{e^{Th^*}}{T^{b/2+1}}e^{-\langle \xi,\alpha\rangle}\left(c_0 + \sum_{n=1}^{N}\frac{c_n(\alpha)}{T^n} + O\left(\frac{1}{T^\delta}\right)\right) \quad \textit{as } T \longrightarrow \infty,$$

for $N < \delta$ where $c_0 > 0$ is a constant which is independent of α. If we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then the constants $c_n(\alpha)$ are in the form

$$c_n(\alpha) = \sum_{l_1+l_2+\cdots+l_b=0}^{2n} c_{l_1 l_2\cdots l_d}\alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_b^{l_b},$$

where $c_{l_1,l_2,\ldots,l_b}$ are constants which are independent of α.

The analysis in this paper is closely akin to that used by Anantharaman in [**1**]. but we give more details of the proofs and more explicit information, necessary to understand the role of the homology class α. In the last section we use an approximation argument to determine how the coefficients depend on the homology classes.

2. *Counting function*

We assume that g has compact support and has C^∞-regularity. Let $g \geqslant 0$. For $\alpha \in H_1(M, \mathbb{Z})$, we define the auxiliary function

$$\pi_g(T, \alpha) = \sum_{\gamma \in \Gamma, [\gamma]=\alpha} g(l(\gamma - T)).$$

Let $\hat{g}$ be the Fourier transform of g, By Fourier's inverse transform formula,

$$\begin{aligned}
\pi_g(T, \alpha) &= \sum_{[\gamma]=\alpha} g(l(\gamma) - T) e^{\sigma(l(\gamma)-T)} e^{\sigma T} e^{-\sigma l(\gamma)} \\
&= \frac{1}{2\pi} \sum_{\gamma \in \Gamma} \int_{\mathbb{R}} \int_{\mathbb{R}^b/\mathbb{Z}^b} \hat{g}(-i\sigma + t) e^{-it(l(\gamma)-T)} e^{\sigma T} e^{-\sigma l(\gamma)} e^{\langle \xi + 2\pi i v, [\gamma] \rangle} e^{-\langle \xi + 2\pi i v, \alpha \rangle} dv\, dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^b/\mathbb{Z}^b} Z(\sigma + it, v) e^{\sigma T + itT} \hat{g}(-i\sigma + t) e^{-\langle \xi + 2\pi i v, \alpha \rangle} dv\, dt,
\end{aligned}$$

where we have defined

$$Z(s, v) = Z(\sigma + it, v) = \sum_{\gamma \in \Gamma} e^{-sl(\gamma) + \langle \xi + 2\pi i v, [\gamma] \rangle} = \sum_{\gamma \in \Gamma} e^{-sl(\gamma) + \langle \xi + 2\pi i v, \int_\gamma F \rangle}$$

for $(s, v) \in \mathbb{C} \times \mathbb{R}^b/\mathbb{Z}^b$. It is well known that when $Res = \sigma > \beta(\xi) = h^*$, $Z(s, v)$ is absolutely convergent.

For the behaviour of $Z(s, v)$ in the neighbourhood of $Res = h^*$, we have the following proposition.

Proposition 1. *There exist $B > 0$, $c > 0$, $\epsilon > 0$, $\beta > 0$, $\rho > 0$ and an open set V_0, a neighbourhood of* 0 *in $\mathbb{R}^d/\mathbb{Z}^d$, such that*:

(1) $Z(s, v)$ *is analytic in* $\{s = \sigma + it \; : \; \sigma > h^* - \frac{c}{|t|^\rho}, |t| > B\}$, *and in this domain* $|Z(s, v)| = O(|t|^\beta)$;

(2) $Z(s, v) + \log(s - \beta(\xi + iv))$ *is analytic in* $\{(s, v) : v \in V_0, \sigma > h^* - \epsilon, |t| \leqslant B\}$;

(3) $Z(s, v)$ *is analytic in* $\{(s, v) : v \notin \bar{V}_0, \sigma > h - \epsilon, |t| \leqslant B\}$.

Proof. The proof of part (1) is analogous to [**8**]. We refer to [**1**] for the proof of (2) and (3). For details see [**5**].

Now in order to estimate $\pi_g(T, \alpha)$, we only need to calculate the integral

$$\frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t) e^{-\langle \xi + 2\pi i v, \alpha \rangle} dt\, dv.$$

In the following let g be of class C^∞ with compact support and $\sigma > h^*$. We will divide $\mathbb{R}^b/\mathbb{Z}^b$ into V_0 and $\mathbb{R}^b/\mathbb{Z}^b - V_0$. Then

$$\begin{aligned}
\pi_g(T, \alpha) &= \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t) e^{-\langle \xi + 2\pi i v, \alpha \rangle}\, dt\, dv \\
&= \frac{1}{2\pi} \int_{V_0} e^{-\langle \xi + 2\pi i v, \alpha \rangle}\, dv \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t))\, dt \\
&\quad + \frac{1}{2\pi} \int_{\mathbb{R}^b/\mathbb{Z}^b - V_0} e^{-\langle \xi + 2\pi i v, \alpha \rangle} dv \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t))\, dt.
\end{aligned}$$

We shall consider the two integrals separately. First we consider the integral over $\mathbb{R}^b/\mathbb{Z}^b - V_0$. For $v \notin V_0$, we have the following estimate.

LEMMA 1. *For $v \notin V_0$ and for all $\sigma > h^*$,*

$$\left| \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt \right| = O\big(\|g^{(m)}\|_{L^1} e^{h^* T}/T^{(m-\beta)K}\big),$$

for any $K, m \in \mathbb{N}$.

Proof. By Proposition 1 and Cauchy's Theorem

$$\int_{\Delta} Z(s, v) e^{sT} \hat{g}(-is)\, ds = 0,$$

where $\Delta = \{Res = \sigma, |Ims| \leqslant R\} \cup \{Res = c(R), |Ims| \leqslant R\} \cup \{c(R) \leqslant Res \leqslant \sigma, |Ims| = R\}$, and where $c(R) = h^* - c/R^\rho$, $R = T^K$ for some K, and $\sigma = h^* + 1/T$. Since g is of class C^∞ with compact support, for any $m \in \mathbb{N}$,

$$|\hat{g}(-i\sigma + t)| \leqslant c\|g^{(m)}\|_{L^1}/|t|^m.$$

So

$$\left| \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt - \int_{-R}^{R} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt \right|$$
$$\leqslant \int_{T^K}^{\infty} e^{\sigma T} c|t|^\beta c \frac{\|g^{(m)}\|_{L^1}}{|t|^m} dt = O\left(\frac{\|g^{(m)}\|_{L^1} e^{h^* T}}{T^{(m-1-\beta)K}} \right).$$

On the other hand,

$$\left| \int_{Res = c(R), |Ims| \leqslant R} Z(s, v) e^{sT} \hat{g}(-is)\, ds \right| = O\left(\frac{\|g^{(m)}\|_{L^1}}{T^{(m-1-\beta)K}} e^{h^* T} e^{-\frac{cT}{T^{K\rho}}} \right),$$

and

$$\left| \int_{c(R) \leqslant Res \leqslant \sigma, |Ims| = R} Z(s, v) e^{sT} \hat{g}(-is)\, ds \right| = O\left(\|g^{(m)}\|_{L^1} \frac{e^{h^* T}}{T^{(m-\beta)K}} \right).$$

Since $cT/T^{K\rho} > 0$, this completes the proof.

Now we consider the integral over V_0. We shall prove the following lemma.

LEMMA 2. *Let $v \in V_0$; then for all $M \in \mathbb{N}$ and for all $\sigma > h^*$, we have*

$$\left| \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt - 2\pi \sum_{j=0}^{M} \frac{1}{T^{j+1}} \frac{d^j \hat{g}}{ds^j} (-i\beta(\xi + iv)) e^{\beta(\xi+iv)T} \right|$$
$$\leqslant c(\|g\|_{L^1} + \cdots + \|y^M g\|_{L^1}) e^{Th^* - \frac{cT}{T^{K\rho}}} + \|g^{(m)}\|_{L^1} e^{Th^*} \left(\frac{c_1}{T^{(m-1-\beta)K}} + \frac{c_2}{T^{(m-\beta)K}} \right)$$
$$+ \frac{c}{T^{M+1}} \left| \int_{C(\sigma)} \log(s - \beta(\xi + iv)) e^{sT} \frac{d^{M+1}}{ds^{M+1}} \hat{g}(-is)\, ds \right|,$$

where $C(\sigma)$ will be defined later.

Proof. When $v \in V_0$, σ is fixed with $\sigma > h^*$. For $2B < R \in \mathbb{R}^+$, let
$C_0 = \{Res = c(R), -2B \leqslant Ims \leqslant 2B\}$; $C_1 = \{Ims = -2B, c(R) \leqslant Res \leqslant \sigma\}$;
$C_2 = \{Res = \sigma, -2B \leqslant Ims \leqslant 2B\}$; $C_3 = \{Ims = 2B, c(R) \leqslant Res \leqslant \sigma\}$;

$C_4^+ = \{Res = c(R), 2B \leqslant Ims \leqslant R\}; C_4^- = \{Res = c(R), -R \leqslant Ims \leqslant -2B\}$;
$C_5^+ = \{Res = \sigma, 2B \leqslant Ims \leqslant R\}; C_5^- = \{Res = \sigma, -R \leqslant Ims \leqslant -2B\}$;
$C_R^+ = \{c(R) \leqslant Res \leqslant \sigma, Ims = R\}; C_R^- = \{c(R) \leqslant Res \leqslant \sigma, Ims = -R\}$.
By part (1) of Proposition 1, $Z(s,v)$ is analytic in $Res \geqslant c(R), |Ims| > B$, so

$$\int_{\{C_3 \cup C_5^+ \cup C_R^+ \cup C_4^+\}} Z(s,v) e^{sT} \hat{g}(-is)\, ds = 0 \tag{1}$$

and

$$\int_{\{C_1 \cup C_4^- \cup C_R^- \cup C_5^-\}} Z(s,v) e^{sT} \hat{g}(-is)\, ds = 0. \tag{2}$$

By part (2) of Proposition 1, we have

$$\int_{-\{C_0 \cup C_3 \cup C_2 \cup C_1\}} (Z(s,v) + \log(s - \beta(\xi + iv)))e^{sT} \hat{g}(-is)\, ds = 0, \tag{3}$$

where the three contours are counterclockwise and $-C_i$ means that the orientation of the path is reversed.
From (1), we have

$$\int_{C_5^+} Z(s,v) e^{sT} \hat{g}(-is)\, ds = \int_{-\{C_3 \cup C_4^+ \cup C_R^+\}} Z(s,v) e^{sT} \hat{g}(-is)\, ds. \tag{4}$$

From (2), we have

$$\int_{C_5^-} Z(s,v) e^{sT} \hat{g}(-is)\, ds = \int_{-\{C_1 \cup C_4^- \cup C_R^-\}} Z(s,v) e^{sT} \hat{g}(-is)\, ds. \tag{5}$$

From (3), we have

$$\begin{aligned} \int_{-C_2} Z(s,v) e^{sT} \hat{g}(-is)\, ds &= \int_{C_1 \cup C_2 \cup C_3} \log(s - \beta(\xi + iv)) e^{sT} \hat{g}(-is)\, ds \\ &+ \int_{C_0} (Z(s,v) + \log(s - \beta(\xi + iv)))e^{sT} \hat{g}(-is)\, ds + \int_{C_1 \cup C_3} Z(s,v) e^{sT} \hat{g}(-is)\, ds. \end{aligned} \tag{6}$$

Let $C(\sigma) = C_1 \cup C_2 \cup C_3$. Adding the three identities (4), (5), (6) we obtain

$$\begin{aligned} &\int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma + it)T} \hat{g}(-i\sigma + t)\, dt \\ &= -i \int_{C(\sigma)} \log(s - \beta(\xi + iv)) \hat{g}(-is) e^{sT}\, ds + i \int_{\{C_4^+ \cup C_4^-\}} Z(s,v) e^{sT} \hat{g}(-is)\, ds \\ &\quad - i \int_{C_0} (Z(s,v) + \log(s - \beta(\xi + iv)))e^{sT} \hat{g}(-is)\, ds + i \int_{C_R^+ \cup C_R^-} Z(s,v) e^{sT} \hat{g}(-is)\, ds \\ &\quad + \left(\int_R^{\infty} + \int_{-\infty}^{-R} \right) Z(\sigma + it, v) e^{(\sigma + it)T} \hat{g}(-i\sigma + t)\, dt. \end{aligned}$$

As we discussed in Lemma 1, for any $m, K \in \mathbb{N}$,

$$\int_{\{C_R^+ \cup C_R^-\}} Z(s,v)e^{sT}\hat{g}(-is)\,ds \leqslant c_1\|g^{(m)}\|_{L^1}\frac{e^{h^*T}}{T^{(m-\beta)K}},$$

$$\left(\int_{-\infty}^{-R} + \int_R^{\infty}\right) Z(\sigma+it,v)e^{(\sigma+it)T}\hat{g}(-i\sigma+t)\,ds \leqslant c_2\|g^{(m)}\|_{L^1}\frac{e^{h^*T}}{T^{(m-1-\beta)K}},$$

$$\int_{C_0}(Z(s,v)+\log(s-\beta(\xi+iv)))e^{sT}\hat{g}(-is)\,ds = c_3\|g\|_{L^1}e^{h^*T-\frac{cT}{T^{K\rho}}})$$

and

$$\int_{\{C_4^+\cup C_4^-\}} Z(s,v)e^{sT}\hat{g}(-is)ds \leqslant c_4\frac{\|g^{(m)}\|_{L^1}}{T^{(m-1-\beta)K}}e^{h^*T}e^{-\frac{cT}{T^{K\rho}}} \leqslant c_4\frac{\|g^{(m)}\|_{L^1}}{T^{(m-1-\beta)K}}.$$

Hence

$$\left|\int_{\mathbb{R}} Z(\sigma+it,v)e^{(\sigma+it)T}\hat{g}(-i\sigma+t)\,dt + i\int_{C(\sigma)}\log(s-\beta(\xi+iv))\hat{g}(-is)e^{sT}\,ds\right|$$
$$\leqslant e^{Th^*}\left(\frac{c_1\|g^{(m)}\|_{L^1}}{T^{(m-1-\beta)K}} + \frac{c_2\|g^{(m)}\|_{L^1}}{T^{(m-\beta)K}} + c_3\|g\|_{L^1}e^{-\frac{cT}{T^{K\rho}}}\right).$$

Next we consider $i\int_{C(\sigma)}\log(s-\beta(\xi+iv))\hat{g}(-is)e^{sT}ds$. Integrating by parts, we have

$$\Bigg|i\int_{C(\sigma)}\log(s-\beta(\xi+iv))\hat{g}(-is)e^{sT}\,ds + \frac{i}{T}\int_{C(\sigma)}\frac{\hat{g}(-is)}{s-\beta(\xi+iv)}e^{sT}\,ds$$
$$+\frac{i}{T}\int_{C(\sigma)}\log(s-\beta(\xi+iv))\frac{d\hat{g}}{ds}(-is)e^{sT}\,ds\Bigg|$$
$$= \left|\frac{i}{T}\log(s-\beta(\xi+iv))\hat{g}(-is)e^{sT}\right|_{s=c(R)+2Bi}^{s=c(R)-2Bi} \leqslant c\|g\|_{L^1}e^{Th^*-\frac{cT}{T^{K\rho}}}.$$

For the integral $\int_{C(\sigma)}\frac{\hat{g}(-is)}{s-\beta(\xi+iv)}e^{sT}ds$, we use the residue formula,

$$\int_{C(\sigma)\cup C_0}\frac{\hat{g}(-is)}{s-\beta(\xi+iv)}e^{sT}ds = -2\pi i\hat{g}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}.$$

So

$$\left|\int_{C(\sigma)}\frac{\hat{g}(-is)}{s-\beta(\xi+iv)}e^{sT}ds + 2\pi i\hat{g}(-i\beta(iv))e^{\beta(\xi+iv)T}\right|$$
$$= \left|\int_{C_0}\frac{\hat{g}(-is)}{s-\beta(\xi+iv)}e^{sT}ds\right| \leqslant c\|g\|_{L^1}e^{Th^*-\frac{cT}{T^{K\rho}}}.$$

Now we have obtained

$$\left|\int_{\mathbb{R}} Z(\sigma+it,v)e^{(\sigma+it)T}\hat{g}(-i\sigma+t)\,dt - 2\pi\frac{1}{T}\hat{g}(-i\beta(\xi+iv))e^{\beta(iv)T}\right|$$
$$\leqslant e^{Th^*}\left(\frac{c_1\|g^{(m)}\|_{L^1}}{T^{(m-1-\beta)K}} + \frac{c_2\|g^{(m)}\|_{L^1}}{T^{(m-\beta)K}} + c_3\|g\|_{L^1}e^{-\frac{cT}{T^{K\rho}}}\right)$$
$$+\frac{c}{T}\left|\int_{C(\sigma)}\log(s-\beta(\xi+iv))e^{sT}\frac{d\hat{g}(-is)}{ds}\,ds\right|.$$

We iterate the preceding operation $M+1$ times, and note that

$$\left|\frac{d^k\hat{g}(-is)}{ds^k}\right| = \left|\int_{\mathbb{R}} y^k g(y) e^{sy} dy\right| \leqslant c\|y^k g\|_{L^1}.$$

We have

$$\left|\int_{\mathbb{R}} Z(\sigma+it, v) e^{(\sigma+it)T}\hat{g}(-i\sigma+t)\,dt - 2\pi\sum_{j=0}^{M}\frac{1}{T^{j+1}}\frac{d^j\hat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}\right|$$
$$\leqslant c_1'\|g^{(m)}\|_{L^1}e^{Th^*}\left(\frac{c_1}{T^{(m-1-\beta)K}}+\frac{c_2}{T^{(m-\beta)K}}\right)+c_2'\left(\|g\|_{L^1}+\cdots+\|y^M g\|_{L^1}\right)e^{Th^*-\frac{cT}{T^{K\rho}}}$$
$$+\frac{c}{T^{M+1}}\left|\int_{C(\sigma)}\log(s-\beta(\xi+iv))e^{sT}\frac{d^{M+1}}{ds^{M+1}}\hat{g}(-is)\,ds\right|. \qquad \square$$

Since

$$\begin{aligned}\lim_{\sigma\to h^*}&\int_{V_0}e^{-2\pi i\langle v,\alpha\rangle}dv\int_{C(\sigma)}\log(s-\beta(iv))e^{sT}\frac{d^{M+1}}{ds^{M+1}}\hat{g}(-is)\,ds\\ &=\int_{V_0}e^{-2\pi i\langle v,\alpha\rangle}dv\int_{C(h^*)}\log(s-\beta(iv))e^{sT}\frac{d^{M+1}}{ds^{M+1}}\hat{g}(-is)\,ds\\ &=O\left(\frac{\|y^{M+1}g\|_{L^1}}{T^{M+1}}e^{Th^*}\right),\end{aligned}$$

by Lemma 1 and Lemma 2, we can prove the following proposition.

Proposition 2. *Let g be class C^∞ with compact support. For all $M, m\geqslant 1$, we have*

$$\left|\pi_g(T,\alpha)-e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{M}\frac{1}{T^{j+1}}\int_{V_0}e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\hat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv\right|$$
$$\leqslant c_1'\frac{\|y^{M+1}g\|_{L^1}}{T^{M+1}}e^{Th^*}+c_2'\|g^{(m)}\|_{L^1}e^{Th^*}\left(\frac{c_1}{T^{(m-1-\beta)K}}+\frac{c_2}{T^{(m-\beta)K}}\right)$$
$$+c_3'(\|g\|_{L^1}+\cdots+\|y^M g\|_{L^1})e^{Th^*-\frac{cT}{T^{K\rho}}}. \tag{7}$$

For $\forall N\in\mathbb{N}$, taking m sufficiently so large such that $(m-\beta)K\geqslant N+\frac{b}{2}+2$ and $M=N+b+2$, we have

Proposition 3. *Let g be class C^∞ with compact support. For all $N\geqslant 1$, we have*

$$\left|\pi_g(T,\alpha)-e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{N+b+2}\frac{1}{T^{j+1}}\int_{V_0}e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\hat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv\right|$$
$$\leqslant c_1'\frac{\|y^{N+b+2}g\|_{L^1}}{T^{N+b+2}}e^{Th^*}+c_2'\frac{\|g^{(m)}\|_{L^1}}{T^{N+1+\frac{b}{2}+1}}e^{Th^*}$$
$$+c_3'\left(\|g\|_{L^1}+\cdots+\|y^{N+b+2}g\|_{L^1}\right)e^{Th^*-\frac{cT}{T^{K\rho}}}. \tag{8}$$

3. *Coefficients of error terms of $\pi_g(T,\alpha)$*

From Proposition 3, in order to estimate $\pi_g(T,\alpha)$ we only need to estimate

$$\int_{V_0}\frac{d^j\hat{g}}{ds^j}(-i\beta(\xi+iv))e^{T\beta(\xi+iv)}e^{-2\pi i\langle v,\alpha\rangle}dv.$$

We shall use the method that Anantharaman used in [**1**]. We first prove the following lemma.

LEMMA 3. *There exist polynomials* $f_j^{(k)}(iv)$ *in* $iv_1, \ldots, iv_b$ *such that the total exponent of each term has the same parity as k and such that*

$$\Big| \int_{V_0} \frac{d^j \hat{g}}{ds^j}(-i\beta(\xi + iv)) e^{T\beta(\xi+iv)-2\pi i\langle v,\alpha\rangle} \, dv - \\ - \frac{e^{Th^*}}{T^{b/2}} \sum_{k=0}^{2N+1} \frac{1}{T^{k/2}} \int_{\|v\| \leqslant \sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)-2\pi i\langle \frac{v}{\sqrt{T}},\alpha\rangle} f_j^{(k)}(iv) \, dv \Big| \leqslant \\ \leqslant c \sup_{n\leqslant 2N+2} \|y^{j+n} g\|_{L^1} \frac{e^{Th^*}}{T^{N+\frac{b}{2}+1}},$$

for some small ρ.

Remark.

(1) Here $\|\cdot\|$ denotes the 2-norm, i.e., $\|v\| = \left(\sum_{i=1}^b v_i^2\right)^{1/2}$.

(2) The proof is similar to that in [**1**]. We denote $(d^j\hat{g}/ds^j)(-i\beta(\xi+iv))$ by $\bar{g}_j(iv)$. For convenience, let V_0 be of the form $\{v \in \mathbb{R}^b/\mathbb{Z}^b : \|v\| \leqslant \rho\}$.

(3) By the first condition, we mean that $f_j^{(k)}(iv)$ can be written in the form

$$f_j^{(k)}(iv) = \sum a_{l_1,l_2,\ldots,l_b} (iv_1)^{l_1} (iv_2)^{l_2} \cdots (iv_b)^{l_b}.$$

In particular,

$$\begin{aligned} f_j^{(0)}(iv) &= \bar{g}_j^{(0)}(0) = \frac{d^j}{ds^j}\hat{g}(-is) \mid_{s=h^*}; \\ f_j^{(1)}(iv) &= \frac{1}{6}\bar{g}_j^{(0)}(0)\beta^{(3)}(\xi)\cdot(iv)^3 + \bar{g}_j^{(1)}(0)\cdot(iv); \\ f_j^{(2)}(iv) &= \frac{1}{72}\bar{g}_j^{(0)}(0)\left(2(\beta^{(3)}(\xi)\cdot(iv)^3)^2 + 3\beta^{(4)}(\xi)\cdot(iv)^4\right) \\ &\quad + \frac{1}{6}\bar{g}_j^{(1)}(0)\cdot(iv)\beta^{(3)}(\xi)\cdot(iv)^3 + \frac{1}{2}\bar{g}_j^{(2)}(0)\cdot(iv)^2. \end{aligned}$$

(4) We can see that $f_j^{(0)}(iv)$ are constants and $f_0^{(0)}(iv) > 0$, since

$$\begin{aligned} f_j^{(0)}(iv) &= \frac{d^j}{ds^j}\hat{g}(-is)\Big|_{s=h^*} = \frac{d^j}{ds^j}\int_{\mathbb{R}} g(y)e^{sy}dy\Big|_{s=h^*} \\ &= \int_{\mathbb{R}} y^j g(y)e^{sy}\,dy\Big|_{s=h^*} = \int_{\mathbb{R}} y^j g(y)e^{h^*y}\,dy. \end{aligned}$$

Next we need to estimate

$$\sum_{k=0}^{2N+1} \frac{1}{T^{k/2}} \int_{\|v\|\leqslant\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)} e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle} f_j^{(k)}(iv)\,dv.$$

We have the following proposition.

Proposition 4.

$$\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\leqslant\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)}e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv$$
$$=\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)}s_j^{(k)}(\alpha,iv)dv+O\Big(\sup_{n\leqslant 4N+b+2}\|y^n g\|_{L^1}T^{-(N+1)}\Big),$$

where

$$s_j^{(k)}(\alpha,iv)=\sum_{l=0}^{k}(-1)^{k-l}\frac{f_j^{(l)}(iv)\langle\alpha,2\pi iv\rangle^{k-l}}{(k-l)!}$$

are polynomials in $iv_1,\ldots,iv_b$ *and we still have that the total exponent of each term in* $s_j^{(k)}(\alpha,v)$ *has the same parity as* k.

Proof. We expand $e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}$ in a neighbourhood of 0,

$$e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}=1-\frac{\langle\alpha,2\pi iv\rangle}{\sqrt{T}}+\frac{\langle\alpha,2\pi iv\rangle^2}{2T}-\frac{\langle\alpha,2\pi iv\rangle^3}{3!T^{3/2}}$$
$$+\cdots-\frac{\langle\alpha,2\pi iv\rangle^{2N+1}}{(2N+1)!T^{N/2+1}}+Z_N(iv/\sqrt{T}),$$

where $|Z_N(iv/\sqrt{T})|\leqslant c\dfrac{\|v\|^{2N+2}}{T^{N+1}}$.

Let

$$s_j^{(k)}(\alpha,iv)=\sum_{l=0}^{k}(-1)^{k-l}\frac{f_j^{(l)}(iv)\langle\alpha,2\pi iv\rangle^{k-l}}{(k-l)!},$$

then

$$\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\leqslant\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)}e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv$$
$$=\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\leqslant\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)}s_j^{(k)}(\alpha,iv)\,dv+O\Big(\sup_{n\leqslant 4N+b+2}\|y^n g\|_{L^1}T^{-(N+1)}\Big).$$

For T sufficiently large, for all $m\in\mathbb{N}$, we have

$$\int_{\|v\|>\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)}\|v\|^m\,dv\leqslant\int_{\|v\|>\sqrt{T}\rho} e^{-\epsilon'\|v\|^2}|v_1v_2\cdot v_d|\,dv$$
$$\leqslant\prod_{i=1}^{d}\int_{\sqrt{T}\rho}^{\infty} e^{-\epsilon' v_i^2}v_i dv_i\leqslant ce^{-\epsilon' T},$$

for some $\epsilon' > 0$. So we have

$$\begin{aligned}\Bigg|\sum_{k=0}^{2N+1} \frac{1}{T^{k/2}} \int_{\|v\|\leqslant\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)} e^{-2\pi i\langle\alpha, v/\sqrt{T}\rangle} f_j^{(k)}(iv)\, dv \\ - \sum_{k=0}^{2N+1} \frac{1}{T^{k/2}} \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} s_j^{(k)}(\alpha, iv)\, dv\Bigg| \\ \leqslant c \sup_{n\leqslant 4N+b+2} \|y^n g\|_{L^1} \big(T^{-(N+1)} + e^{-\epsilon' T}\big).\end{aligned}$$

Now the proof of the proposition is complete. □

In the following lemma, we will see that the coefficients of $T^{-\frac{k}{2}}$ vanish.

LEMMA 4. *If k is odd, then*

$$\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} s_j^{(k)}(\alpha, iv)\, dv = 0.$$

Proof. Since

$$\begin{aligned} s_j^{(k)}(\alpha, iv) &= \sum_{l=0}^{k} (-1)^{k-l} \frac{f_j^{(l)}(iv)\langle\alpha, 2\pi iv\rangle^{k-l}}{(k-l)!} \\ &= \sum_{l=0}^{k} (-1)^{k-l} f_j^{(l)}(iv)(2\pi i)^{k-l} \left(\sum_{i=1}^{b} \alpha_i v_i\right)^{k-l} \\ &:= \sum a_{l_1,l_2,\ldots,l_b}(\alpha)(iv_1)^{l_1}(iv_2)^{l_2}\cdots(iv_b)^{l_b}, \end{aligned}$$

where $l_1 + l_2 + \cdots + l_b$ is odd and $a_{l_1,L_2,\ldots,l_b}(\alpha)$ are constants which depend on α and g. Let $v' = -v$, i.e., $(v'_1, v'_2, \ldots, v'_b) = (-v_1, -v_2, \ldots, -v_b)$, then

$$\begin{aligned} &\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} s_j^{(k)}(\alpha, iv)\, dv \\ &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} \sum a_{l_1,l_2,\ldots,l_b}(iv_1)^{l_1}(iv_2)^{l_2}\cdots(iv_b)^{l_b}\, dv_1 dv_2\cdots dv_b \\ &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(-v',-v')} \sum a_{l_1,l_2,\ldots,l_b}(-1)^{l_1+\cdots+l_b}(iv'_1)^{l_1}(iv'_2)^{l_2}\cdots(iv'_b)^{l_b}\, dv'_1 dv'_2\cdots dv'_b \\ &= -\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v',v')} s_j^{(k)}(\alpha, iv')\, dv'. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} s_j^{(k)}(\alpha, iv)\, dv = 0. \qquad \square$$

By Lemma 4, for k odd the coefficients of $T^{-\frac{k}{2}}$ vanish. So we only need to calculate the coefficients when k is even. Let $b_j^{(k)}(\alpha)$ be the coefficient of T^k in

Proposition 4. If we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then

$$\begin{aligned} b_j^{(k)}(\alpha) &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} s_j^{(2k)}(\alpha, iv)\, dv \\ &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} \sum_{l=0}^{2k} (-1)^l \frac{f_j^{(l)}(iv)\langle \alpha, 2\pi iv\rangle^{2k-l}}{(2k-l)!}\, dv \\ &= \sum_{l_1+l_2+\cdots+l_b=0}^{2k} b^{(j)}_{l_1 l_2 \ldots l_b} \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_b^{l_b}, \end{aligned}$$

where $b^{(j)}_{l_1 l_2 \cdots l_b}$ are constants. More precisely,

$$\begin{aligned} b^{(j)}_{l_1 l_2 \ldots l_b} = \frac{(l_1+l_2+\cdots+l_b)!}{l_1! l_2! \ldots l_b!} \int_{\mathbb{R}^b} & e^{-\frac{1}{2}\beta''(\xi)(v,v)} (2\pi i)^{l_1+l_2+\cdots+l_b} \\ & \times v_1^{l_1} v_2^{l_2} \ldots v_b^{l_b} f_j^{(2k-(l_1+l_2+\cdots+l_b))}(iv)\, dv. \end{aligned}$$

Thus we have proved the following proposition.

Proposition 5.

$$\begin{aligned} & \left| \sum_{k=0}^{2N+1} \frac{1}{T^{k/2}} \int_{\|v\|\leqslant\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(\xi)(v,v)} e^{-2\pi i\langle\alpha, v/\sqrt{T}\rangle} f_j^{(k)}(iv) dv - \sum_{k=0}^{N} \frac{b_j^{(k)}(\alpha)}{T^k} \right| \\ & \qquad \leqslant c \sup_{n\leqslant 4N+b+2} \|y^n g\|_{L^1} \left(T^{-(N+1)} + e^{-\epsilon' T} \right), \end{aligned}$$

where $b_j^{(k)}(\alpha) = \sum_{l_1+l_2+\cdots+l_b=0}^{2k} b^{(j)}_{l_1 l_2 \ldots l_b} \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_b^{l_b}$ *are polynomials in* $\alpha_1, \alpha_2, \ldots, \alpha_b$ *and the degree of* $b_j^{(k)}$ *is* $2k$ *and* $b^{(j)}_{l_1 l_2 \cdots l_b}$ *are constants which depend on* g.

Let

$$\begin{aligned} c_n(\alpha) &= \sum_{i=0}^{n} b_i^{(n-i)}(\alpha) = \sum_{i=0}^{n} \left(\sum_{l_1+l_2+\cdots+l_b=0}^{2(n-i)} b^{(j)}_{l_1 l_2 \ldots l_b} \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_b^{l_b} \right) \\ &:= \sum_{l_1+l_2+\cdots+l_b=0}^{2n} c_{l_1 l_2 \cdots l_b} \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_b^{l_b}, \end{aligned}$$

where $c_{l_1 l_2 \ldots l_b}$ are constants and

$$c_0(\alpha) = b_0^{(0)} = f_0^{(0)} \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} dv = \frac{(2\pi)^{b/2}}{\sqrt{|\det \beta''(\xi)|}} \hat{g}(-ih^*) > 0,$$

so that $c_0(\alpha)$ is independent of α. On the other hand, for $n \geqslant 1$, $c_n(\alpha)$ is polynomial in $\alpha_1, \ldots, \alpha_b$ whose degree is $2n$ by Proposition 5. From the expression of $b_j^{(k)}(\alpha)$, we have

$$\begin{aligned} c_n(\alpha) &\sim \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} f_0^{(0)}(iv) \frac{\langle \alpha, 2\pi iv\rangle^{2n}}{(2n)} dv \\ &= (2\pi i)^{2n} \frac{f_0^{(0)}}{(2n)!} \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(\xi)(v,v)} \langle v, \alpha\rangle^{2n} dv \\ &\sim (-1)^n c_g \|\alpha\|^{2n} \qquad (c_g > 0 \text{ and } c_g \text{ is dependent on } g), \end{aligned}$$

where $A \sim B$ means that $\lim_{\|\alpha\|\to\infty} A/B = 1$. In order to make the result positive, we assume that $K < \frac{1}{\rho}$ so that

$$e^{-\frac{cT}{T^{K\rho}}} \leqslant c\frac{1}{T^N},$$

for any $N > 0$. Now we obtain the following theorem.

THEOREM 1. *Let M be a compact manifold with first Betti number $b > 0$ and let ϕ_t: $M \to M$ be a homologically full transitive Anosov flow. Let g be of class C^∞ with compact support and $g \geqslant 0$. There exist $\xi \in \mathbb{R}^b$, $h^* > 0$ such that for $\alpha \in H_1(M, \mathbb{Z})$, we have*

$$\pi_g(T,\alpha) = \frac{e^{Th^*}}{T^{b/2+1}} e^{-\langle \xi,\alpha\rangle} \left(\sum_{n=0}^{N} \frac{c_{n,g}(\alpha)}{T^n} + O\left(\frac{1}{T^{N+1}}\right)\right) \quad as\ T \to \infty, \tag{9}$$

for any $N \in \mathbb{N}$. If we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then the constants $c_{n,g}(\alpha)$ are in the form

$$c_{n,g}(\alpha) = \sum_{l_1+l_2+\cdots+l_b=0}^{2n} c_{l_1 l_2 \ldots l_d} \alpha_1^{l_1} \alpha_2^{l_2} \cdots \alpha_b^{l_b},$$

where $c_{l_1,l_2,\ldots,l_b}$ are constants which are independent of α.

Remark. If g is of class C^m with compact support and m satisfies $(m-\beta)/\rho \geqslant N + \frac{b}{2} + 2$ for some $N \in \mathbb{N}$, (9) still holds.

4. *The proof of the main result*

In this section we use an approximation argument to obtain $\pi(T,\alpha)$. Let g be a characteristic function. For all T we take g_T^- and g_T^+ of class C^∞ with compact supports such that:

(1) $g_T^- \leqslant g \leqslant g_T^+$;
(2) $\|g_T^+\|_\infty \leqslant 2$ and $\|g_T^-\|_\infty \leqslant 2$;
(3) for $0 \leqslant n \leqslant M$, $\|y^n g_T^\pm\|_{L^1} \leqslant c\|y^n g\|_{L^1}$;
(4) $\sup_{n\leqslant M} \|y^n(g_T^\pm - g)\|_{L^1} \leqslant T^{-\lambda} \sup_{n\leqslant M} \|y^n g\|_{L^1}$ for some $\lambda > 0$.
(5) $\|g_T^{\pm(m)}\|_{L^1} \leqslant cT^{m\lambda}\|g\|_{L^1}$.

These can be done by a convolution argument.

By (7), we have that

$$\left|\pi_{g_T^\pm}(T,\alpha) - e^{-\langle\xi,\alpha\rangle} \sum_{j=0}^{M} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i\langle v,\alpha\rangle} \frac{d^j \widehat{g_T^\pm}}{ds^j}(-i\beta(\xi+iv)) e^{\beta(\xi+iv)T} dv\right|$$
$$\leqslant \frac{c\|y^{M+1} g_T^\pm\|_{L^1}}{T^{M+1}} e^{Th^*} + c\frac{\|g_T^{\pm(m)}\|_{L^1}}{T^{(m-\beta)K}} e^{Th^*} + c\big(\|g_T^\pm\|_{L^1} + \cdots + \|y^M g_T^\pm\|_{L^1}\big) e^{Th^* - \frac{cT}{T^{K\rho}}}$$
$$\leqslant c_0 \frac{\|y^{M+1} g\|_{L^1}}{T^{M+1}} e^{Th^*} + c_1 \frac{\|g\|_{L^1}}{T^{(m-\beta)K - m\lambda}} e^{Th^*} + c_2\big(\|g\|_{L^1} + \cdots + \|y^M g\|_{L^1}\big) e^{Th^* - \frac{cT}{T^{K\rho}}}.$$

By condition (4),

$$\sum_{j=0}^{M} \frac{1}{T^{j+1}} \left| \int_{V_0} e^{-2\pi i\langle v,\alpha\rangle} \left(\frac{d^j \widehat{g_T^+}}{ds^j} - \frac{d^j \widehat{g}}{ds^j}\right) (-i\beta(\xi+iv)) e^{\beta(\xi+iv)T}\, dv\right|$$
$$\leqslant c_3 \sup_{n\leqslant M} \|y^n g\|_{L^1} \frac{e^{Th^*}}{T^{\frac{b}{2}+1}} \frac{1}{T^\lambda}.$$

Hence,

$$\pi_g(T,\alpha) - e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{M}\frac{1}{T^{j+1}}\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\widehat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv$$

$$\leqslant \pi_{g_T^+}(T,\alpha) - e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{M}\frac{1}{T^{j+1}}\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\widehat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv$$

$$\leqslant \pi_{g_T^+}(T,\alpha) - e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{M}\frac{1}{T^{j+1}}\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\widehat{g_T^+}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv$$

$$+ e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{M}\frac{1}{T^{j+1}}\left|\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\left(\frac{d^j\widehat{g_T^+}}{ds^j}-\frac{d^j\widehat{g}}{ds^j}\right)(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv\right|$$

$$\leqslant c_0\frac{\|y^{M+1}g\|_{L^1}}{T^{M+1}}e^{Th^*} + c_1\frac{\|g\|_{L^1}}{T^{(m-\beta)K-m\lambda}}e^{Th^*}$$

$$+ c_2(\|g\|_{L^1}+\cdots+\|y^Mg\|_{L^1})e^{Th^*-\frac{cT}{T^{K\rho}}} + c_3\sup_{n\leqslant M}\|y^ng\|_{L^1}\frac{e^{Th^*}}{T^{\frac{b}{2}+1}}\frac{1}{T^\lambda}.$$

Taking $M\geqslant\frac{b}{2}+1+\lambda$ and $g=\chi_{[-T^{1/4M+2},0]}$, then

$$\left|\pi_g(T,\alpha) - e^{-\langle\xi,\alpha\rangle}\sum_{j=0}^{M}\frac{1}{T^{j+1}}\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\widehat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv\right|$$

$$\leqslant e^{Th^*}\left(\frac{c_0}{T^M}+\frac{c_1}{T^{(m-\beta)K-m\lambda-1}}+c_2e^{-\frac{cT}{T^{K\rho}}}+\frac{c_3}{T^{\frac{b}{2}+\lambda}}\right).$$

Let $\zeta=\min\{\lambda-1,(m-\beta)K-m\lambda-\frac{b}{2}-2\}$, where $K<1/\rho$. Since we can take M sufficiently large, the best error term is $1/T^\zeta$. By $\lambda-1>0$ and $(m-\beta)K-m\lambda-\frac{b}{2}-2>0$, we have $\lambda<K<1/\rho$ if we let $m\to\infty$. We assume that $\rho<1$ and let $\delta=[\frac{1}{\rho}]-1$. We do the same as was done in the last section for

$$\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}(d^j\widehat{g}/ds^j)(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv$$

and obtain

$$\sum_{j=0}^{M}\frac{1}{T^{j+1}}\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\widehat{g}}{ds^j}(-i\beta(\xi+iv))e^{\beta(\xi+iv)T}dv$$

$$=\frac{e^{Th^*}}{T^{\frac{b}{2}+1}}\left(c_0+\sum_{k=1}^{N}\frac{c_{k,T}(\alpha)}{T^k}+O\left(\frac{1}{T^N}\right)\right).$$

Since for $g=\chi_{[-T^{\frac{1}{4M+2}},0]}$, $\hat{g}(-i\beta(\xi+iv))=g(\xi+iv)+O(e^{-T^{\frac{1}{4M+2}}h^*})$, hence

$$c_{k,T}(\alpha)=c_k(\alpha)+O\big(e^{Th^*-T^{\frac{1}{4M+2}}h^*}\big).$$

Hence

$$\pi_g(T,\alpha)=\frac{e^{Th^*}}{T^{b/2+1}}\left(c_0+\sum_{k=1}^{N}\frac{c_k(\alpha)}{T^k}+O\left(\frac{1}{T^N}\right)\right).$$

It is well known that

$$\pi(T - T^{\frac{1}{4M+2}}, \alpha) = O\big(e^{Th^* - T^{\frac{1}{4M+2}} h^*}\big),$$

and

$$\pi\big(T - T^{\frac{1}{4M+2}}, \alpha\big) + \pi_{\chi_{[-T^{\frac{1}{4M+2}},0]}}(T, \alpha) = \pi(T, \alpha).$$

However, $e^{-h^* T^{1/4M+2}} \to 0$ faster than $1/T^n$ for any n. We have the following theorem.

THEOREM 2. *Let M be a compact manifold with first Betti number $b > 0$ and let $\phi_t: M \to M$ be a homologically full transitive Anosov flow. There exist $\xi \in \mathbb{R}^b$, $h^* > 0$ and $\delta > 0$ such that for $\alpha \in H_1(M, \mathbb{Z})$, we have*

$$\pi(T, \alpha) = \frac{e^{Th^*}}{T^{b/2+1}} e^{-\langle \xi, \alpha \rangle} \left(c_0 + \sum_{n=1}^{N} \frac{c_n(\alpha)}{T^n} + O\left(\frac{1}{T^\delta}\right)\right) \quad \text{as } T \to \infty \tag{10}$$

for $N < \delta$, where $c_0 > 0$ is a constant which is independent of α. If we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then the constants $c_n(\alpha)$ are in the form

$$c_n(\alpha) = \sum_{l_1+l_2+\cdots+l_b=0}^{2n} c_{l_1 l_2 \ldots l_d} \alpha_1^{l_1} \alpha_2^{l_2} \cdots \alpha_b^{l_b},$$

where $c_{l_1, l_2, \ldots, l_b}$ are constants which are independent of α.

Remark. As we see in [**5**] for closed geodesics, we still have that

$$c_n(\alpha) \sim (-1)^n c \|\alpha\|^{2n} \quad (c > 0) \text{ as } \|\alpha\| \to \infty.$$

Acknowledgements. (1) After obtaining the results of this paper, the author learned that Motoko Kotani [**4**] had studied the same question for closed geodesics and also obtained how the coefficients $c_n(\alpha)$ of the error terms depend on the homology class α for geodesic flows.
(2) The author would like to thank Richard Sharp, for his suggestion of the field of study, fruitful discussions and encouragement, and Mark Pollicott for his advice and encouragement. The author also wishes to thank CVCP and Mathematics Department of Manchester University for their financial support.

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